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Phase field modeling;
the materials science,
mathematics and
computational aspects
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Module No.8
Lecture No.32
Diffusion equation:

analytical solution II

Welcome so we have been looking at solving the diffusion equation using variable separable method and we have solved one of the ODs and the second OD that we want to solve.

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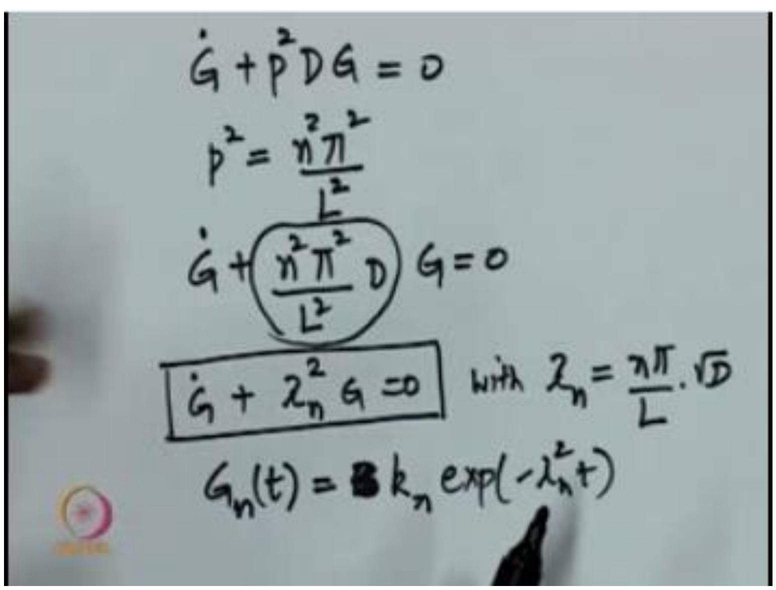

So let me write it down $\dot{G} + P^2 DG = 0$ but from our solution for the first OD we know that P^2 is nothing but $n^2 \pi^{2/} L^2$ so I am going to put it here so $\dot{G} + n^2 \pi^{2/} L^2 D G = 0$ I am going to call this as some λn^2 okay $\dot{G}\lambda n^2 G = 0$ with λn being given by $n \pi / L \sqrt{D}$ right so λ^2 will become $n^2 \lambda^2 / L^2$ d so that is what is here so let us solve this equation $\dot{G} + \lambda^2 n^2 G = 0$ of course you know you can integrate this equation and the solution will be Gn(t) will be some constant B.

So let us not call it as D let us call it some constant k_n times $\exp(-\lambda n^2)D$ so this is the solution for this equation because $DG/Dt + \lambda n^2 G = 0$ so you will take it to the other side and DG/G will become $-\lambda n^2 t$ and DG/G will give $\ln G \lambda n^2 G$ so you will integrate it that will give you will exponential it that will give you G_n as exponential $-\lambda n^2 t$ + that constant and if you assume that constant to some ln k and that will become k_n okay.

So that is what the solution is so OD is you know how to solve so from the solution of this very simple ordinary differential equation we get this solution so we know how to solve the second equation now so let us put together the solution now so we have the solutions for both eth parts we have solved both the OD so we know what f is we know what G is so remember.

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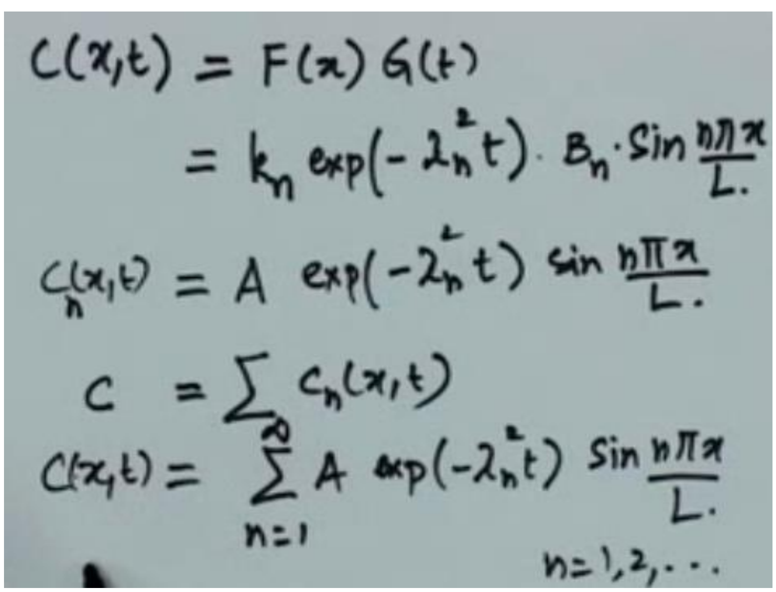

$C(x, t)$ was assumed to be $F(x)$ times $G(t)$ right so now I know what the $G(t)$ is that's I some k_n times $\exp(-\lambda n^2 t)$ and we what the $F(x)$ is so that is some other constant B_n times we had the solutions $\sin n\pi x/L$ okay so let us merge all this constant so we have some constant say $A \exp(-\lambda n^2 t) \sin n\pi x/L$ so this is the solution but the solution is different for each n so the most general solution that you can write.

So let us call this solution as Cn because this is for a particular n so C is a summation of all Cn (x, t) that is then given by summation n = 1 to $\propto$ some $A \exp(-\lambda n^2 t)$ and then $\sin n\Pi x/L$ okay so we have C for all x for all times given by this expression because there and n goes from 1, 2, 3……so that is why you have this summation. We have the solution we have not used our initial condition yet which will help us then determine what this constant is, so and we said that at time T = 0 the C(x) is a function f(X) right so we can substitute that in the assumption, so let us take the quantity.

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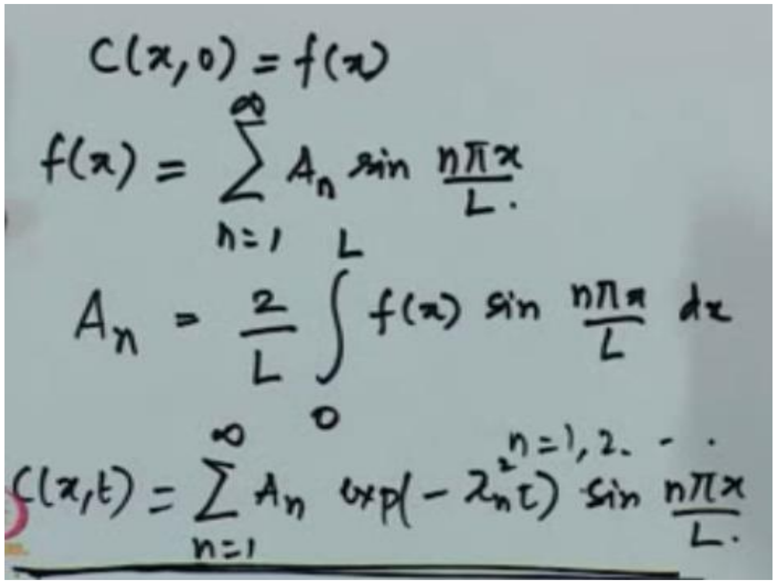

So a C(x, 0) = f(x) which means f(x) = summation n = 1 to $\propto$ A T0 so $\exp(-\lambda n^2 t)$ is going to be U unity so you have sin nΠx/L, okay. Now from this expression let me also write this An we have not written for each n is going to an A so now all this An can be determined because if you multiply by another sin nΠx/L and integrate it over Dx over length 0 to L then you will get at some constant An and which means you are going to do this same thing on the other side.

So you can find An to be 2/L so we can do this as a tutorial come in out as that this is V be so this becomes 0 to L f(x) Sin nΠx/L dx, this happens because you know for Sin nΠx unless the Sin suppose if you multiply by Sin nΠx/L and this N and this N are the same. Only then this will light up all the other cases it will be 0 which mean An will be chosen and when that $\sin^2$ nΠx/L with n some value 1, 2, 3… that integration will give you 2ΠL I mean L/2.

So that is why you are having 2/L here so that quantity and so its integral f(x) sin nΠx/L dx which means this is integral An nΠx/L and Sin mΠx/L, right. Only when n=m you are going to get some value. And all the other cases you are not doing so, so this is how we determine this constant and N goes from 1, 2 etc, so you have determined the transfer function, so we now finally have solution C(x, t) is nothing but summation n = 1 to $\propto$ An $\exp((-\lambda n^2 t)$ Sin nΠx/Ls and

An is determined here so this is the solution that we have applied using the variables separable, okay. Now at this point I want you to go back and remember one of the solutions that we wrote down for the diffusion. When we were looking at the classical diffusion equation and its solution and what happens when diffusivity becomes negative we wrote down a solution so I want to remind you of that solution.

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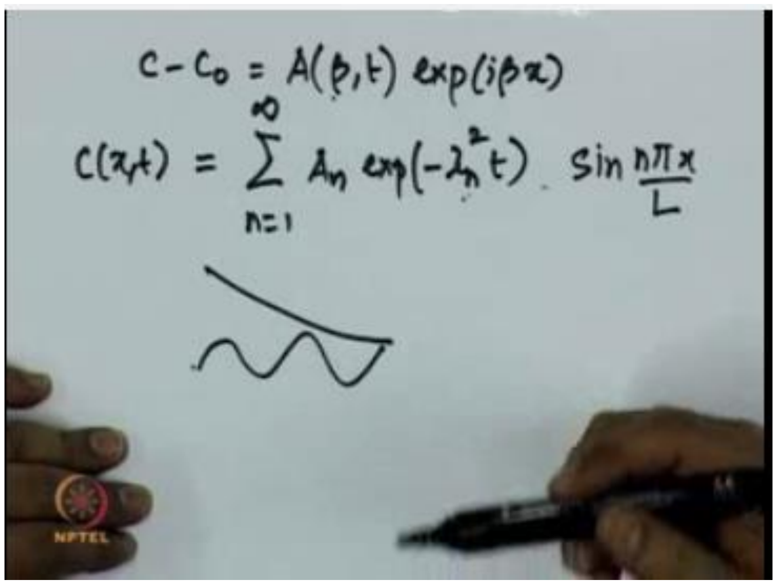

We wrote that $C\text{-}C_0$=sum $A(\beta,t)\exp(i\beta x)$, okay now compare this with the solution that we are getting you are saying that $C(x,t)=\Sigma n=1$ to ∞ $A_n \exp(-\lambda_n^2 t)$ λ and β are related, right β we said is $2\pi/\lambda$ okay, so this λ so this is a function of λ and here also we said that this is a function of λ and time, so you can see that this is just a function of λ and $\exp(i\beta x)$ instead we have here sin $n\pi x/L$ so basically $\exp(i\beta x)$ is a sinusoidal solution and here you have a sinusoidal solution, okay.

And but for a constant, right so you can add any arbitrary constant that is only going to shift so your solution is a sinusoidal like this and with time it is going to decay so there is a decay which is involved with this sinusoidal solution and all this can be shifted by some constant so that is all that we assumed in the other solution. So now you probably understand where the solution came from this solution we did not derive the solution at that point we said that this suppose this is a solution to this equation we substituted it back and we proceeded with the analysis and here I am trying to show you why this assumption that this is a solution for this equation is correct, because you can take the partial differential equation and you can use variable separable method, you can reduce it to two ordinary differential equations, you can solve them one by one, you can use the corresponding boundary conditions and initial conditions.

And when you finally collect the solution together you get the form of the solution which is the same as what we assumed in the other analysis, okay. So in other words so we have looked at the solution to the diffusion equation and before doing that we actually also showed that the so called Fourier law of heat conduction is the same mathematically speaking as our diffusion equation, and so the solutions for both expect for symbol differences is the same mathematically, so heat conductions solutions are also the solutions for diffusion classical diffusion equation.

So and then when we went to the classical diffusion equation we use the method called variable separable method sp we assumed that the solution can be split into a part which depends only on position and another part which depends only on time and we proceeded to derive this form of this time dependent part and position dependent part we see that the position dependent part is a sinusoidal one and we see that the time dependent part is an exponential decay, okay and this is what we were also trying to plot and show.

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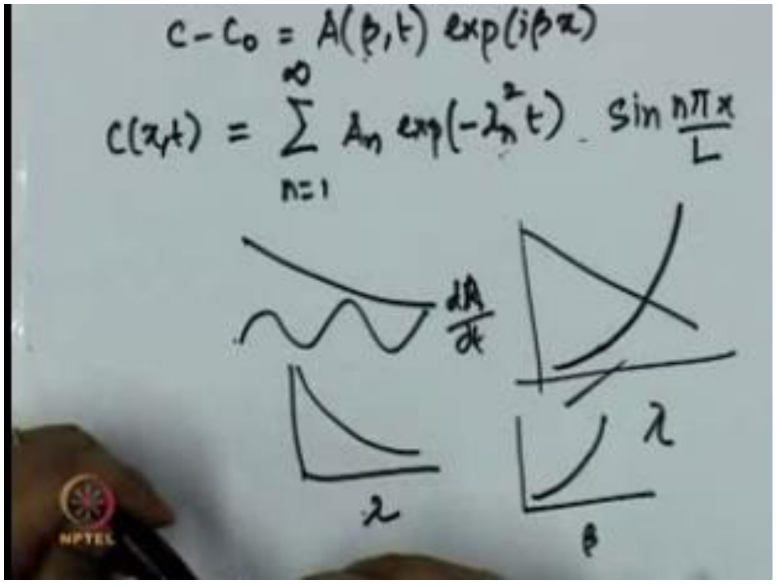

So if you look at this solution so there is this exponential λ_n^2 remember λ_n was involving diffusivity and the diffusivity has some negative value then this is going to be positive so this exponential is going to grow, so this is what we actually plotted when we did this and we try to show that if you take λ and if you let take a look at what happens to the R as we define you know dR/dt as a function of time how this changes.

We called dA/dt and they dependent on some R and that actually grew exponential right so we saw that so that sorry this is not the way it that was with β so with δ we found that it grew exponentially has δ it mean towards it is 0 if it was β it went like that okay β is $1/\delta$ so very small δ they also grew very fast and that is understandable when diffusion have to take place over shorter distance it can happen much faster.

This is idea which is related to the non - dimensionalization which is what we will use to again show you another form of solution to the diffusion equation I like set there is no single solution to the diffusion equation it depends on what non de condition they have going to assume so we have assume some boundary condition for which we have obtain some solution which is what we have discuss still now.

So there is another way of deriving similar solution a similar kind of solution but with different boundary conditions so which is what we will take it up in the next part of the lecture after we do that, that involve some non - dimensionalization and some neat ideas from non dimensionalization o we are going to do that after we do that we will go back to solving this equation numerically and using finite difference technique first and then using the furrier transforms okay.

So you can see that already there is some indication of sinusoidal solutions here and historically also furrier was looking at is hit conduction equation and as part of the solution is how we derived so is furrier series the analysis came from this trying to find solution form this problem so obviously furrier techniques can be use to solve and numerically because there are ways of doing very fast furrier transforms and those algorithms help us actually do these spectral solution for this equations quite easily and quite fast and quite efficiently.

So we will do both ways of solving at numerically using final difference and some spectral methods but before we do that we will do one more analytical solution which is obtained using a transformation of variable and that transformation variable depends on non dimensionalization to some extent so which is what we will take up in the next lecture, thank you.

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