Optical Methods for Solid and Fluid Mechanics
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Lecture – 05
Mathematical Preliminaries II

So welcome back. Last time we were discussing the topic of Eulerian and Lagrangian points of view. So, the reason it is important is simply because of the nature of deformation that we are tracking. And the complexities that are inherent in continuum system as opposed to the simple Newtonian system, where we were tracking just point masses. So, life is much more complicated when you are dealing with an infinite number of points.

And that necessiated a slightly more rigorous look at how we do bookkeeping for displacements what we saw is there are two ways. One is the Lagrangian way which is the where you are already familiar with in Newtonian mechanics and the other is the Eulerian way of keeping track of displacements and velocities. There is a big difference between the two and the way the reason the difference comes is simply because how we track the velocities.

And because of that we now had to look at how to do the derivative of variables that are Eulerian in nature. And what we realize is when we have to take such a derivative we have to evaluate the Lagrangian rate of change.

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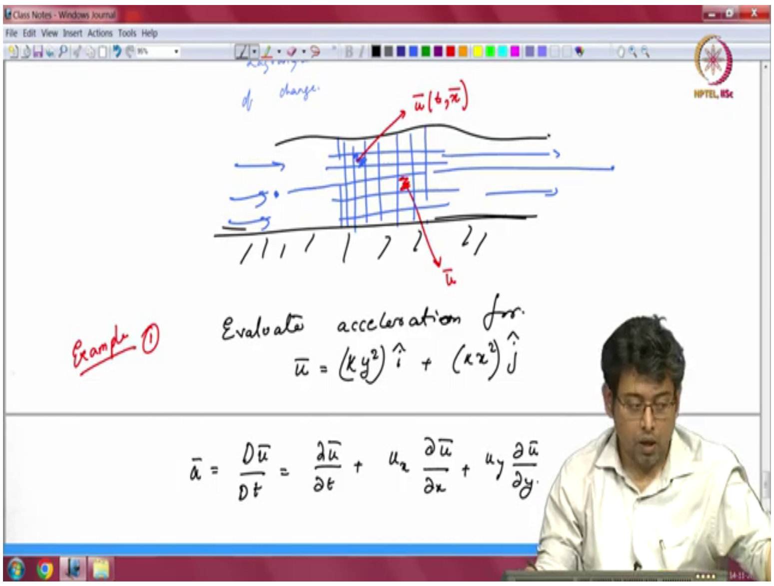

Then it looks like this expression that we saw last time where we had just left off and this bracket dot dot dot here denotes any variable which we are measuring in that particular format. So, now, here in fluids what we have to do is? We always do a Eulerian analysis. So, let us say you have there is a river and the river is flowing in this direction, a particular point of mass. Let us say, switch originates here.

This point of mass will undergo a very fast displacement and end up at a very, very different location. So, in order, if you were to do the Newtonian mechanics, you would have to or the Lagrangian point of view of doing things. Then you would have to track the motion of this particle and as it goes and as it goes through now imagine yourself sitting. maybe in a in Haridwar or next to the river Ganges.

And then some mass is flowing by that mass will flow by very fast and it will flow down to a very, very long distance or in order to understand the velocity. You now have to run after the particle as it moves in the river. That is a very difficult proposition for fluids, where the deformations are very, very large. So, there what we do is? We cannot do that so, what do we usually do?

Well, I mean when you are, as I said, maybe in Haridwar city to the next to the Ganges. You focus your view onto a small and limited section of the river and then you just notice the patterns that come through. So, we do the same thing and here what we will do is? In fluid mechanics we are used to doing the same thing. We divide this entire thing into small grid points and then say that the velocity here.

So, there is a some velocity that is associated with this. Let me be consistent with the notation so, this is u bar. So, this velocity is there is some other point this has some other velocity maybe. And I am just drawing some vectors at random here now we identify this velocity u as the velocity of the mass occupying the grid, point at that exact, instant. So, in the next instant, this mass would have moved to some other window some other mass would have come into this.

So, this velocity it will now be reflective of the velocity of the new mass that has come into this. So, it is no longer the Newtonian way of doing things where we are keeping track of velocity of a single particle here now the velocity of this grid point fluctuates as new mass comes in and enters and old mass leaves the system. So, as flux mass flux occurs in this box, your velocity keeps on changing.

And so, this vector u is now representative of something slightly different. So, this is my Eulerian point of view and when we do a derivative this we can no longer do a simple time derivative. So, this now u is a function of time and location and I have to take care of my derivative in this particular manner. So, this creates an interesting scenario, for example let us try to do example 1.

Let us, say evaluate acceleration for let us say there is a velocity field which is given to you as k times y square i hat + k times x square j hat. Now, if you had to calculate this displacement, sorry this acceleration, you have to do what we did earlier. You have to take this Lagrangian derivative and this now would simply be my.

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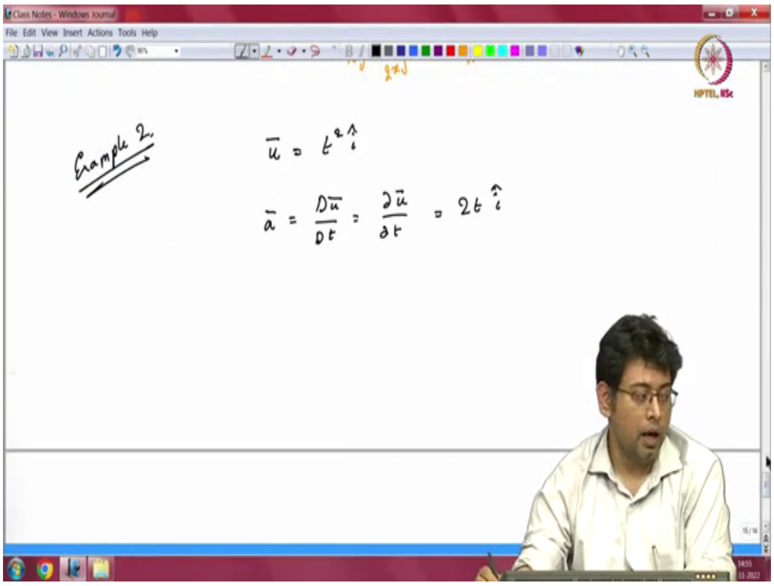

So now, we already can see that this term is 0. This velocity is not fluctuating with time, so, you only have contributions from these two terms and you can very easily identify what these are. So, this is, for example = ky square and this is kx square and if you have to evaluate this it would simply be 2 xj hat. Similarly, you can do it for the other one. I am not going to complete this.

The reason I am trying to you are the one I wanted you to look at this example is because interestingly we are used to an idea of acceleration when velocity is changing in time. But in this example there is no change with type. There is only change in the spatial direction and yet you have an acceleration. This is what changes in the Eulerian view of things. Similarly, if I one more quick example, example 2.

Let us, say my u is nothing but some t square i hat then you now have, if you have to calculate acceleration in this case which again just like the before you just end up with this which is 2t i hat. So, in this case my because I do not have any special terms, my acceleration, in the Eulerian, the Lagrangian, if whichever derivative I used to take, would have been sort of the same. So, here things are slightly simpler.

So, this is a key difference, so the take home message is there are because in container mechanics the way we look at because we the way we look at position and how we do a bookkeeping of what and how? So, what are we doing a bookkeeping off? And how we are doing a big book bookkeeping off? In Newtonian mechanics you only have to track a point or some simple finite number of points.

And then this Lagrangian way of doing things sufficed and there your velocity now is a function only of time. Whereas here it is something is quite different and the reason is now you are doing a bookkeeping of this huge infinite number of points in container mechanics. So, this is an important mathematical preliminary to be doing for the sake of this course. Because we will be needing to evaluate these in our experiments.

So, it is important to realize how the experiments are being done and also to make sure that when you are measuring velocities in experimental systems, they are what you intended them to be or what they should be. So that is why we are covering all this. These are again all preliminary requirements for the flow of part of this course. In flow visualization we will use these equations that we had developed in order to understand what a streamlined streak line and path line is:

We will come to that in a few lectures there is one more mathematical preliminary that I would like to do right now and that concerns correlation and covariance. So, it is honest it would appear that it is slightly different from what we have been doing but you will see that in PIV these concepts are very, very important and are very much required for flow quantification.

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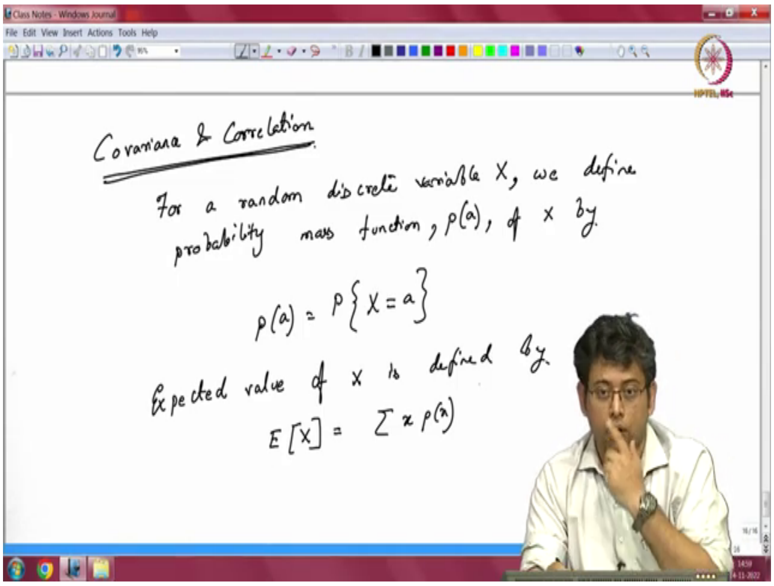

So, let us look at covariance and correlation functions. Now, in order to do this, we will introduce the idea of covariance and correlation from the perspective of probability theory. So, in probability theory we have, if you have done a course in that you would have experienced the concept of a random variable that takes on different values at different instances or it can take a range of different values.

So, let us say we have a discrete random variable x so, for a random discrete variable let us say x we define something called the probability mass function. We will call it, let us say p of a of x by this equation. So, the probability mass function p bracket a is basically nothing but the probability that your discrete variable X takes the value a. So, this is how my probability mass function is defined.

And if once you have something like this then you can also define an expected value x is defined as so, my expected value often written as E of X. This is defined as a sum of x into px, where x is a value that the variable capital X takes so, this capital x is the way, is the discrete variable and x is the value that this variable takes. So, this is my expected value now, you will recognize this is more or less like doing an average in some sense and we will use this idea of expected value later on.

For our analysis, I am defining now everything in the way you might be accustomed to if you have done a course in probability theory and then we will extend this to time series data or a series data.

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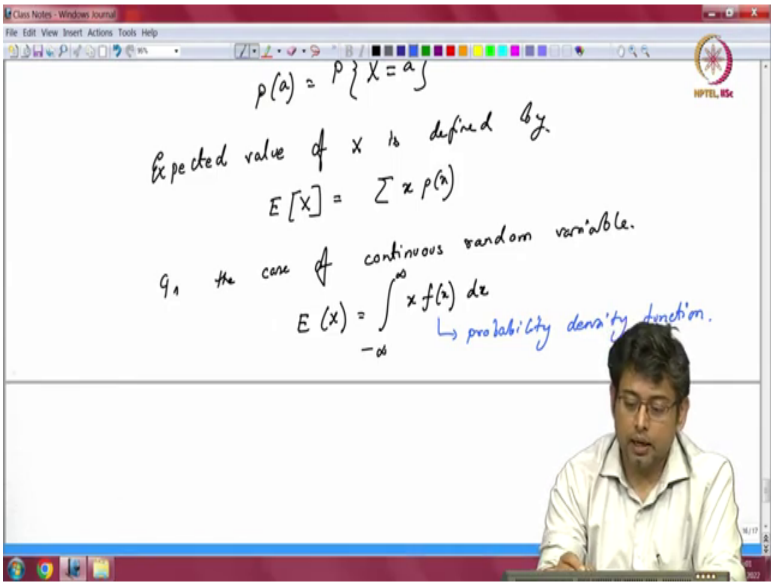

And in the case of continuous functions continuous, sorry, not functions continuously random variable, I am sorry continuous random variable. We define something called as the your expectation is now defined in terms of so, minus infinity to so, this variable x can take, let us say any value between minus infinity to plus infinity and you have x and then we have an fx which is also known as the probability density function.

So, the probability density function is a function such that integral of just fx dx = 1. So, the probability of that random variable occurring would be just 1. Similarly, here the just the summation of px over all possible values of x that is going to be 1. So that the variable is going to occur is going to take at least some one of the values in the given set. So, this is just one particular variable.

Now you can extend this idea to two related variables, x and y. So, if you have two related variables then we are going to define what is known as the probability the joint probability mass function. And from joint probability mass function we are going to define a function called covariance. And we will see how correlation is related to that?

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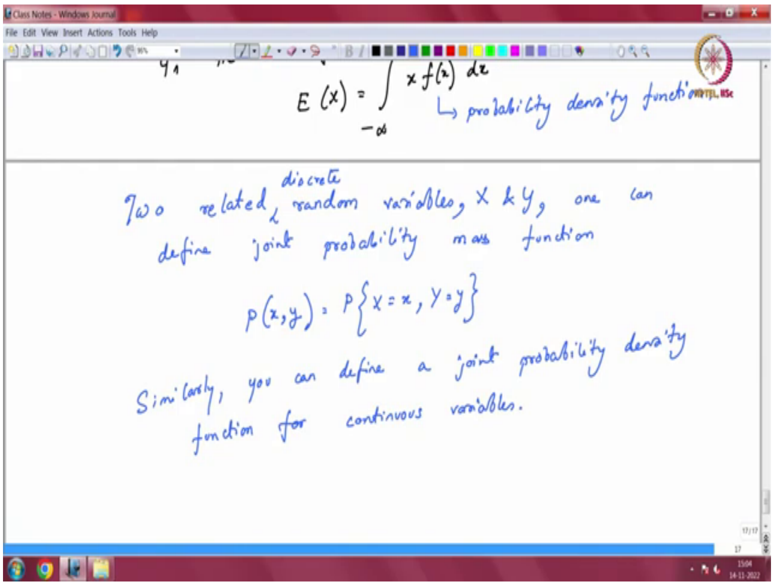

So, let us say you have two related random variables which will call x and y one can define the joint probability mass function as let us say p of xy is; oh sorry, this should be small, so, in probability theory usually the capital letter is, is reserved for the variable and the small letter is reserved for the value it takes. So, here where I am writing it as small x. It basically means that the capital X takes a value of small x.

So, this means that this is equal to the probability that X takes the value small x and capital Y takes the value small y. So, this is my joint probability mass function. And similarly, you can define the probability density function for x and y. So, for continuously, this is for stew, related. Here I have missed the term discrete. So, this is for a discrete case and similarly you can define a joint probability density function for continuous variables.

Not A for continuous variables X and Y, I am not writing X and Y It sort of understood that it is for X and Y.

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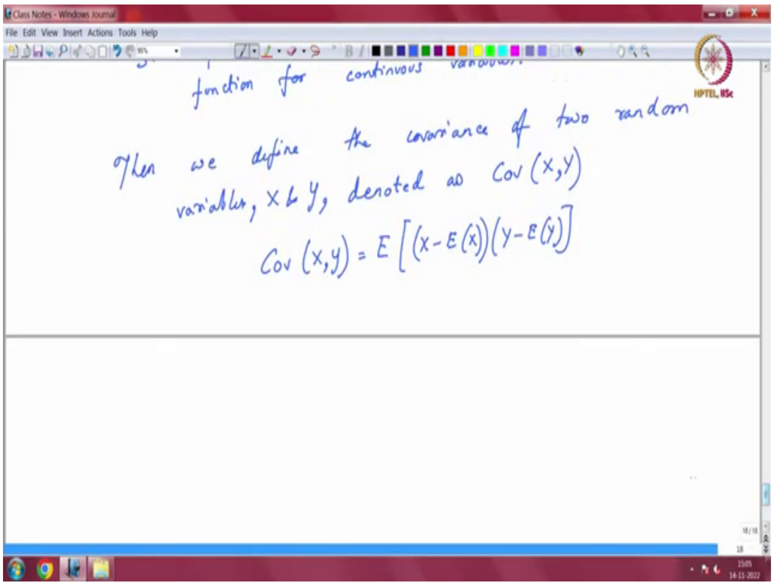

So, now, we want to define something called covariance. And so then we define the covariance of two random variables capital X and capital Y denoted as cov. So the function covariance is denoted as the letters cov. And so, cov now, x, y is nothing but expectation of X – expectation of X and Y – expectation of Y. So, this is the definition and you will see that I am writing in capitals.

Because this is not yet taken any this is consistent with what we have defined before. So, this is expectation of capital X and on the hand side you had the small letter. So, my expectation is in all capitals.

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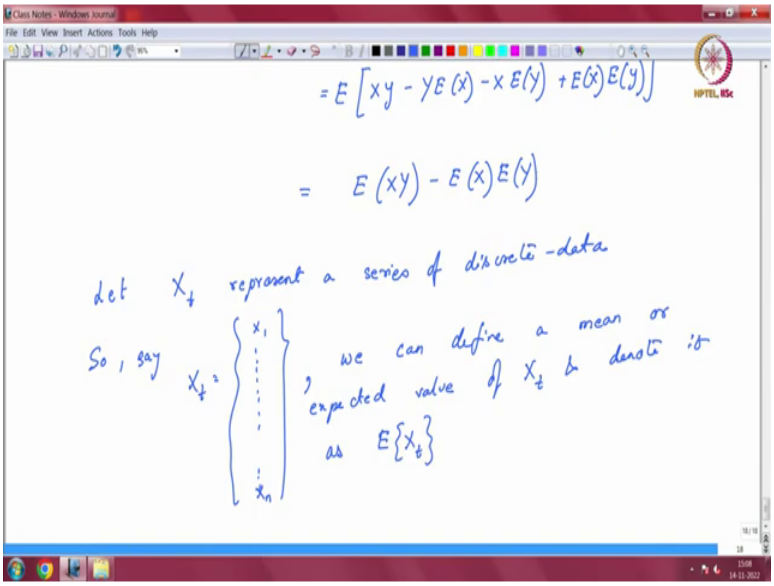

So, this you can simplify also you can now write it as expectation of Xy – Y extraction X – X expectation Y and + expectation X expectation Y. Now, if you simplify this now expectation

X that if you take expectation X is already a fixed value. So, if you are taking going to take expectation of Y expectation of X it is nothing but expectation of Y multiplied by expectation of X

So, if you were to simplify this overall thing here, you essentially end up getting this. So, this defines a function called covariance which is a very, very important function. So, let capital X t represent a series of discrete data. So, basically say X t is nothing but a list of values, x n small value. So then, taking our q from the previous expressions that we saw, we can define a mean or expected value of capital X t and denote it as.

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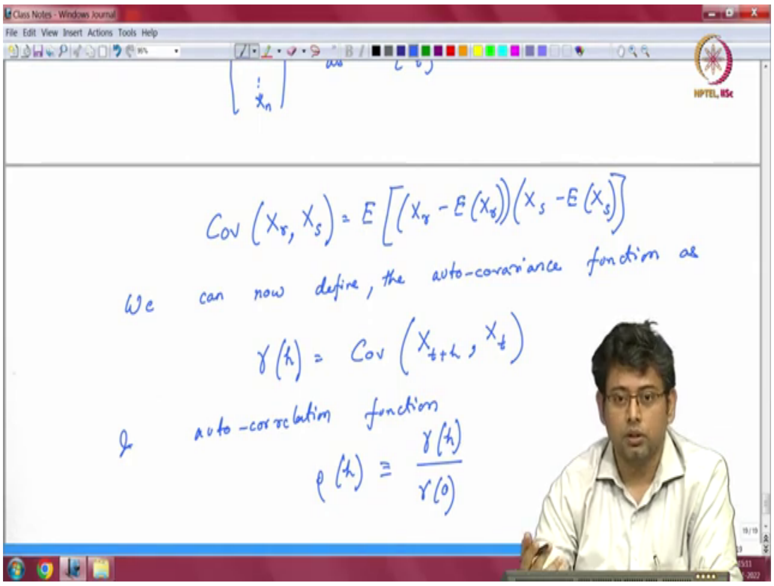

So basically, my subscript is telling you that it is a series data and you can have two different series if you want and you can do a covariance of X r let us say X r and X s and this you will define it again. Just using my previous so, here or sometimes people just use mu and this is; so, now, if you have to define something called the auto covariance function then the word auto implies that you are using the same series data.

So, if I had to define now the auto covariance function which is also called which is called ACVF in short. So, we can now define the auto covariance function as let us say this is some value h. As you are taking now data from the same series, so, this is X t + h and let us say this is X t where H is variable. So, you can vary it from across different values and you can now define an auto correlation function and use this same.

Now, this gamma H to define an autocorrelation function which is nothing but a covariance function which is normalized. So, the auto covariance function which often is written as rho h is nothing but is defined as my gamma h by gamma 0. So, basically, 0 means my h = 0 here. And here you will vary h again at different values. So, my h is varying so that I can hop from one discrete value to the other.

So, this is my auto correlation function and cross correlation function is nothing but you are doing the same thing now with two different series. So, you have some series data here and you have another series data xs. Then, if you do the covariance function of two different then two different series, you will call it a cross correlation. You will not call it just called a covariance and when you do the correlation, this is called cross correlation.

Just simple mnemonic to help you remember. So, now these are two important functions that we are going to use. We now want to understand what is the physical implication of this? So, why do not we try to understand with a couple of examples here?

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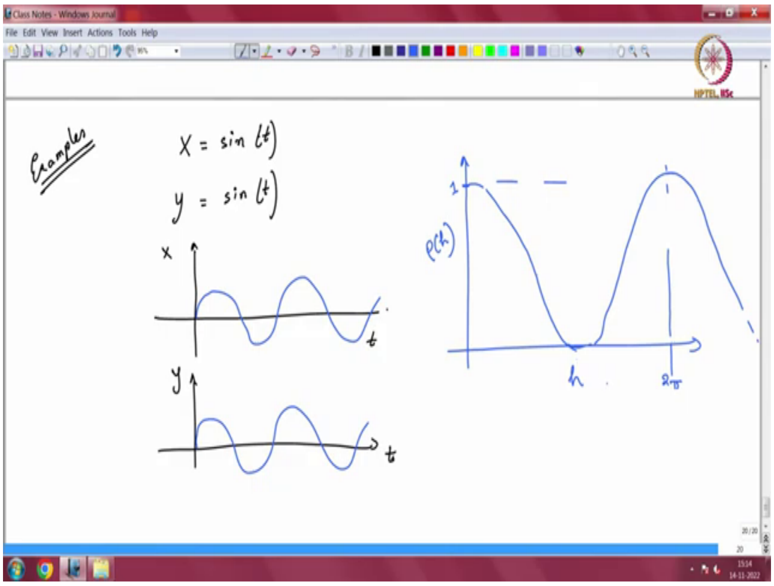

So, let us say we want to understand, what this really signifies? So, let us say you have a function X which is nothing but some sin of t. Now, here I was defining it for discrete data. You can define the whole thing again for continuous data also but I am here I am just going to show you. I am not going to derive or explain everything in too much detail or I am sorry rather, I would not do too much extensive calculations.

I just want you to understand the physical implication of what his correlation function means. So, let us say you have two discrete sets of data, one coming from sin t and again another one also coming from sin T. So, what you have you have nothing but two data sets both are resulting from the same type of graph which is a sin t. So, now what I want to understand is how will my variation in rho look like rho h as a function of h why?

This is essentially now become a autocorrelation. Now, you see you are going to be multiplying the way. things are you are now going to be multiplying here, expectation of X Y– expectation of X – Y. What you will see is in the autocorrelation function. You will get a maximum here when your h as you change it over time, h is the maximum at sorry rho h rho h is maximum when h = 0 which is at 1.

And then it decreases and and then finally, it keeps on repeating itself. So, it again reaches a value of 1 at a distance of 2 pi and keeps on repeating. So, this function rho h is now my looks like a periodic function, just like the original function. But basically at h = 0 it is saying that these two functions have the maximum overlap in some sense or these two are the maximum alike.

So, because these are the two same functions, they are obviously going to be the most alike when they are just overlapping each other. So, there is no, so that in that sense your autocorrelation here becomes a maximum h = 0.

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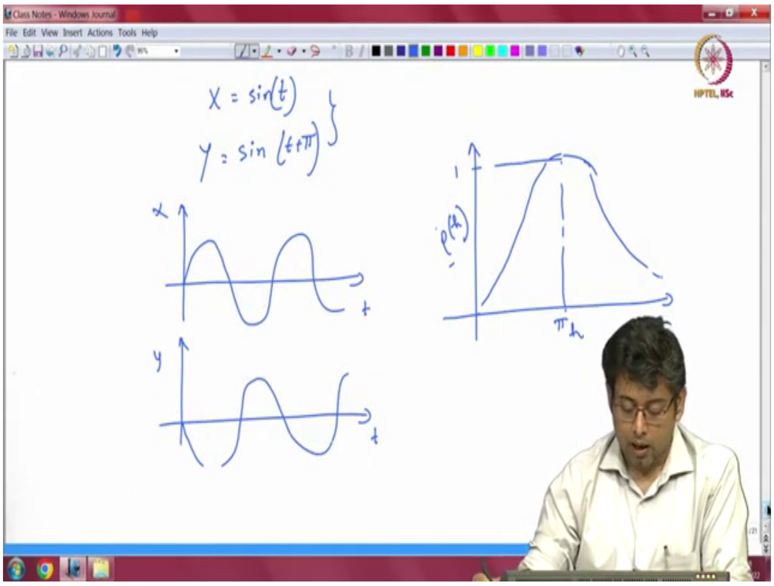

And you can do this calculation for yourself and see. I am not doing it here but if you now change this so that my X = sin t and my Y was now coming from sin t + pi. So, my X now looks like this, whereas my Y is going like that: So, you see, here the 2 are almost in opposition at when they sort of when they are. If you just took this particular Y and moved it upwards, when there is no, when the h is 0 or the displacement between the two function is 0.

My rho is is now going to be 0 here and when you move these 2 by pi again, you will have the maximum overlap between the two. So, these two functions become similar when you translate the one of the functions by pi. So, your rho now reaches your maximum at of a value of 1 at pi. And everywhere else it has a lower value and this keeps on repeating. So, this is again a periodic function.

So, my correlation function here essentially tells me that if you have two different waveforms, by what value I can shift them in order to get the maximum overlap. If the two functions are not the same here, I have deliberately taken sin and sin which is? Why it is reaching a value of 1 if it were not the same function? It would reach a value lesser than one whenever there is a maximum overlap.

But basically what this is telling you is? By how much can I displace one function? So that the match between the two functions is the maximum. So, if you did not know these two functions were not given to you a priori or you are just given a set of discrete data. You could calculate the auto correlation function or the sorry, the cross correlation function here and you would be able to tell.

By what value should you displace one of the functions in order to get a good match? So, this is an idea that will use quite a bit when we do particle image velocimetry. And you will see that we will have unknown displacements that we will try to back calculate by moving one image over the other, displacing it by various amounts. And seeing when the cross, when the cross correlation function becomes a maxima.

And that way we will be able to get very accurate measurements of unknown displacements. So, this is another mathematical preliminary that you might want to go into by yourself and acquaint yourself with because we are going to use these ideas. So, we will stop here today and we will take up a slightly new topic in the next class. Thank you.