Dealing with Materials Data:
Collection, Analysis and Interpretation
Professor Hina A Gokhale
Department of Metallurgical Engineering and Materials Science
Indian Institute of Technology, Bombay
Lecture 70
Hypothesis Testing III

Hello, and welcome to the course on dealing with materials data. Last few sessions we have been working with statistical process called hypothesis testing, it is a part of statistical inference in which we try to inform more about the population that we are trying to understand.

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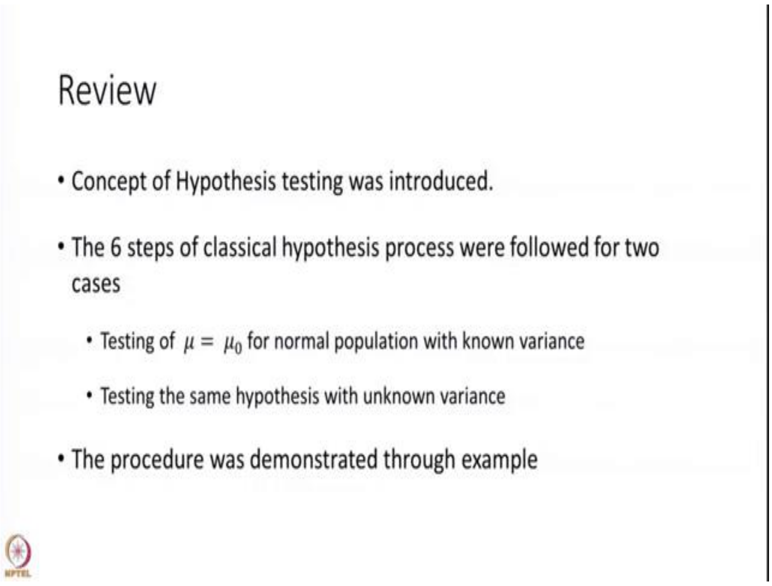

So far, we introduced the concept of hypothesis testing begins six steps of classical hypothesis process, and we followed the steps to derive the hypothesis testing procedures under the assumption that the population is normal be considered two cases. One is when the variance of the population is known, and the other when the variance for populations unknown. And both the times we tried to test the hypothesis, the null hypothesis that the mean of the sample, mean of the population. I am sorry mean of the population is equal to a fixed value Mu 0.

In both the cases, we also demonstrated it through an example. And we found that when variance is known it is the standard normal deviate Z, which is equal to the sample mean minus Mu 0 divided by standard deviation over square root N is the test statistic. And when sigma square is unknown X bar that is the sample mean minus the Mu 0 divided by sample standard deviation over square root N, which is distributed as T with N minus 1 degrees of freedom is the test statistic.

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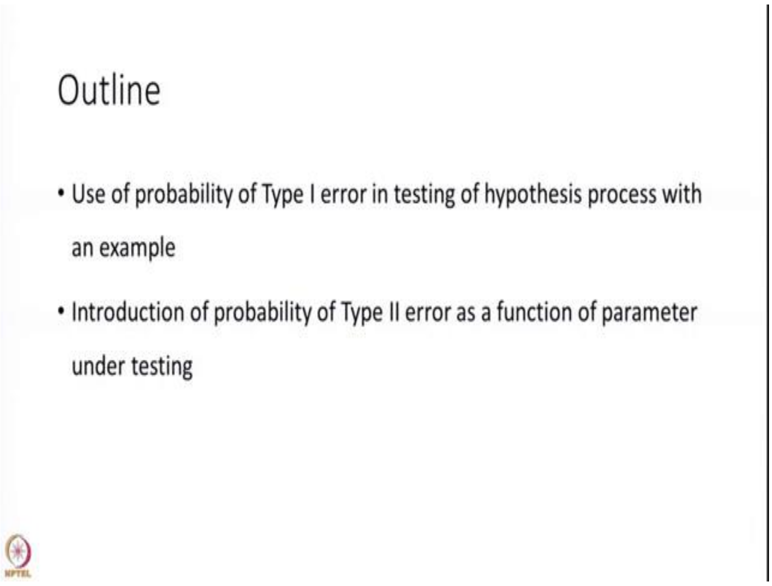

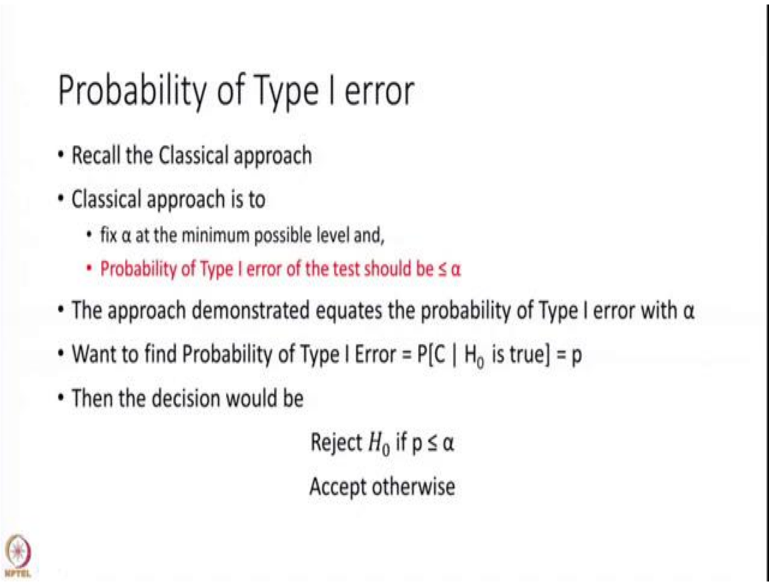

In this particular session, we would like to talk about the type one error in testing of hypothesis process and talk about type two error as a function of parameter under testing. So, far, what we have done is we have set up the procedure, the six steps of classical testing, we define the critical region by equating it to the fixed value type one error alpha. But if you look at the classical approach, it says that fix the level alpha at minimum possible level. And then the probability of type one error should be actually less than or equal to alpha.

So far, we have equated it with the alpha, it does not make any difference, but we would like to set up a test procedure in which we actually calculate the probability of type one error and we tested with alpha. So, you remember, we had a critical value in the previous approach, and we were comparing the test statistic with the critical value. Here we would like to compare the probability with the alpha value that is with the fixed value for which we want to have your type one errors smaller than that fixed value.

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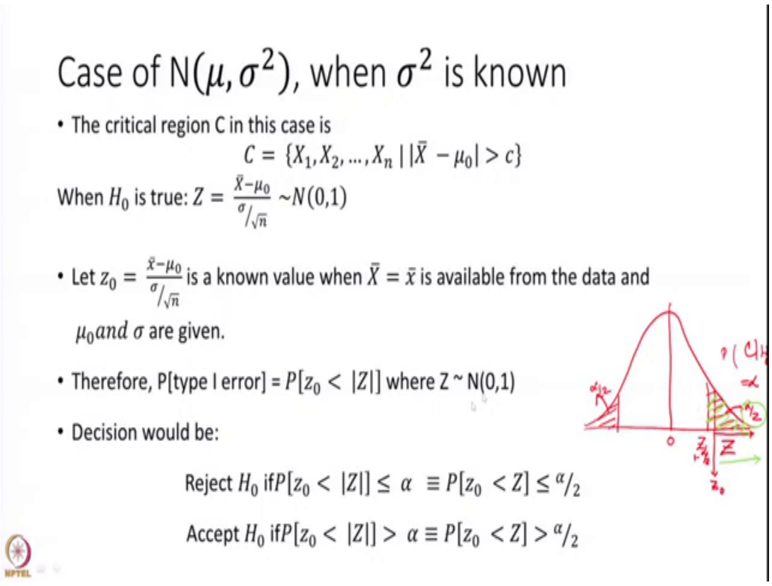

So, let us start, we again take the same two cases, we assume that the population is normal mean is unknown, sigma square, that is the variance of the population is known, this is one case, when it is unknown, it is a second case.

The critical region C in this case is

$$C = \{X_1, X_2, \dots, X_n \mid |\bar{X} - \mu_0| > c\}$$

$$\text{When } H_0 \text{ is true: } Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

And all the side-lines, I would like you to tell you that this assumption of normality gives a very beautiful closed form solution.

So, it is very easy to understand this procedure, but we will also consider a case in which it is not a normal distribution and still, the same statistic can also be utilized with some more calculations to arrive at the similar testing of hypothesis procedure. So, we start our test statistic is Z which is a $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

It is a parameter without any unknown parameter, it is a normal distribution without no with known parameters, 0 and 1. Let us call $z_0 = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ is a known value when $\bar{X} = \bar{x}$ is available from the data and $\mu_0 and\ \sigma$ are given. So, if known value of X bar is small x bar is available from the data and Mu 0 and sigma are given to you. Then this is a known quantity, it is a number.

Therefore, type one error we actually = $P[z_0 < |Z|]$ where Z ~ N(0,1) this critical region will be then we will say that reject H_0. If probability that z_0 is smaller than absolute, random variate Z is less than or equal to alpha or equivalently we can say that probability of z_0 less than Z the random variable is less than or equal to alpha by 2. Shall we repeat this once again, why it becomes alpha by 2? Because here is where I find many people tend to get confused.

So, I am sorry if it is being too much of a reputation, but my experience is that it is worth repeating it. So, we will have Z is the standard normal distribution with a means 0 and here is what we are looking for this area and this area to be the critical region. Since this is symmetric, if the probability of critical region under H 0 is alpha, then each of this region has to be alpha by 2, this is first part.

So, that is why we have come that if you want to take only Z greater than that is the random variable Z greater than small z_0 This is what we would like to have alpha or smaller. Remember that it will have alpha value if this is Z alpha by 1 minus alpha by 2 if you recall the previous session. So, this whole area maybe I should change the color of the pen. Let us make it green. So, this whole area is alpha by 2 and this is the area we are saying that should be smaller than alpha by 2. So, then we continue.

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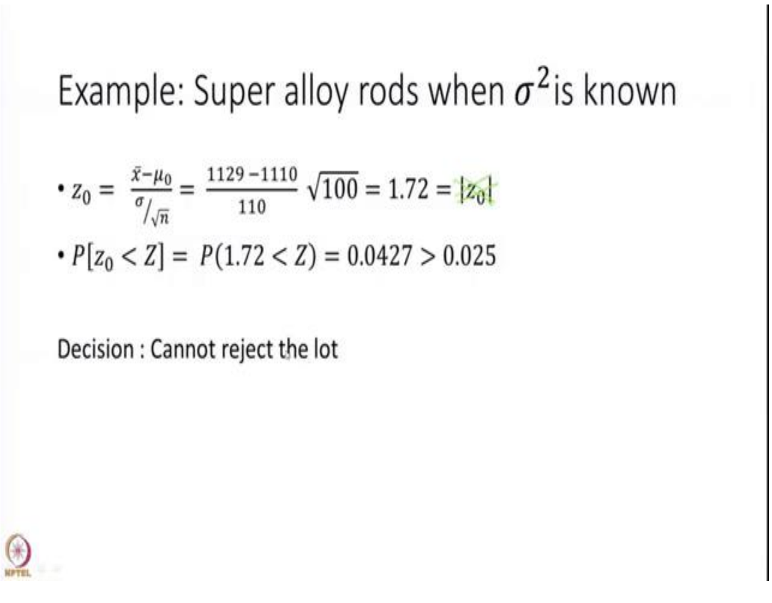

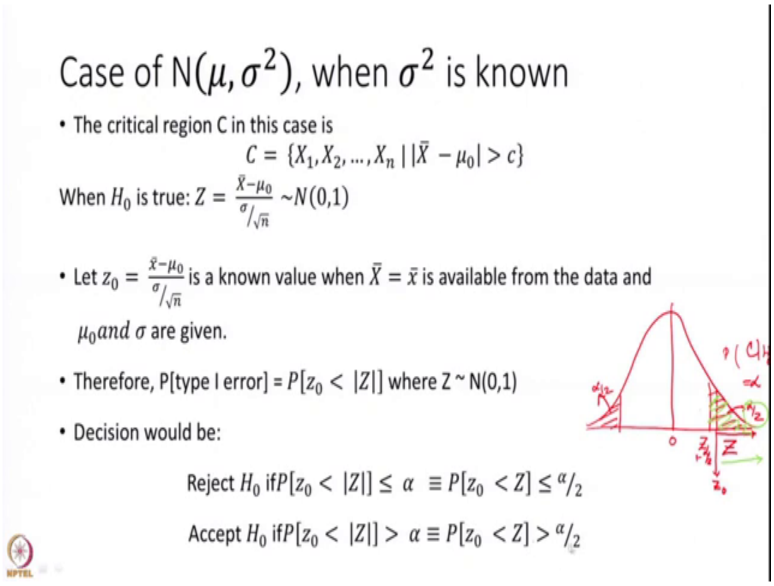

So, if we take the example, I am not going to repeat the statement of the example you can take it up from the previous session, but then z_0 turns out to be 1.72. And I am sorry this is the miss error. I do not want to write this. So, please remove this. And so, we have probability to have Z 0 smaller than Z, which is probability to have 1.72 smaller than Z.

And that probably turns out to be 0.04727, which is much greater than 0.025, which is alpha by 2. And therefore, we cannot reject the lot, we have not enough evidence to reject the lot, we have to accept it. This comes out from the previous slide

Reject H_0 if $P[z_0 < |Z|] \le \alpha \equiv P[z_0 < Z] \le \alpha/2$

Accept H_0 if $P[z_0 < |Z|] > \alpha \equiv P[z_0 < Z] > \alpha/2$ and that is what we find here.

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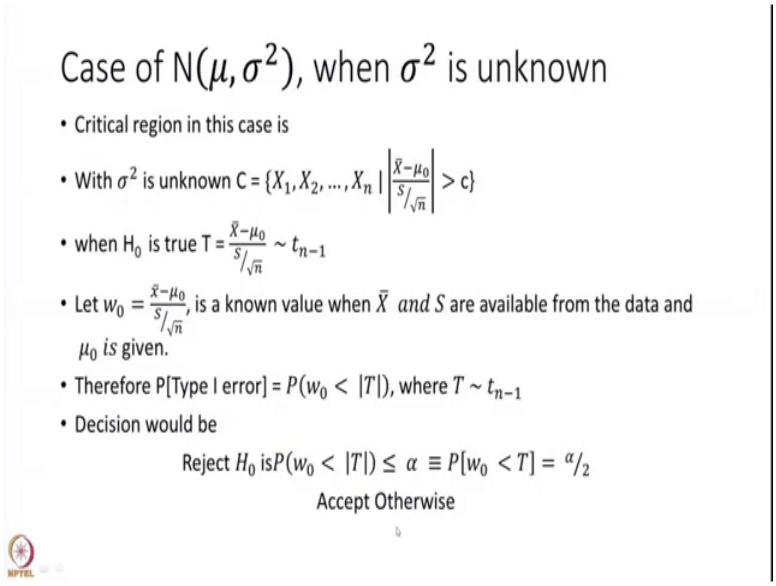

If sigma square is unknown, it is the same thing except that it becomes a T statistic. Because the sigma other way standard deviation gets replaced by sample standard deviation. Then it follows a T distribution and it becomes a T random variable. And therefore, if you take a T 0 or W 0, as it is a known quantity given. So, type one error becomes that W 0 because we have already assumed H_0 is true I mean sorry, null hypothesis to be true. So, we have already put a Mu 0 here, and therefore this is the type one error probability. And that has to be is equal to alpha or this is the type one probability, where T is distributed as a T distribution with the N minus 1 degrees of freedom.

With σ^2 is unknown C = $\{X_1, X_2, \ldots, X_n \mid \left|\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\right| > c\}$

- **when H_0 is true T = $\frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim t_{n-1}$**
- **Let $w_0 = \frac{\bar{x}-\mu_0}{S/\sqrt{n}}$, is a known value when $\bar{X}$ and S are available from the data and μ_0 is given.**
- **Therefore P[Type I error] = $P(w_0 < |T|)$, where $T \sim t_{n-1}$**
- **Decision would be**

Reject H_0 is $P(w_0 < |T|) \leq \alpha \equiv P[w_0 < T] = \alpha/2$

Accept Otherwise

So, we say that reject the null hypothesis if the W 0 is less than absolute value of random variable T. That probability has to be less than alpha equivalently in the same argument you can say that the probability of W 0 less than the random or the let us put it the other way around. The random variable T has to be greater than W 0. So, probability that random variable T is greater than W 0 is one minus alpha that is the correct way to put it, otherwise, you accept the hypothesis.

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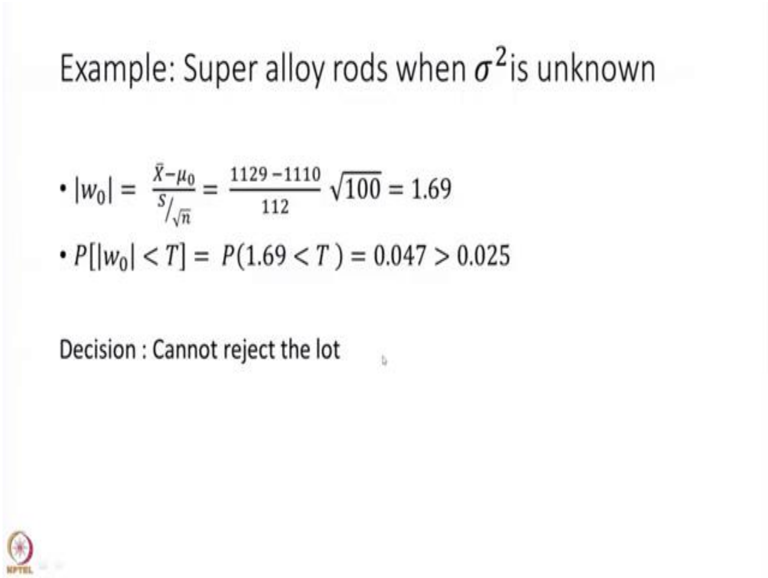

Once again, if we go to the super alloy rods example. Now, you know that in the second case, we had taken sample variants or sample standard deviation as 112 MPA. So, this value turns out to be 1.69. So, probability that a T randomly variable with N minus 1 degree of freedom is larger than 1.69 is 0.047, which is definitely greater than 0.025. And therefore, we cannot reject a lot, we do not have sufficient evidence to reject a lot.

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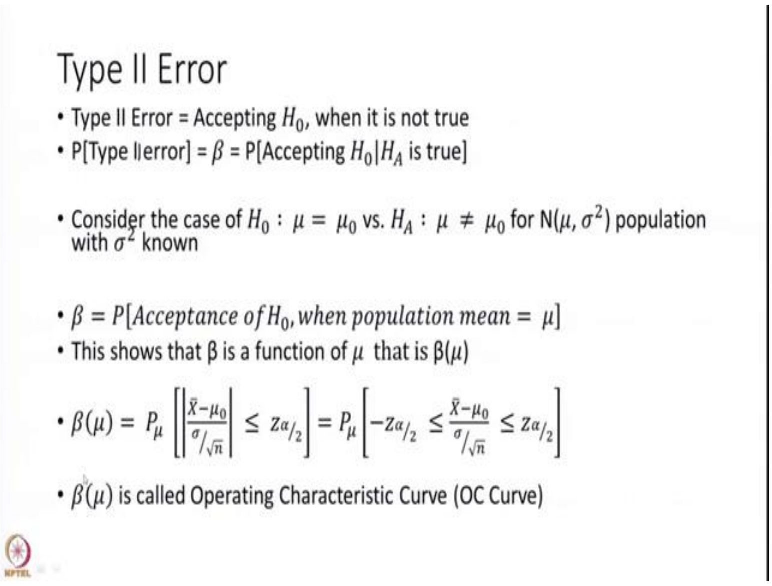

Let us talk about type two error, type two error is accepting the null hypothesis when in reality it is not true. So, probability that type one, sorry it should be type two. Let us correct it here and change the colour of the pen to black this has to be type two error, please correct it, it has to be a type two error. So, that lead then we again consider the case of normal population with sigma square known.

Consider the case of $H_0 : \mu = \mu_0$ vs. $H_A : \mu \neq \mu_0$ for N(μ, σ^2) population with σ^2 known

$$\beta = P[Acceptance\ of H_0,\ when\ population\ mean = \mu]$$

So, beta is probability of acceptance of H 0, when population means not Mu naught, it is some other Mu. It means that beta is a function of Mu because please remember, Mu is equal to Mu 0 completely defines everything. Here, when you say that Mu is not equal to Mu 0, it does not completely define and therefore, it is, it becomes the, type two error becomes a function of mean and mean value which is not equal to Mu naught.

And therefore, it can be expressed

$$\beta(\mu) = P_\mu \left[\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| \leq z\alpha_{/2}\right] = P_\mu\left[-z\alpha_{/2} \leq \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \leq z\alpha_{/2}\right]$$

This shows that β is a function of μ that is β(μ)

$\beta(\mu)$ is called Operating Characteristic Curve (OC Curve)

Because of our critical region is coming through test statistic and you are making it acceptance. This is called an operating characteristic curve, which has argument Mu and the response meter.

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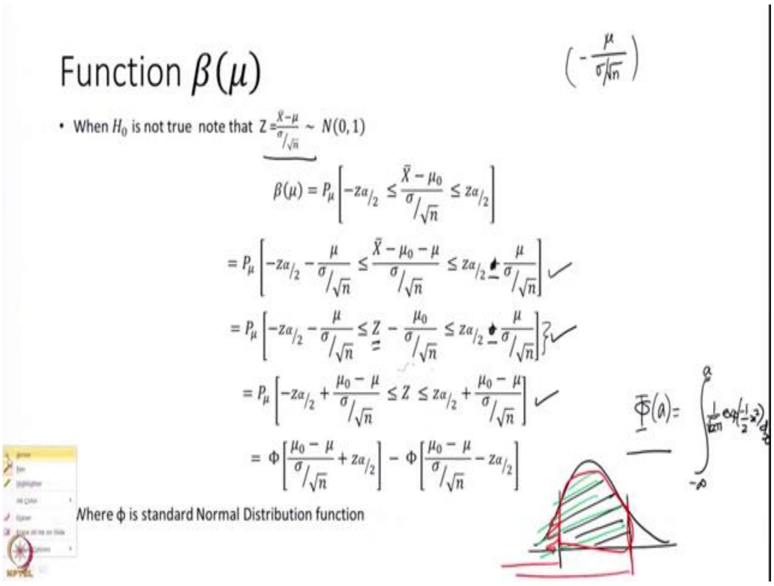

If you look at this function beta of mu, when H naught is not true, Z is equal to X bar minus Mu over sigma over square root n is distributed, where Mu is not equal to Mu naught, it is distributed as a standard normal variate. So, we can derive it from here, this is the probability as derived earlier, we have added or rather we have subtracted from every side. The Mu divided by sigma square root N and I think there is an error here and let me correct it. Yes, there is an error here, there are lots of errors. So, let us start correcting it here.

We have to subtract it in all the sites. So, this has to be minus. So, we have subtracted everywhere Mu over sigma square root n, this is subtracted from all the sites. So, this has to be minus and then what, then what we find is I think this step there is an error, this step there is an error let us do it here.

When H_0 is not true note that $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

$$\beta(\mu) = P_\mu\left[-z\alpha/_2 \le \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \le z\alpha/_2\right]$$

$$= P_\mu\left[-z\alpha/_2 - \frac{\mu}{\sigma/\sqrt{n}} \le \frac{\bar{X}-\mu_0-\mu}{\sigma/\sqrt{n}} \le z\alpha/_2 - \frac{\mu}{\sigma/\sqrt{n}}\right]$$

$$= P_\mu\left[-z\alpha/_2 - \frac{\mu}{\sigma/\sqrt{n}} \le Z - \frac{\mu_0}{\sigma/\sqrt{n}} \le z\alpha/_2 - \frac{\mu}{\sigma/\sqrt{n}}\right]$$

$$= P_\mu\left[-z\alpha/_2 + \frac{\mu_0-\mu}{\sigma/\sqrt{n}} \le Z \le z\alpha/_2 + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}\right]$$

$$= \Phi\left[\frac{\mu_0-\mu}{\sigma/\sqrt{n}} + z\alpha/_2\right] - \Phi\left[\frac{\mu_0-\mu}{\sigma/\sqrt{n}} - z\alpha/_2\right]$$

Where φ is standard Normal Distribution function

So, you get this so, this is correct, this is correct with this correction, this is correct with this correction and therefore, it is Phi, remember Phi is, Phi of A is another notation. Where minus infinity to A 1 over square root two Pi, exponential one-half X square dx is called Phi of A is a function, it is a cumulative distribution function of standard normal variate.

So, this shows the cumulative distribution function of standard normal variate. So, what it says is that, it says that the area under this curve can be shown as area in this curve minus the area in this curve. So, that is what is being shown here. It takes the full area here, minus this area is this central area is what this equation shows.

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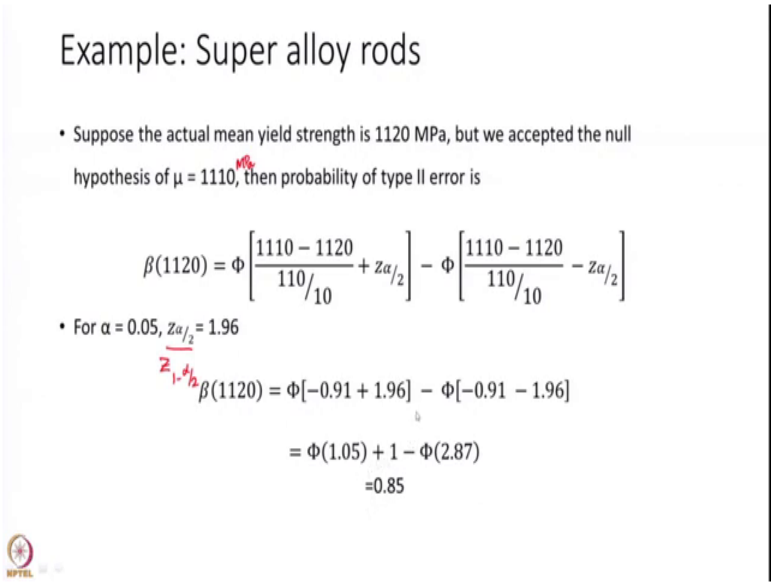

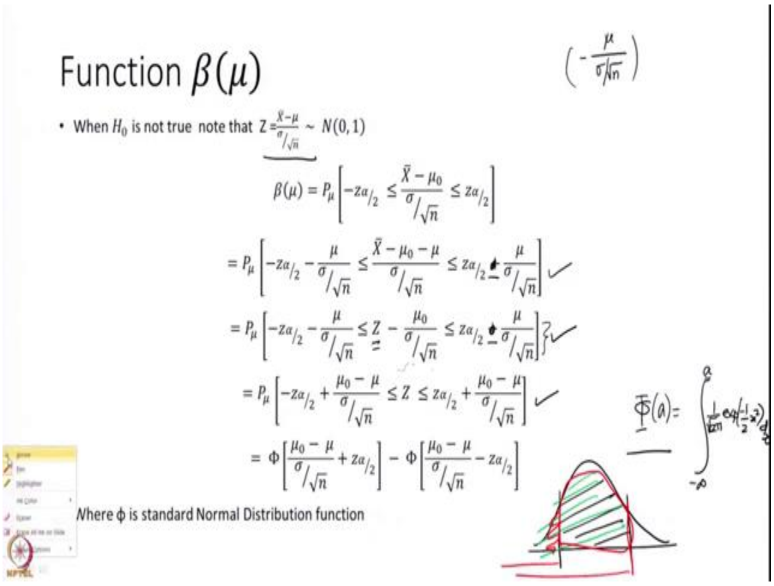

So, if you take the super alloy rods area, I mean the case and you say that suppose the actual mean or yield strength is hundred and 1,120 MPa and but we accepted the null hypothesis, you remember in the previous case we said that the null hypothesis is acceptable it means that you have accepted that Mu 0 is 1110 MPa I have forgotten to put the unit here please make sure this is MPa.

So, in this case what is the type one error we have committed. So, we calculated

- Suppose the actual mean yield strength is 1120 MPa, but we accepted the null hypothesis of $\mu = 1110$, then probability of type II error is

$$\beta(1120) = \Phi\left[\frac{1110-1120}{110/10} + z\alpha/_2\right] - \Phi\left[\frac{1110-1120}{110/10} - z\alpha/_2\right]$$

- For α = 0.05, $z_{1-\alpha/2}$ = 1.96

$$\beta(1120) = \Phi[-0.91 + 1.96] - \Phi[-0.91 - 1.96]$$

$$= \Phi(1.05) + 1 - \Phi(2.87)$$

$$=0.85$$

And therefore, we get the type two error is 0.85 which is actually is very high. One of the reasons could be you will see that this yield strength is not near, really normal and we are comparing it with the standard normal variate and therefore, it might be giving us this value anyway.

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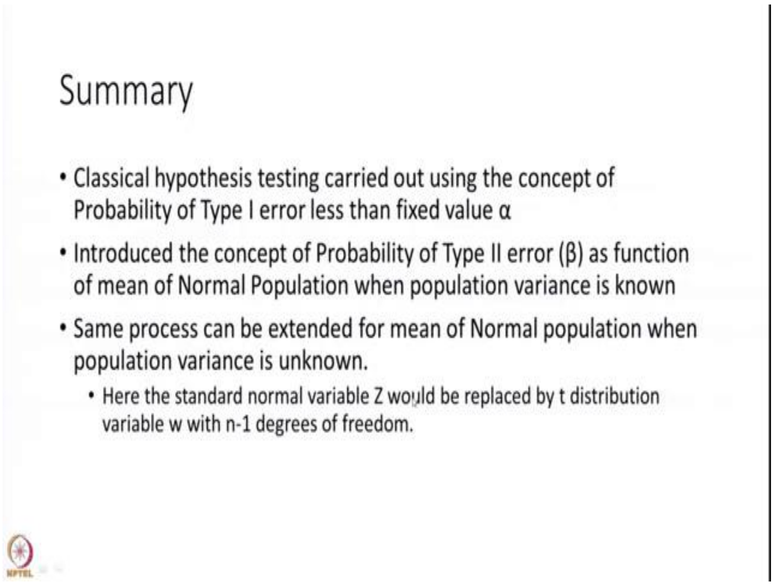

This was just an example to demonstrate how the type two error is to be calculated. So, let us summarize. We carried out the classical hypothesis testing procedure by fixing the type one error to alpha. In this case, what we did is we actually calculated out the type one error and we showed it that if we set up the decision procedure that you reject the null hypothesis if the type one error is less than or equal to alpha.

And you accept the null hypothesis if it is greater than alpha. We introduced also I mean reintroduce the concept of type two error as a function of mean of normal population when variance is unknown, variances known. Please note that this becomes, this mean is not the same as Mu 0, this is under the alternate hypothesis and the same process you can extend it for the case when the population variance is unknown instead of dealing with standard normal deviate Z, you will be working with a T deviate with degrees of freedom n minus 1. Thank you.