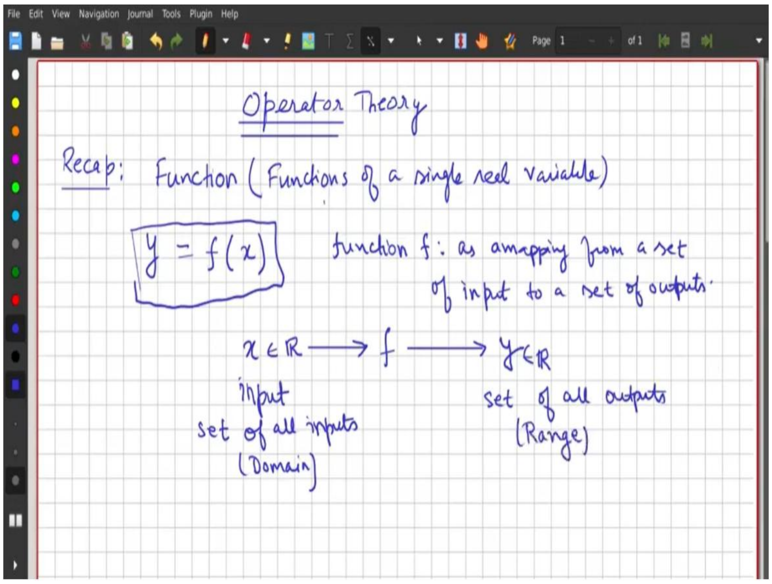
Statistical Thermodynamics for Engineers
Professor Saptarshi Basu
Indian Institute of Science, Bangalore
Lecture 09
Supplementary Video 1 Operator Theory 1

Hello, everyone. My name is Durbar Roy. I am a PhD student working under Professor Saptarshi Basu. And I am one of the course TAs(())(0:08) for the course Statistical Thermodynamics for Engineers. So, and what we are doing here is we are providing a set of parallel lecture materials, supplementary lecture materials that will go as a side and I will say provides the mathematical basis and mathematical preliminaries that will be required to strengthen our understanding of statistical thermodynamics in general.

So, here we will be covering a lot of mathematical concepts and topics that we will be using in the main lecture series quite a lot. So here we will be discussing ideas like operator theory, various kinds of coordinate systems, eigenvalue problems. So, overall, the flavor of these discussions that we will be having is like, it will be very much intuitive. And what we will be trying to do here is to rule out the mathematical and physical intuition required to understand statistical thermodynamics in general.

Further, we will be also doing a lot of problems on different sections of the course materials, the concepts are understood better and overall, I hope that what we are trying to provide here is to better understand the main course content, the main course statistical thermodynamics, as we will see, a statistical thermodynamics has lot of mathematical concepts, especially in probability theory, operator calculus, in general. So, we will be beginning our session by a series of introductory supplementary videos on operator theory in general. So let us start by understanding what operators are in general.

(Refer Slide Time: 02:19)

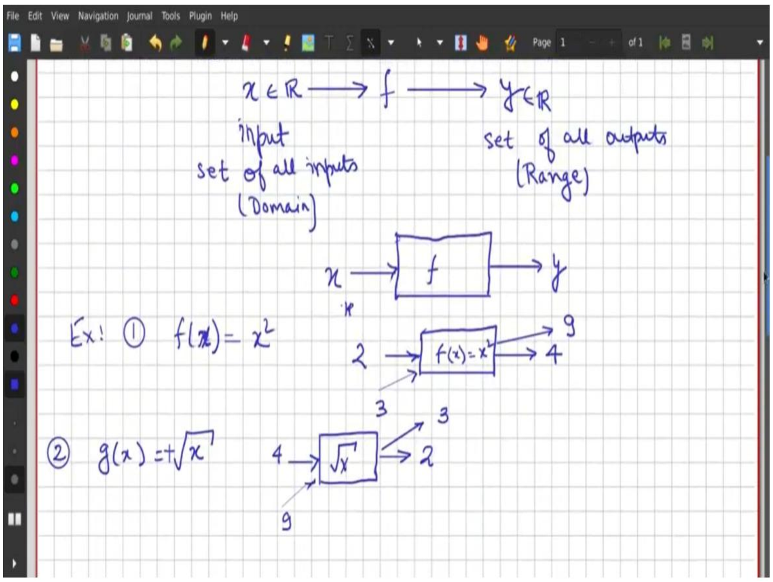


So, let us start by having a basic understanding of operators in general. So, the topic that we will be discussing here, Operator Theory. So, to have a better understanding of what operators are in general, let us start with something more basics which we already know. So, we do recap from our previous mathematical studies of let us recap what we mean by a function.

So, let us say what we are, when we studied our single variable calculus, we studied functions of a single variable, of a single variable, of a single real variable, let us say, real variable. So, we write an expression of the form let us say y as a function of x. So, what the statement was all about? So, this said something like you can think of this function, function f as a mapping, as a mapping from a set of input to a set of output.

So in case of real variable, what is the input? So, the x, x belongs to the real piece, the real line you can think about, the one-dimensional real line. So, this goes as an input an is a real number, you can take as an input to the function and that provides an output y, the input, the set of inputs is called the domain, if you remember, all inputs is the domain and it produces real numbers and this y also belongs to a real number and set of all outputs, what we call as the range of the function f.

(Refer Slide Time: 04:59)



Another way of understanding this the same idea is of thinking the function as some kind of a machine. Let us say we have this function f as a machine, f say takes an input x and output, provides an output y, that is another way of thinking about this one. So, it takes a real number. Let us say takes, let us take a specific example. So, let us say, we have a function example 1. We have square.

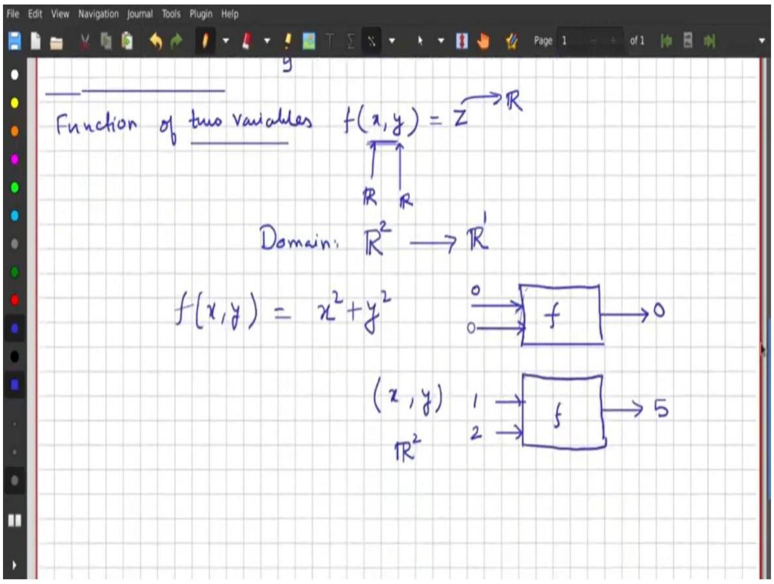
So, this is a squared function. So, what it does is there is a function f x equal to x squared. What it does is whatever input you give, squares it and produces an output. So let us say we put a number 2 as an input, output is 4, provide the number 3 as an input, output is 9. So, it is squaring. So, you see it is doing some kind of an operation.

Here we will see how, see how the general idea of what do we mean by operation and hence, operators will come in handy. So, we will be realizing later that and think about functions also some kind of operators. But will be building the actual concept of what we dreaming by operators are in the most generic fashion.

Let us take another example, where let us say we have another function, g of x, let us say is the square root function. What it does? Again, let us think about in terms of some kind of a function machine, it is the root. So, it takes an input, let us say takes the input 4, and let us say what it does is provides a positive value of it. So, here is the output.

Similarly, what we can do, let us say give it an input line, it gives us an output. So, these are an examples of what we know as single variable functions which we studied in our single variable calculus codes, where you take one single input x from the domain, and it generates an output a single variable again, in the range, in the range set. That is, that is what we mean by function of a single real variable.

(Refer Slide Time: 07:47)



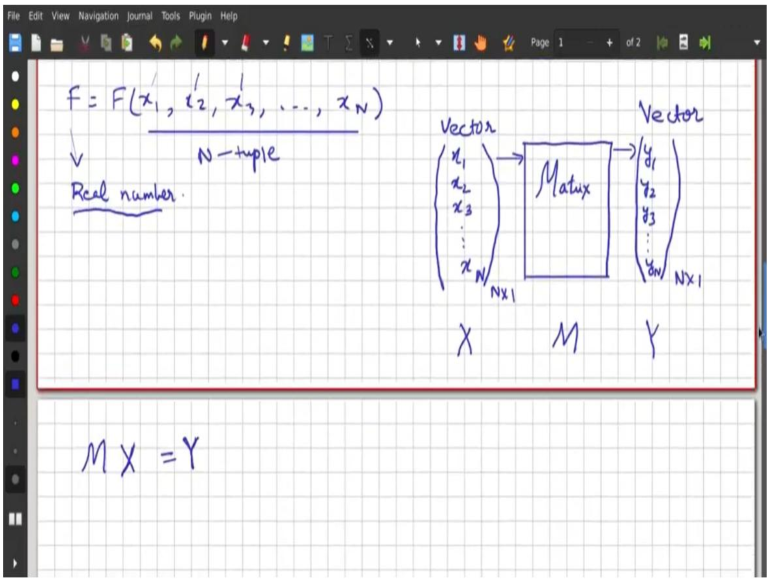
But this idea of functions can be as, it will be generalized to higher dimensions. So, let us say we have a function of two variables f of comma y and that provides an output. Now here we see the domain as x can be any let us say again we are dealing with let us say real variables, and be any real number, y can be any real number.

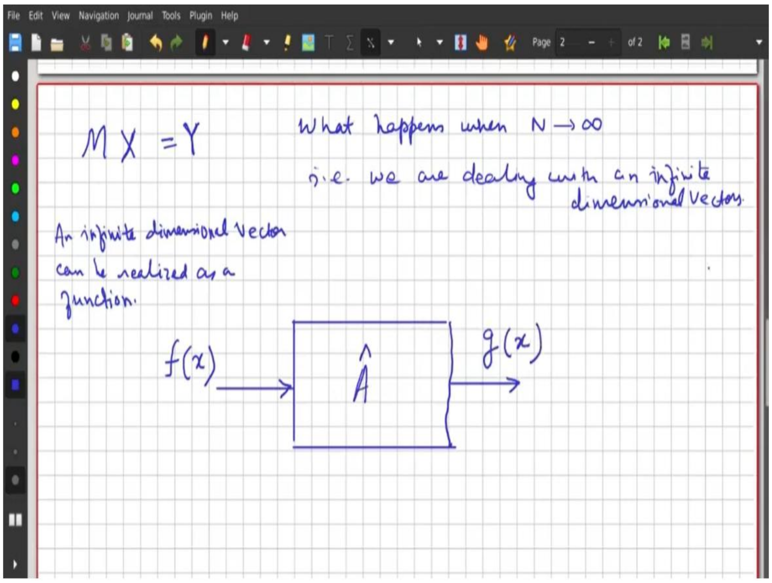
And the output can also be any real number let us say. So, so, if you see the domain, the domain is a tuple. So, the domain if you see it is, so the set so, what we are providing is a set as an input is it is a tuple, it is a set x y. So that is the domain belongs to the R2 space but the output the range as you can see is in R1 space.

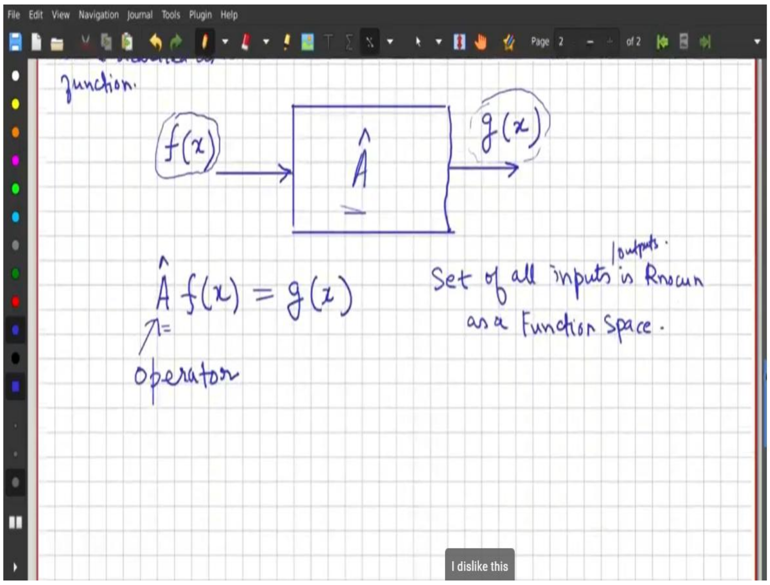
So, that is what the idea of let us say a two-variable function of two variables. Take the two numbers as an input and it gives, provides, the single output. Example of this as for example, f of x, y is let us say a distance function at a squared distance function from the origin. We can think about this as some kind of, again some kind of a machine f, which calculates the square distance from the origin of coordinates, cartesian coordinates let us say. What it does it takes two inputs and y and it provides squared distance as an output.

So, let us say we give it 00 provides output 0, that is telling the distance of, so, the distance is 0 because you are right at the origin itself. Distance is measured with respect to the origin. Let us say use the same function and you give it the point 1, 2 so, you see the input is a set x, y that you are giving so, it is a, it is coming from a two-dimensional space R2. And so, once you give the point 1, 2 it will give you the output. It is one squared plus 2 squared so, that is 5 that is the square distance from the origin of the point 1, 2. That is the generalization of function in two dimension and this generalization can be done in higher dimensions.

(Refer Slide Time: 10:34)







For example, the functions can be f of x1 x2 x3 dot dot, N, this is a function N that same N variables. And the same idea holds or you give it a N tuple, so this is known as N tuple. And it provides a single real variable as an output so, this is a real, real number. So, this idea of a function of where you are taking N inputs, let us say can be thought of in a slightly different manner where let us say now, we have so let us write this tuple in the form of column vector, let us say we write it x1 x2 x3 dot dot dot xN. This is N cross 1 column vector.

And what we do with this is, we do some operation on this. That means you pass it to some kind of a Blackbox, some kind of machine, we do not know what that is. And what it is, what it generates is not a single number as before, but it generates series of it generates N tuple let us say, that again generates an vector, y1 y2 y3 dot dot dot yN in general.

So, what it does is takes a vector as an input, this is an input to this function, and it generates a vector as an output. And from our previous understanding and previous studies of linear algebra, we know what kind of mathematical object transforms a vector into another vector it is a matrix. Or matrix transforms one vector to another.

We can rewrite this in a different mathematical form. Let us say this input vector we call by x, a matrix by m and the output vector by y, we can write this as the matrix m acting on the vector x gives us the vector y. We can see that we can think this also as the matrix m operates on the vector x, operates on the vector x to provide the vector y.

So, you see how this generalization is happening, right from a single variable system to two dimensional to N dimensional and to matrix vector operations. And if we continue the simulation further, we come to a realization, so let us say what happens when we ask the question what happens when the N goes to infinity, or that is, we are dealing the infinite dimensional vectors, infinite dimensional vectors.

So, as you see the logic remains the same. We have some kind of so, function box, kind of a black box. It takes an infinite dimensional vector. So, now we need a very, very useful mathematical realization that infinite dimensional vector can be realized as a function. Given that this realization we have so we can think about operators that take infinite dimensional vectors or to say let us say takes function as an input, this is no continuous now, you see what it is taking, it is taking a continuous object so, we can say infinite dimensional in that sense.

It gives out a function as an output g of x. And that is what we mean by an operator. So, this is the kind of operator that we are, we wanted to like come up with the main concept, so, this is a generalization that we were looking for. So, an operator A is an object that acts on a function f to produce another function as a result. That is what here A, this is the operator that we were looking for.

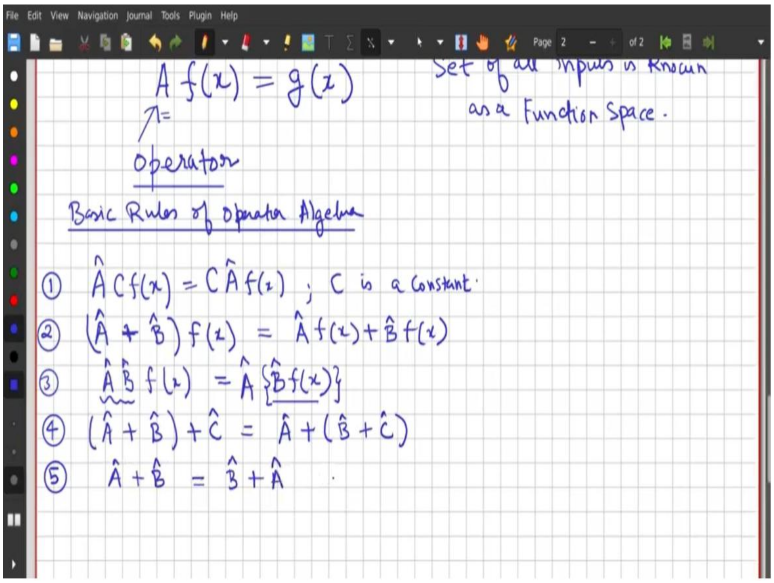
So, that is a generic idea of what we mean by operators in general. So, that is the general idea of operators that we wanted to come up with, so, an operator is an object A, here that is written as A hat and it acts on a function f of x to provide us another function g of x. So, you see how we came up with the generalization right from the idea of from a single variable function to multiple variables, to let us say N variables and through the idea of a matrix vector operations to the idea of a continuous operator that we are discussing.

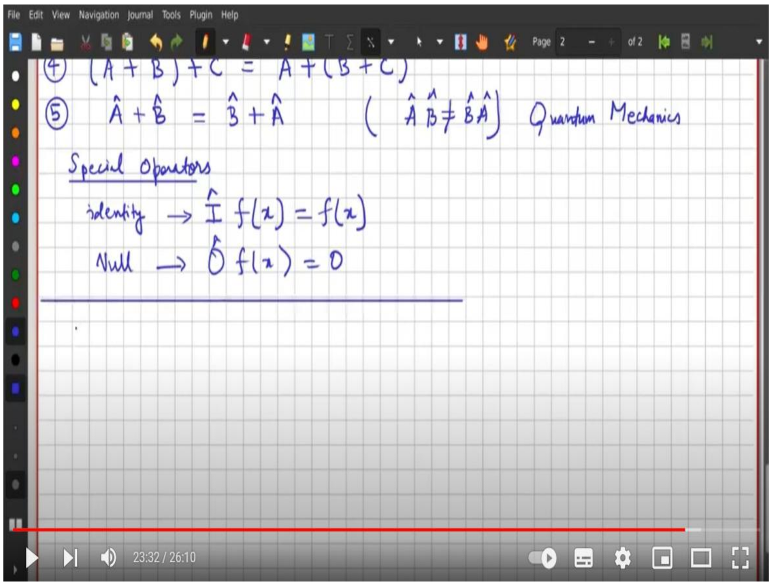
A very important point to keep in mind, for example here is, here when we are dealing with functions where the inputs were discrete, and non-continuous objects, like let us say a single variable two variable and invariable, the domain, the domain set we see that a set of all numbers and hence it belongs to let us say, R1 space, R2 space, R3 space and so on.

On the other hand, now for the idea of the operator that we are discussing now for A hat so you see the set of all inputs, set of all inputs, set of all inputs, is known as, is known as function space, function space. Not just all inputs, but also outputs, the set of like, so what we are dealing with is we are picking functions from a space of functions.

And one of that elements of that function space is, let us say f of x, and you pass it through the operator A, and it provides another function g of x from g of x. And that g of x can be, can be on the same function space or it can be on a different function space and we will be seeing all these things as we require different ideas related. So, but that is a generic idea of operators that we wanted to come up with.

(Refer Slide Time: 18:47)





So, now we will be discussing some basic rules, rules of operator algebra, some basic rules of operator algebra and this I think so we will be familiar from a previous understanding of let us say, of thinking about let us say derivatives and integrals as operators. So, let us write down the rules. So, let us say operator A hat acting let us say on C time f of x is same as C A hat f of x where C is a constant and C is a constant.

Similarly, let us say we have an operator, we have two operators A hat, B hat and the sum, the sum of both the operators. And when we are thinking about sum that sum has to be very, very specific, that means these two operators needs to be, should be able to add to each other, that is one way of putting it. And when you put it and operate on the function f of x that is same as the operators acting on f individually and then summing the result. That is another property.

Third, there is a property, let us say you two the product of an operator and you generate a new operator, upgrade that on a function f of x. That is the same as let us say first you take operator B act on the function f of x to generate some kind of an operation and then, then take that as an input to the operator A and you should get the same result.

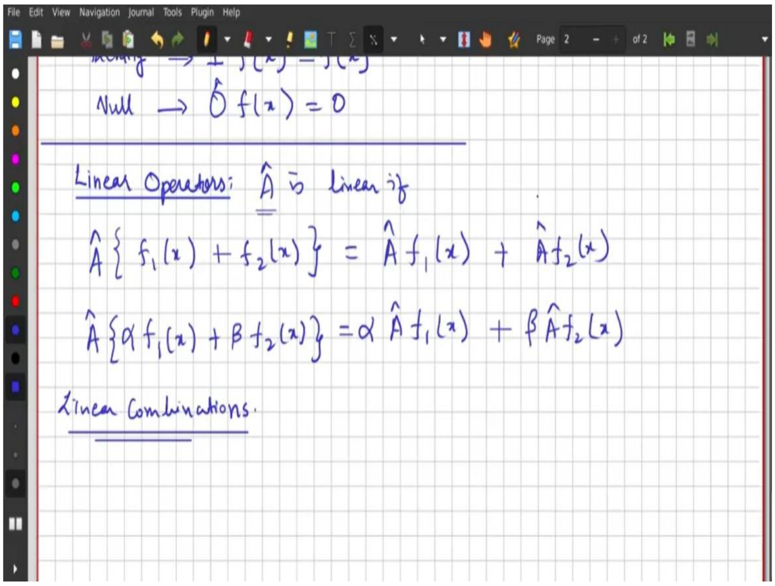
Similarly, this property that now we will be writing let us say we have an operator A plus B plus operator C, this is same as operator A plus operator B plus operator C. So, this is the associative law for addition, and that we have let us say A plus B equals B plus A. That is the you can think about the commutative law of, commutative law with respect to addition and you see A plus B same as B plus A.

But as we will see later that this is not true for multiplication operator, that means A keep in mind we will be discussing this later but A hat B need not be equal to B hat A. And this will come in very simply a very important role and this plays a very important role in quantum mechanics as we will see later. This is called a non-commutative properties of quantum operators, some of the quantum operators.

Then we have some special operators, special operators that we can draw parallels with from linear algebra as which we call let us say the identity operator I hat. So we can think about, this is the parallel, the identity matrix, it acts on a function f of x and it returns the original function back f of x. So, this is the identity operator, identity operator and then we have the null operator, where what it does is act on a function to produce 0 as an output.

So, those are some of the very important properties of our I will say the basic rules of operator, operators that we will be discussing. Extension of these ideas and a very important property and a very special class of operators that we will be dealing with in specifically in quantum mechanics is what are known as linear operators, linear operators.

(Refer Slide Time: 23:32)



So, what are linear operators? So, linear operators are those operators that satisfy the following rule. A hat is linear if, write A hat is linear if it satisfies the following, A hat acting on the function f1 of x plus f2 of x is the same as it acts on individual functions and you sum up the results and you should get the same result, that is called the idea of linearity.

And in basic extension of the idea of this linear and combining this rule one, generalizes the idea of linearity. And just by thinking about we can write this in terms of rather than just know the function, we can think about its multiplying by some kind of constant x. Let us see another constant beta f2 x and what we should get as alpha A1 f1 x plus beta A f2 x. As you can see, what we are doing is linearly combining, is a linear combinations, linear combination, that is what we mean by linear operators in general.

So, what we have covered in this lecture is what we saw the idea of what a general operator is what a general operator is, we have understood what do we mean by an operator acting on an object we can get an understanding of what our function space is, we saw some basic rules of operator algebra and we understood what do you mean by linear operators. So, we will be covering the remaining things in the next lecture. Thank you.