Statistical Thermodynamics for Engineers
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Lecture 22
Supplementary Video 7 Coordinate System 2

Welcome, everyone to another segment of supplementary videos, and today we will be discussing about the various ways of writing the different vector fields in various kinds of coordinate systems. So, in the previous segment, we introduced the various types of coordinate systems, and the volume elements in the respective coordinate systems. So, today we will be starting and try to build a framework to write the various vector fields in a generic coordinate system.

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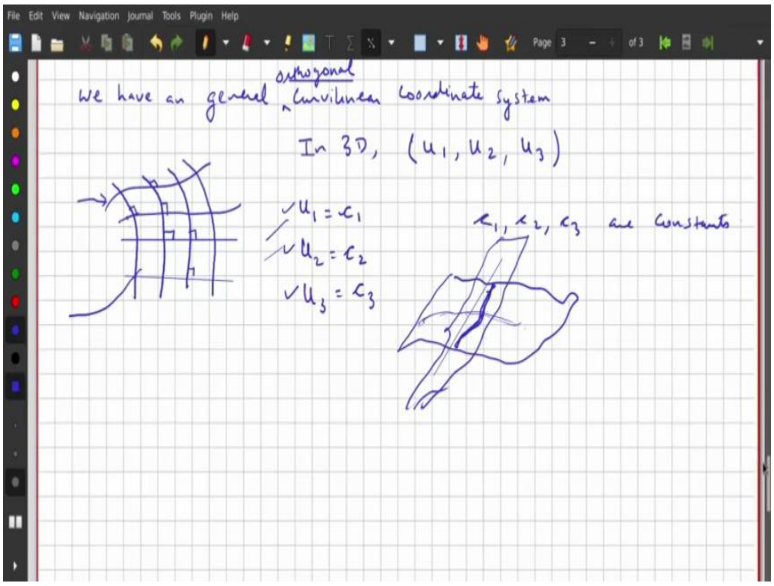

So, let us start. What we will do is let us start with let us assume we have a general curvilinear, let us say orthogonal curvilinear coordinate system to begin with. In two-dimensions, the coordinate systems could look something like this. This is any arbitrary and then we also have the third dimension. So, that is the generic coordinate system that we are thinking about. So, this is a 2D, in 2D.

So, in 3D any generic coordinates; in 2D the isolines as you can see, are represented by curves. So, these are the isolines. Let us call in 3D the general (())(2:03) are represented by u1, u2, and u3. Those are the three orthogonal directions in the curvilinear coordinates. So, u1 equals a constant, u2 equals constant. So, these curves in 2D represent these curved lines, these constraint relationships, and in 3D, this will require curved planes. So, these are set of, these 3 systems of equations define all the iso surfaces basically in that coordinate system.

So, here c1, c2, and c3 are constants. So, if you take each of these let us say this and this or this and this they will intersect at a curved, they will intersect at a curve basically. Like so, so it is like a so here this is one surface and then we have some other surface. So, the intersection of this is basically a curve in general.

So, and these two, these two surfaces, these two surfaces are represented by one of these constraint relationships. And one important thing that we need to be kept in mind is since we want to work in only in orthogonal coordinates, that means this, the isolines or the isosurfaces always intersect at the right angle wherever possible. So, these all are 90 degrees. It does not look like 90 but these are 90 degrees. So, that is, that is what we mean by orthogonal curvilinear. So, here also at the point of intersection, at the point of intersection, it intersects at right angles.

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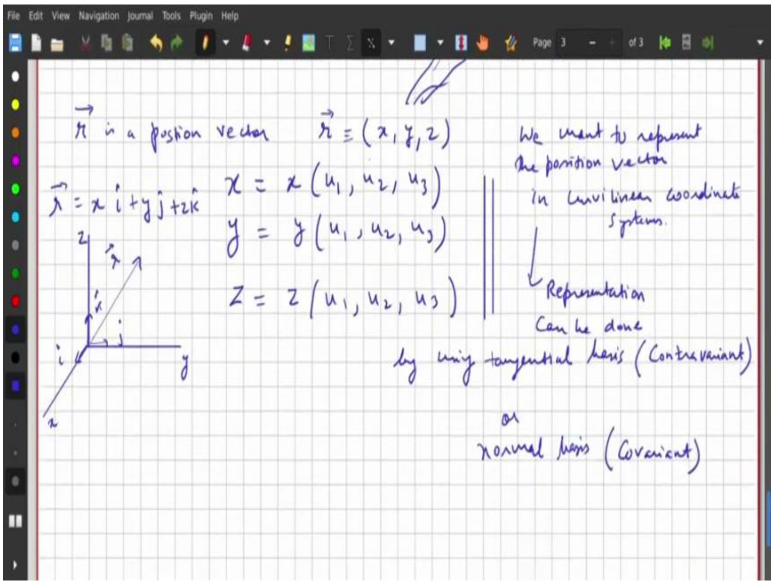

So, let us say we now have any general vector. r is a position vector. And in Cartesian coordinates, r is; the components of r is given by x, y, and z. Correct? So, let us say we want to find out a relationship between let us say how between, let us say we want to represent a single point in both Cartesian coordinates as well as let us say in this curvilinear coordinates. That means we are seeking a relationship of sort x equals x some function of u1, u2, u3.

That u1, u2, and u3 is the new curvilinear coordinate. y is a function of u1, u2, u3, and similarly, z is a function of u1, u2, u3. That is the general relationship that so these things, so these are basically, equations of the form when we were discussing coordinate systems and we were discussing the transformation rule. So, something like this basically. So, that is generic; so we are trying to write the generic form of these set of rules. The coordinates that we did before. So, something like this- from Cartesian to cylindrical, from Cartesian to spherical.

So, the general transformation rules are written in this way. So, that is the generic way you want to seek a relationship of that form. So, in Cartesian, this r is given by xi cap plus yj cap plus zk cap where i, j, k denotes the unit vector in the x, y, z direction results. So, this is i, j, and this direction we have k. So, now, so let us see how we will do it here.

So, what we want to do is try to represent any generic arbitrary vector r in this curvilinear coordinate. So, that is our task at hand. So, what we want to do is, so, let us say we want to represent the position vector in curvilinear coordinate, in a curvilinear coordinate system. And what we will do is we will try and this could be done in two ways and those from two different systems.

So, this representation, there are, we can use representation can be done by using the tangential components or by the normal components and tangential and are normal, what we mean we will discuss what do you mean by those things. So, whenever we represent a vector in these curvilinear coordinates, in terms of the tangential basis, tangential basis instead of components, let us write basis, tangential basis, normal basis. So, when we do it in tangential basis, this is known as contra-variant basis and when we do it in the normal basis that is known as the covariant. The covariant basis.

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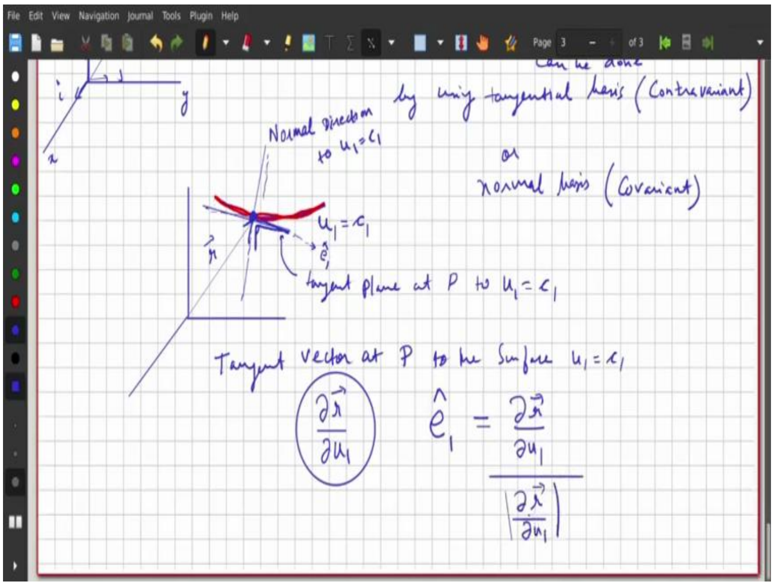

So, let us say we have a vector, let us say r at a point P and at that point we let us say, we have some isosurface. This is some, this is point P and this is the isosurface let us say given by u1 equals c1, equal. This is, let me draw this is in a different colour. This is the u1 equals c1. And on top of that we have the point P. So, you see at point P, at point P we can define a tangent to, we can define a tangent, tangent plane basically. So, this is a tangent plane. This is the tangent plane at P to u1 equals c1.

Similarly, we can define another direction which is the normal direction. This is the normal, normal direction. This is normal direction to u1 equals c1. That means, let us see how we can write a tangent vector at point P. So, any tangent vector at point P will be in this direction, this direction.

So, the tangent vector at P, at P to the surface or let us to the surface u1 equals c1 is given by partial r partial u1, and this is the tangent vector in this direction and in that direction let us call the unit vector to be e1, this direction; and this direction the unit vector we are calling let us say e by e1 hat; so e1 hat, this is the vector, this is a tangent vector. So, the unit vector in that tangential direction is e1 hat which del r del u1 divided by its magnitude which is del r by del u1.

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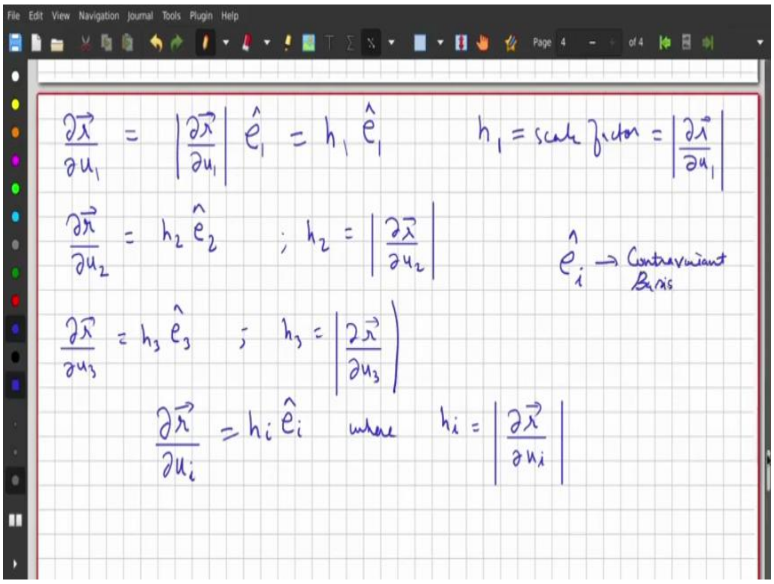

So, this we can write in a different way in which you have del r del u1 is equal to partial r partial u1 magnitude times the unit vector. And this we can write h1 e1 where we call this, see this is just a scalar. This is known as a scale factor and that is partial r partial u1 magnitude. That is h1. Similarly, we can write in other two directions, del r del u2 is h2 e2, where h2 is the magnitude of partial r by partial u2. And partial r partial u3 is h3 e3 where h3 is the magnitude of partial r partial u3.

So, in a generic index notation, we can write this, all these 3 equations as del r delta u is equal to hi ei cap where hi is the magnitude of partial r partial ui, a scale vector; a generic scaling factor. That is the representation, that is the and think about these e, e sub i cap, these represent the contravariant basis, contravariant basis.

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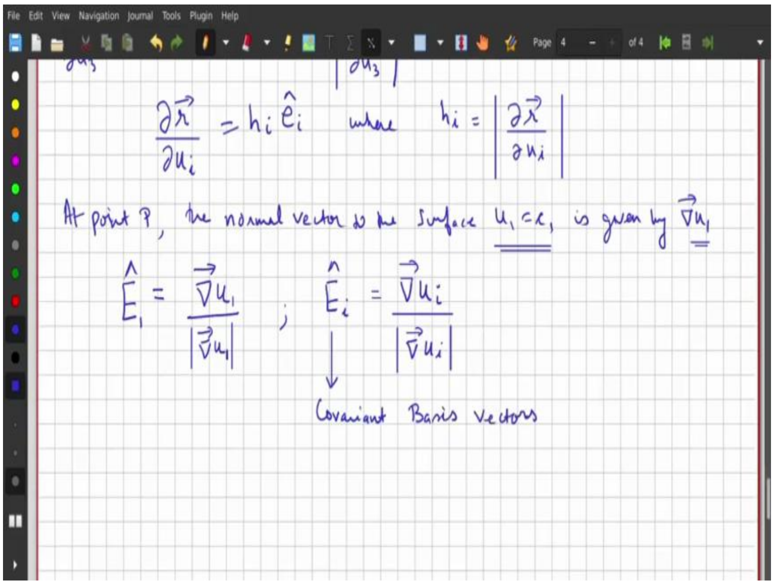

So, now let us do the, let us try to represent in terms of the normal. So, at P the normal, at point P the normal vector to the surface u1 equals c1 is given by a gradient, that we know. So, this, in this direction basically the normal direction, we will have the gradient that is the direction of the gradient, u1. So, in the normal direction, the unit vector let us call that capital E1 cap and let us write that as a gradient of u divided by the magnitude of u, , and that is E1.

So, like that in a similar fashion, we can write the generic components which are given by Ei that is grad u divided by or let us grad ui because this will be u1 basically. So, this is normal to the curve u or surface u1 equals c one. So, this is E. Therefore, E1 should be grad u1 basically, grad u1. grad ui divided by the magnitude of grad ui. That is, these are the covariant basis, covariant basis vectors.

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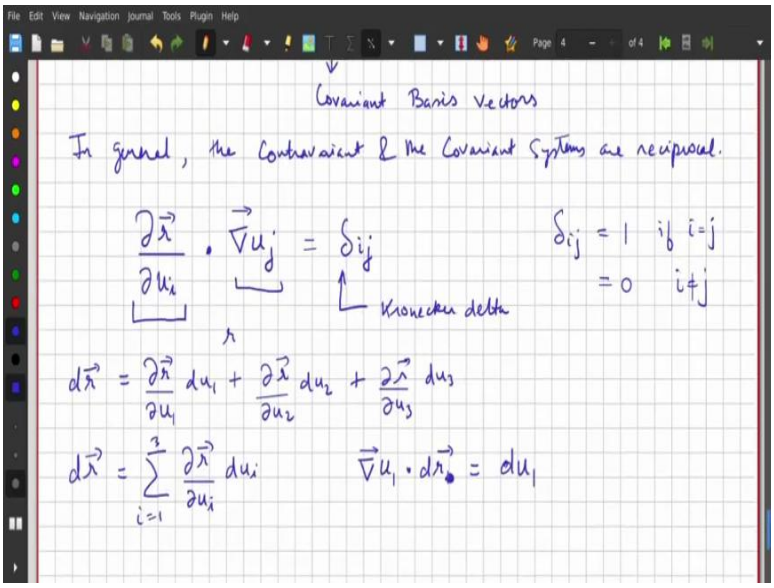

And in general, what happens, is in general, the contravariant and the covariant systems are reciprocal. That means that so the generic which is, the generic tangent vector. And if you take the dot product of this, this tangent vector with the contravariant uj. This in general is the Kronecker delta. This is the Kronecker delta where delta ij equals 1 if i equals j and this is equal to 0 if i is not equal to j. That is the relationship between let us say you can think about the contravariant with the covariant basis. They form some kind of reciprocal systems with each other.

So, let us say a differential element dr vector in curvilinear coordinate could be represented something like this where remember r is a function of u1, u2, u3. So, write this as u1, d u1 as partial r partial u2 d u2, and then we have partial r partial u3, d u3. And therefore, we can write this in a little bit different way which is using the summation notation. It becomes sum over partial r partial ui d ui, where i goes from 1 to 3.

And let us take the gradient of u1, and our earlier studies of vector calculus you know grad u1 dot d r1 is nothing but or let us say grad u1 dot d r is the d u1. That is from where the idea of that at u1 equals constant grad u gives the normal direction.

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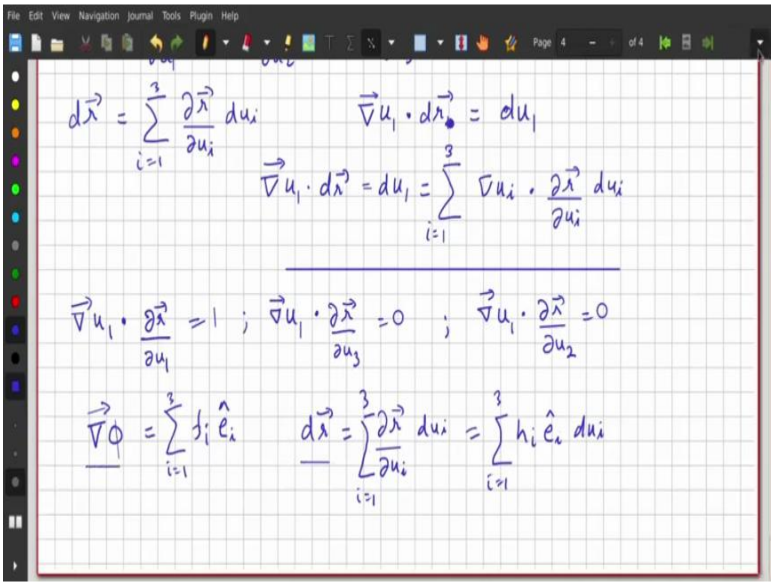

So, if you write this product grad u1 dot d r, this is d u1 and this could be written using the summation already have which is grad ui dot partial r partial ui d ui and i goes from 1 to 3.. So, that is the dot product of these two systems. If you see due to since only the dot products in the same direction will survive, we will have grad u1 dot partial r partial u1 that has to be 1.

On the other hand, grad u1 dot r partial u3 equals 0, and grad u1 dot partial r partial u2 is also 0. So, let us write any arbitrary gradient of any let us say function e that could be written in terms of the contravariant components as sum i equals 1 to 3 f sub i ei hat and the dr vector similarly could be written as partial r partial u sub i times d ui sum over i equals 1 to 3 and this could be written using the scale factors, i equals 1 to 3.

Partial r partial ui can be written as hi ei cap. So, that is using the definition of scale factors if you remember where we were writing this or this using this expression, partial r partial ui is h ei cap. That is what we are doing, partial r partial ui is h ei cap. This becomes d ui and so, this is a grad phi, this is dr.

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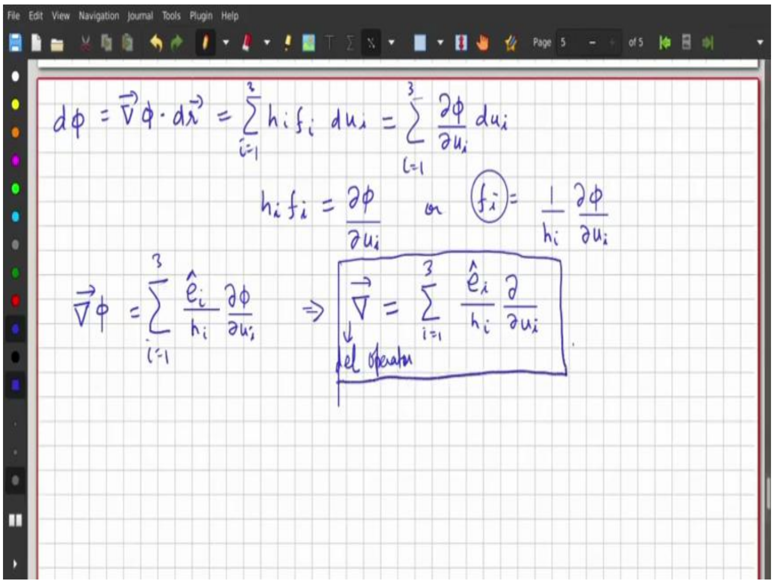

And if we take the dot product of the two at phi dot dr we will get the elementary change in phi. So, that is d phi. d phi is grad phi dot dr and that is equal to sum over i equals 1 to 3 at hi fi d ui. And that is the same as summation of partial phi partial ui d ui in because you remember phi can be represented in terms of the curvilinear coordinates, u1, u2, u3. If you see from here, we get a very interesting and useful entity between the scale factors. So, this becomes hi phi is partial phi partial ui, or fi is 1 over hi partial v partial ui, partial phi partial ui.

So, let us write a grad of phi, grad of phi was, the fi, so that is what we found out what fi is. So, we can that thing here- grad of phi is sum over i equals 1 to 3, ei hat by hi partial phi partial ui.. That is a way of writing grad phi.

And if you see from here, we can write a generic version of the gradient operator using the scale factor and the contravariant basis that is so the grad, this del operator basically. So, this is the del operator, del operator, could be written in a generic way which is sum from 1 to 3 ei cap by hi partial partial ui. That is the generic definition of the del operator using the scale factors and ei cap.

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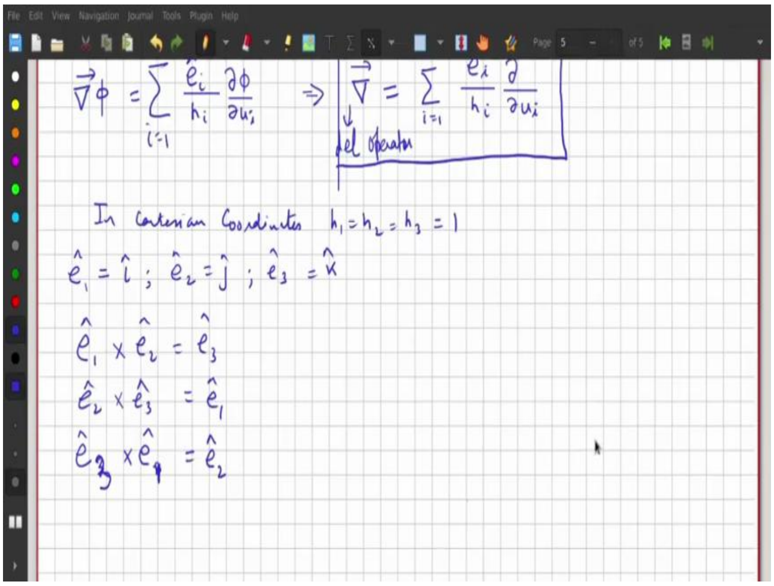

In Cartesian coordinates, as you saw before, we have shown h2, h3 all the scale factors goes to unity and hence the individual contravariant basis vectors are the unit vectors in the x, y, z directions. e2 is j and e3 is k. So, as i, j, k satisfies the rules of cross, the cyclic rules in cross product, same is apply applicable for e1, e2, and e3. That means we will have e1 cross e2 that will have e3.

Similarly, we will have e2 cross e3 that is e1, and finally we will have e3 cross e1; e2, e3, e1 and then we will have one e2 cross 3, e2 cross e3 and that is. So, e2 we already have it. So, 3-1. This will be 3e e1 that is e2. So, that is all about the cyclic behaviour of the contravariant basis vectors. So, that is for the segment. We will pick up from here in the next segment. Thank you.