NPTEL
NATIONAL PROGRAMME ON
TECHNOLOGY ENHANCED LEARNING

IIT BOMBAY

CDEEP
IIT BOMBAY

Phase field modeling; the materials science, mathematics and computational aspects

Prof. M P Gururajan
Department of Metallurgical Engineering
And materials Science, IIT Bombay

Module No.11
Lecture No.46
Transformation laws

Welcome we are looking at the definition of scalars and vectors and we are trying to see how this notion can be generalized, scalars are quantities which are just represented by magnitude they are just numbers vectors are quantities which have magnitudes and directions so this is the traditional definition which is correct of course but it has the and it is also very straightforward easy to understand but it makes it very difficult to define what a second rank tensor ease.

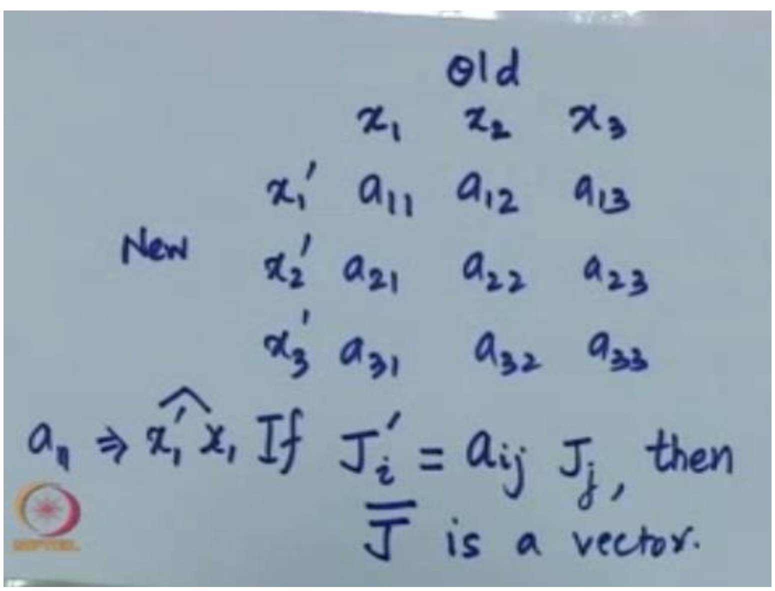
Because it is difficult to explain kind of these kind of terms as to what a second rank tensor is, so it is possible to generalize and for doing that the first thing we do is to define the components of a vector so we now consider that vector is a quantity with magnitude and direction and you can represent this information of its magnitude and direction by resolving it along the coordinate axis so we define an axis a system of axis according it frame of reference.

And we take this vector we resolve it along each one of those directions and we give the components and that can now be represented by either in a matrix form as a column vector or in a symbolic form by giving an index, so now we can define scalars as quantities with no indices vectors as quantities with one index then that naturally generalizes to second-rank tensors are quantities with two indices third-rank tensors are quantitative three indices and so on, so you can define an N^{th} rank tensor which has n indices for example, now this also is problematic because we know that vectors exist without reference to any reference frame by frame of reference is our way of describing a vector.

So we do not want our description to change the way we look at any vector or we want to demand that different ways of representing the vector using different frames of reference they should be consistent with each other this is possible only because the vectors have their own existence so they do not change and that helps us that define vectors in it another way which again is very favorable for the generalization.

That says that take the vector take your first frame of reference and resolve it along the directions and it has some components, now consider the new frame of reference take the same vector and to find the components along the new coordinates and take this quantity see if these two are related through what is known as a transformation matrix so we define the transformation matrix so I am going to redo it once more just to make clear.

(Refer Slide Time: 03:29)



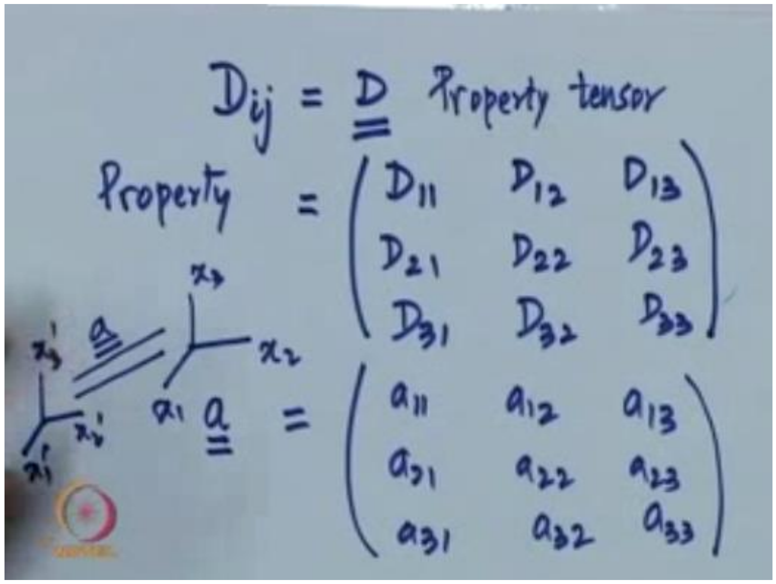
So you have a old frame of reference which is defined as x1 x2 x3 so we are going to mostly deal with a three dimensional space and Cartesian vectors and there is a new frame of reference which is defined as x1' X2 'and X 3' the direction cosine a_{11} for example a_{11} is nothing but the direction cosine of X1' with X 1 right and so one first one represents this prime quantity and the second one represents the unprimed or old quantity.

So in a similar fashion we define a_{12} a_{13} a_{21} $a._{22}$ a_{23} a_{31} a_{32} a_{33} this we call as a transformation matrix and now if you have a vector J which is represented in the new frame of reference let us say that its components are J_i 'then you know the components of this vector in the old frame of reference I think yesterday I made a mistake here it should be J_j these are the components in the old frame of reference these are the components in the new frame of reference.

If they are connected through this relationship $a_{i\,j}$ J_j is $j_j^{'}$ then if that is the condition we say then we say J is a vector, remember this is possible only because J has its own existence so we take two different frames of reference we take the same J we find out its components in these two frames of reference and once we know the two frames of reference we also know how they are connected through this transformation matrix so we see if the components we get by resolving in the new frame of reference the same vector.

They happen to have this relationship with the components from the old frame of reference if it says then we call j as a vector okay, so this definition automatically helps us generalize to higher order tensors also we will see an example especially with respect to the diffusivity and flux and concentration gradient scenario that is taking the ficks first law we will try to understand, but while doing that it is also important because we said that the diffusivity which is a second-rank tensor right.

(Refer Slide Time: 06:21)



We said that D_{ij} is nothing but a second rank tensor called diffusivity and this also had a matrix representation which was like D_{11} D_{12} D_{13} D_{21} D_{22} D_{23} and D_{31} D_{32} D_{33} now if you look at so this is also a matrix but this is representation of a tensor and the transformation matrix the A is also a matrix it also has this same structured but these three these two are completely different entities.

This is a property, okay diffusivity is a property so that D is a property tensor okay there are also tensors which are not property tensors we will discuss them by now you might have guessed for example Flux is a tensor is a first rank tensor but that is not a property concentration gradient is the first rank tensor but that is not a property okay but the diffusivity is a property right, so a tensor which represents a property is a property denser.

Because this is a property tensor this exists on its own okay now when we are referring to some particular frame of reference we might be able to write this diffusivity in this matrix form but the existence of D is independent of whether we are able to represent it in this form or some other form on the other hand the transformation matrix A necessarily depends on having two different frames of reference.

The way to think about it is that I have a frame of reference that is X_1' X $_2$' X_3' and I have another frame of reference x_1 x_2 x_3 A is a quantity which is like the bridge between the two, now if you remove this A does not have any resistance if you remove this A does not have an existence, okay if you do not have any frame of reference A does not have any existence A is something that straddles one frame of reference and another frame of reference.

Its existence is completely dependent on first being able to define two different frames of reference only when they are defined you can define A, on the other hand D is a property tensor that is completely defined for a given particular system now whether it becomes this or some D_{11}' D $_{12}$ 'D 21' etc depends on the frame of reference that you choose, even then for example I might have only this frame of reference.

D is completely defined in that frame of reference or I might have only this frame of reference and D is completely defined in that frame of reference when you have these two then you have 2 D's defined and they are related to each other through our relationship which is what this A is all about, so we need to think of A as some quantity which connects the two frames of reference so it acts like a bridge or an entity which talks to these two frames of reference.

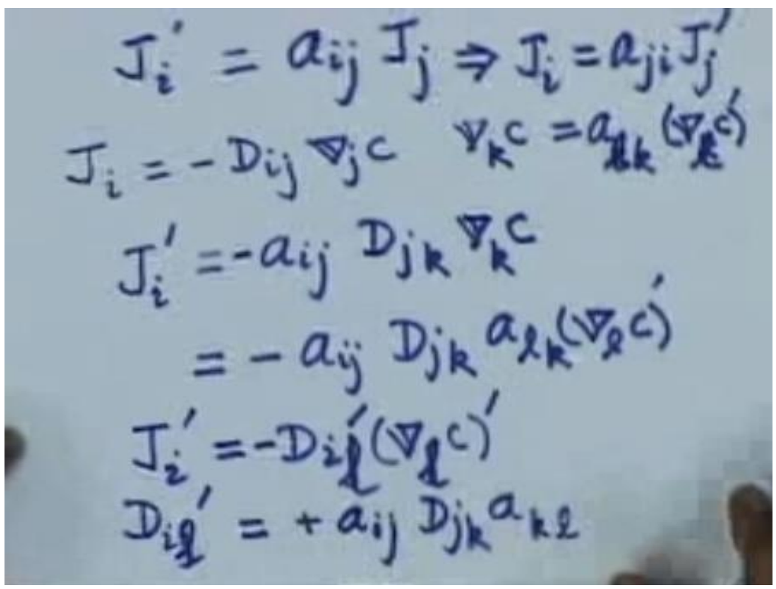
And make sure that things that we define in one place can be translated into things that you define in the other place but it does not have an existence of its own on the other hand property tensors have an existence of their own and that existence is independent of your even defining a frame of reference, so this Is a very important distinction that we need to make transformation matrices are different from property tensors.

Transformation matrices are matrices irrespective of whether you are defining second rank tensor, third-rank tensor, fourth rank tensor they are just matrices which have this form in three dimensions they are three by three matrices on the other hand if you have a third rank tensor for example it does not have a matrix representation in the same way as a second rank tensor has, so second-rank tensors have a matrix representation.

Which in three dimensions looks as similar to that of a transformation matrix but these two are very different types of entities they just happen to have a matrix representation in both the cases okay so it is very important not to confuse between a property tensor which is represented as a 3 by 3 matrix and a transformation matrix which is used to define the relationship between two different frames of reference which is also defined as a 3 by 3 matrix, okay.

So now that we have done it let us try to see how to generalize the notion of vector to a second rank tensor, okay. What is our new definition, our new definition says that a vector is a quantity which transforms under a coordinate transformation according to a given set of rules what did that rule say.

(Refer Slide Time: 11:48)



Suppose if you had the Ji’ as the representation of the j vector in the prime the frame of reference it is related to the unprimed components of the same vector J through the transformation matrix AJ so anything that obeys this would be called as a vector, now if that is so what is a second rank tensor in terms of a_{ij} can we define what is a second rank tensor in terms of the transformation matrix.

The answer is yes we can do and that is what we are going to be, okay so we know that in the regular frame of reference ji is nothing but $–D_{ij} \nabla_j C$ so now we are going to replace for the JJ please remember I mean this dummy index thing you have to keep in mind so you can because this is ji you cannot put D_{ij} here because this I this I then will get a summed over according to Einstein summation convention.

Whereas in this equation I is to be a free index because in the left-hand side I is a free index so that has to be retained, so this is where you have to use some other dummy index when you replace for J let us do it, okay a_{ij} jj is nothing but so $-D_{jk} \nabla_k C$ right $D_{J\,k} \nabla_k C$ is nothing but JJ right this will get summed over and that is JJ and JJ with a_{ij} will give you ji, right so this is so we are we are looking at this quantity ji ‘.

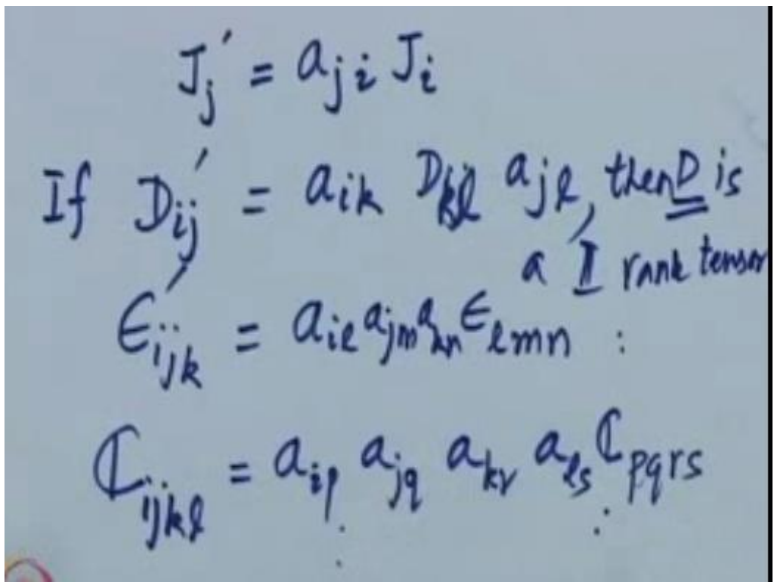
Which is a_{ij} JJ and JJ is getting replaced by this quantity and if Ji’is a_{ij} JJ you can also represent by the same token j_i in terms of okay so like the components in the new frame of reference are related to components in the old frame of reference through a_{ij} you can also relate the components in the old frame of reference to the components in the new frame of reference using this.

So now $\nabla_K C$ I am going to use this kind of relationship right so I am going to say that $\nabla_K C$ in terms of new frame of reference will is going to be so it should connect to $\nabla_K C$ ‘ it cannot be k because k is going to come here in AK so I am going to say $a_{lk} \nabla_L C$ okay L and L will get summed over k is what will be left so I am going to replace this DK by - $a_{ij}\ D_{jk}\ a_{lk} \nabla_L C’$ right that is the prime quantity.

Now you can see what is it that I have got ji’ let us say is equal to $D_{ij} \cdot \nabla_J C’$ right if I have to write it like this then I can see that this quantity okay, now I need to be careful it is so these are dummy indices you can write it in whichever way you want but k is summed over L is summed over so L is the free index that is left and j is summed over so it is $i_L \nabla_L$ right remember this is a dummy index it could be any then make up.

So what does this say this says that d_{ij} lie L' for example is nothing but $-a_{ij}$ D_{ij} a_{kl} right JJ will be summed over kk will be summed over so you will have il left, so you now define second-rank tensors as quantities know that this minus sign is not effect so this is because J is minus this so minus is absorbed so D_{il} is nothing but this quantity, so now our definition so generalized definition they were to say. That scalars are quantities which do not change under a coordinate change.

(Refer Slide Time: 17:14)



Now vectors let us say like Jj' or quantities which change like this second-rank tensors like $D_{ij'}$ are quantities which change like this D_{ij} I have to say D_{KL} for example a_{ik} a_{JL} right LL will get summed over kk will get summed over ij will be left behind so second-rank tensors are quantities which transform when you do a coordinate frame change in such a way that you have to take the old components and you have to multiply by the transformation matrix twice.

Similarly I am going to define a third rank tensor let us say some ε_{ijk} as one which transforms according to the following rule a_i so this is prime ε_{lmn} a_{il} a_{jm} a_{kn} right so depending on how many times you multiply by the transformation matrix if you multiply by 0 times that is the scalar if you multiply by one time that is a vector if you multiply by 2times that is the second rank tensor if you multiply by three times that is the third rank tensor.

Now I can define a fourth rank tensor ijkl = a a a a C_{PQRS} so it is LS, KR, JQ, IP right so PQRS will get some jkl will come so this now defines a fourth rank tensor and so on okay, so this is the third definition that we have for vectors which is again generalizable so first definition said that vectors are quantities which have magnitude and direction same information but was put in a different language when we said that vectors are quantities with the single index when you refer to a specific frame of reference.

Now we are defining vectors as quantities which have its components transformed in a particular fashion when a coordinate transformation is done, both this index definition and this coordinate frame change definition help us generalize to higher order okay, so we have now shown that for example second-rank tensors transform according to this law so we go back and then we make it a definition.

If this then D is a second rank tensor okay and so on so for everything now we can have a definition which is based on this coordinate changes right transformation matrices, okay. So this is how we generalize the notion of a vector to first rank tensor so that it can then help us define higher order tensors okay so we have defined it in two different ways one definition says that the rank of a tensor is defined by the number of indices that is required to represent that quantity in any given frame of reference.

And the second one says that under coordinate transformation these quantities transform in a particular fashion, if they transform like that then we call them as tensors and that also helps us generalize to higher order tensors, okay. So the next step is to see what these transformations mean or if there is a way we can understand the second rank tensors a little bit more better that is what we will do in the next part of this lecture thank you.

NPTEL

Principal Investigator

IIT Bombay

Prof. R.K Shevgaonkar

Head CDEEP

Prof. V.M Gadre

Producer

Arun Kalwankar

Digital Video Cameraman

&Graphics Designer

Amin B Shaikh

Online Editor

&Digital Video Editor

Tushar Deshpande

Jr. Technical Assistant

Vijay Kedare

Teaching Assistants

Arijit Roy

G Kamalakshi

Sr. Web Designer

Bharati Sakpal

Research Assistant

Riya Surange

Sr. Web Designer

Bharati M. Sarang

Web Designer

Nisha Thakur

Project Attendant

Ravi Paswan

Vinayak Raut

NATIONAL PROGRAMME ON TECHNOLOGY

ENHANCED LEARNING

(NPTEL)