Optical Methods for Solid and Fluid Mechanics
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Module No # 09
Lecture No # 43
Cone Beams, Parallel Beams and the Feldkamp Algorithm

In the last session I introduced this technique called the filtered back projection algorithm and we went through a sequence of steps that you have to typically follow to implement a very simple version of the filtered back projection algorithm. Now this algorithm itself is fairly old it is been around for the 30, 40 years at least and so there are very sophisticated implementations from a, programming point of view.

There are parallel implementations there are filtered black projection schemes with minimal data with data loss with limited projections and things like that. We will not have time unfortunately to discuss many of these interesting ideas some of them actually constitute areas of research even to this day because x-ray tomography general is a very active area of research.

Today will be our last session on tomography and what I will do is I will talk a little bit about more realistic geometries. We spoke about this parallel beam projection basically assuming all the beams that were coming were parameterized by the same theta and the same and of course varying t right for a given projection and that is not always the case right. Especially when you go to, 3 dimensions you have what is called a cone beam so you have a point source and you have light coming from a point.

So the x-ray is coming from a point source and so you typically have a cone with its apex at the source and going all the way to the director right. Now and you know on the face of it looks like a very different proposition because everything we have done so far has dependent on, this Fourier slice theorem and that seems to be the cornerstone for our inversion scheme. And of course the Fourier slice theorem itself was obtained for a parallel beam projection.

But it turns out that the corresponding calculations for cone beam or in 2 D what is called a fan beam are little bit more algebraically complicated.

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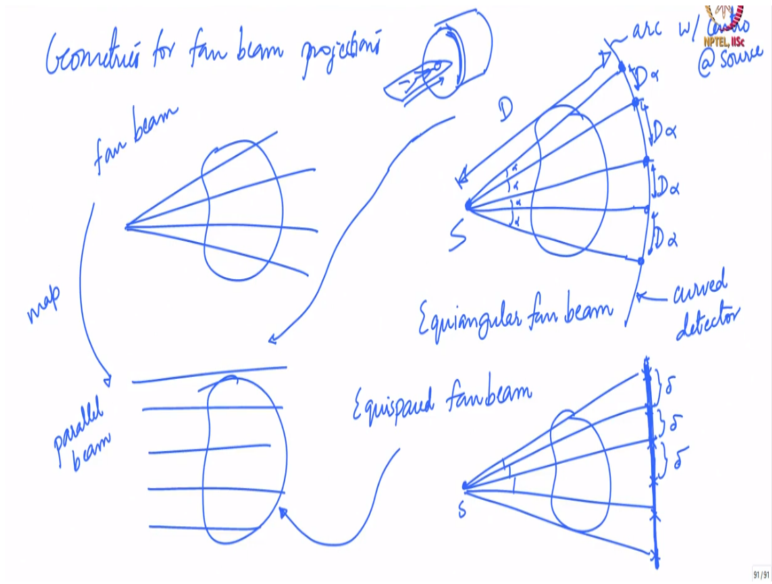

In the sense that if we go back and look at the geometries for 2 D fan, beam projections. Now this is single source point source and you have your object like this and then rays are going out like this behind inverting data that you obtained from this versus data that you obtained from this for the same object. The whole idea is to find the map for raise for rays in the fan beam find a map to get rays from the fan beam to an equivalent parallel beam because we know how, to reconstruct from a parallel beam.

So if you can map somehow the rays from here to this then we are done right it is from here to here and that really is the crux of the matter right. The whole algebra the complication comes up in establishing this map from here to here right. Now one of the things we want to keep in mind is that if you have a cone beam now there are 2 different possibilities if, you have a cone beam like this you can obtain data let us say this is our source.

We will talk about fan beams right in 2 D not cone beams are in 3 D we talk about 2 D for now when we come to 3 D later you can obtain data at equal angular intervals. Which means your detector is actually a circle or you know part of a circular arc with its origin at the center right where the source is. So, this is arc with center at source this is the source s if you do this then the detectors are spaced let us say this distance is d then the directors are all spaced equidistant but on a curved line right.

This distance is d t alpha this is the alpha and so on so equal space pattern line so this is an equiangular what is called an equiangular fan beam and this will only work when you have a, curve detector. So scheme for mapping this situation to this will be applicable when you have a curve detector. And typically when you go and do x-ray micro city let us say for you know imaging the brain or somebody's skull or something like that you will see that they go into this cylindrical structure in the person is placed like this and fed inside sitting on a bed.

And here the vectors are, all on a curve so the curved map is a little bit different you have to take into account the fact that the director is on a curved line or in this case a curved surface. The other option is if you have the same object and your source is still the same and you are passing rays again as before. But now your detector is actually a flat panel like this typically in your cameras you know even your, DSLR cameras or you know phone cameras and so on your sensor is flat your sensor panel is flat.

And the sensors are placed at equal distance call this delta for example right then array it is a 2 D array not in this case it is a 1d array we are looking at a 2 D version and the 1 D array has equal spacing. So the sensors are always based in line at equal distance if you take the corresponding angles, they will be different. So they will subtend different angles of the source but they will have same distances between detectors.

So this is called an equi-spaced fan beam and the map from this to the parallel beam will be really different from the map from this to the parallel beam. So you have to worry about which detective geometry you have now which little geometry you are dealing with in this case cool. So how do we do this?

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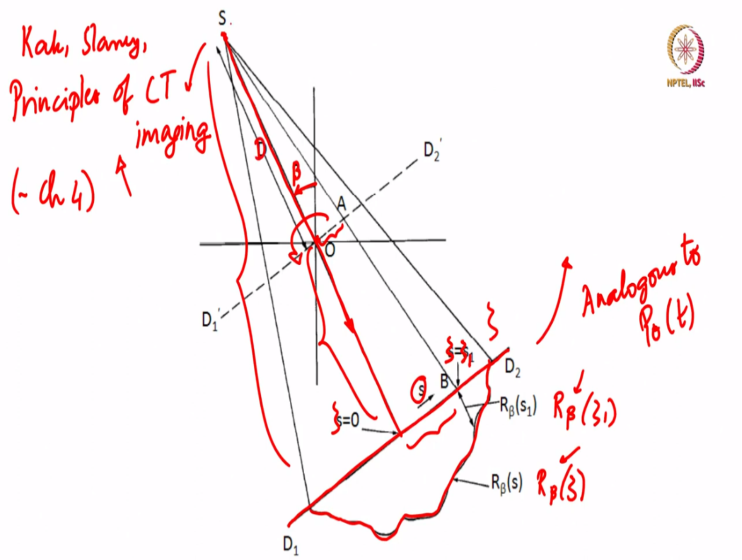

So I am going to work out some of this geometry very quickly for the equi-spaced fan beam not the equiangular case. We will not be able to work through all the details so I will give you a reference where you can find much of this information book by Kak and Slaney I might have mentioned it at the beginning it is called principles of, computerized tomographic imaging. And you will find much of this information in chapter 4 of the book including for cone beams and so on.

For now I will just discuss the equi-spaced fan beam so that we are on the same page the figure that I am going to show you here and the figure on the next slide are both adapted from the same book. So I would highly recommend this book because it, is a very readable introduction it is very self-contained and it is in my view it is a very excellently written book on computer tomographic image it is one of the first books that came out that discussed many of these things in explicit details so implementations could be done very easily.

So if you are interested in implementing some of these things I would highly recommend this book. So, here is a typical equi-spaced detector so a detector is d1 d2 that is right here and our source is here ok if you take a look at the razor I mean we are assuming the source is a point source right. So the source does not have any dimension it is a single point everything is converging to that point all the rays that are coming here are converging or diverging from the point whichever way, you look at it.

Now the book uses the variable s I will use the variable zeta on the next page because we have used s for the variable along the ray direction. So just to avoid confusion I am going to use zeta so this is let us say zeta = 0 this is zeta = zeta 1 and this is the intensity that you measure now this is not the same as p theta of t because p theta of t, was obtained for a parallel beam right.

This is something else let us call this R beta of zeta 1 and R beta of zeta in general zeta being the variable along the iterated direction. Now we do not have a single angle in this case like we had theta in the previous because we do not have a single angle because the center line from the source the center point of the vector let us say passes, through the origin. So we will make that assumption the center line passes through the origin and the angle subtitle by the center line is this angle beta.

So that is the only angle we can fix because the source in the vector are kept together and either the detector the object is rotated about its axis or the source and the regulator together about the object's origin as being the center. The, only thing that is fixed is this line right the other lines are all in relation to this line but you can get the relative position a relative parameter describing one projection by specifying beta right.

I could as well take some other line here and then say I am going to specify that angle it is fine it will just be algebraically more complicated right this makes our life easier because it passes through the origin of coordinates. So we will use r beta of zeta this is analogous to p theta of t but only remember that beta is only for the center line not for every single ring. So here is what we will do so we will move the detector from where it was if you go back detector was here so we are going to move the director to the origin.

So we will place it at this d one prime d 2, prime just to make our algebra simpler so our zeta will not be measured on this distance but will actually be measured on the translated directive. So you have to scale it accordingly you will have to take this distance and this distance and then use the ratio to scale it which you can do simple using simple trigonometry resistance is denoted d the distance from the source to the detector can, be something else and you have to use that ratio to scale zeta.

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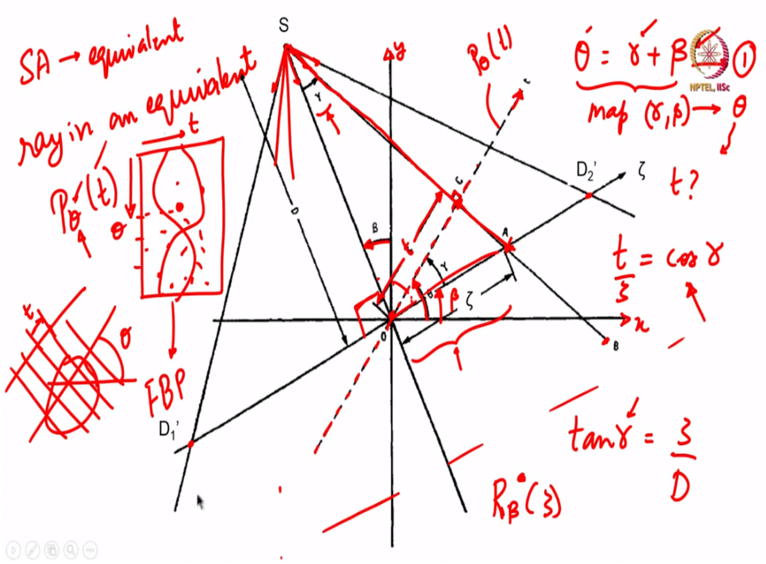

So here is our zeta here and beta is as before this is the source again and then these are the beams that are going out it is an exaggerated version of the previous schematic. And this is our translator detector the actual director was much further down you go down and actually it was somewhere here but we have taken it out. Now what is our aim now? Our aim is let us say we have one ray coming like this that has some value of the parameter zeta we call the zeta naught whatever.

And its origin from the same source when; the source is inclined at an angle beta to the vertical the y axis remember there is the x y axis for the object. This is our line this line itself is parameterized by 2 quantities one of, course is zeta and the other is gamma right if you change gamma you will get this line or you get this line you get whatever line right. So that is determined by gamma m z does the other line as the other variable now gamma and zeta are of course are not independent.

Because if you choose some gamma then you draw a line it will come and intersect at a corresponding value of zeta. So you can, pick either one either gamma or zeta in the case of the equiangular distribution you will use gamma in the case of the equispaced detectors you will use zeta that is the only difference. Now the aim is we take this line let us take the line, SA and we will map it to an equivalent ray in an equivalent p theta of t.

That is what we have to do once we do that and if you can do this for, all SA's parameterized by beta and zeta beta being the inclination of the source detector pair and zeta being the intercept then you have a map from the fan beam to the parallel beam. And if you have a fan beam data you can generate parallel beam data from that and then do the usual filter back position that we already discussed.

So that is the logic that we are going to use here now how do we do, that? The first step is to use some geometry here so if you look at this dashed line that is here this line is perpendicular to the line SA right this angle is 90 degrees. So if you were to think of this as being some type of parallel projection this line will be the normal to the parallel projection. If you remember the parallel projections for the object an object is, somewhere here of course not drawing the object because it will interfere with the lines here and make a mess.

But if you remember in the parallel beam case the orientation theta for the parallel beam was given by this angle right this angle whatever correct this angle is theta that is exactly the same theta that we have marked here. So this is a hypothetical theta for an equivalent parallel beam, which determines the theta in the p theta. So we know theta or we need to know theta that is what we need to know right and how do we get that well?

If you look at the simple geometry you know this angle is beta you know that this s line the middle line is perpendicular to the director right by definition because the midpoint of the detector. So this angle is 90 degrees so this angle is 90, degrees as a result so this angle will be beta. Because this angle is also 90 degrees and this angle is also 90 degrees this will be 90 – beta.

This will be beta so the first relation you have is theta = gamma + beta so you know theta. So if you are given a particular projection you know beta a particular ray in the projection has a particular gamma. So from that you can get an, equivalent theta so your first parameter in p theta of t is known you know which projection in the p theta of t you need to populate with the information coming from this ray with the r beta of zeta from this ray.

But you do not know in what at what point along the t axis this data has to go right. So now we have mapped from gamma beta or a single ray in the r bit of zeta we mapped to the theta, projection that we need for the parallel beam case. But we do not know what; is the t that we need to map to that single data point that is coming at this electrical point A. You can get that by doing somewhat simple trigonometry which is basically looking at this angle.

So this is now t right because this is your hypothetical p theta of t plane the line whatever so this is t from the origin to the, point at which it intersects the plane just like we had here right this was t for this ray this was t right. So now this for this ray SA this is t so you need know what t is and how do you do that? You write t by zeta t by zeta this Cos of gamma in this triangle right you have t you have zeta t by zeta is Cos gamma.

So now you need you know which t to map to right how do you get gamma now gamma's, gamma if you knew equiangular you knew what gamma was. But now if you have an equal space detector you can get gamma simply by writing tan gamma is zeta by d tan gamma in this triangle is zeta here divided by d distance of social relator. So if you know r beta of zeta then you can first get gamma of course using zeta by d is a fixed value because it depends only on the distance between the, source and the detector.

So you can now calculate gamma once you know gamma you add gamma and beta you will get theta. For a given zeta value you do t by zeta is going to cause gamma you get gamma you add it to it you get theta. So you know in what if you remember our discussion the sinogram business right you had t going like this and you have theta going like this. Remember you had this, thing that looked like this I showed you an example in the last session maybe a couple of sessions ago.

You know the theta coordinate from this equation from one you know the theta coordinate so you know where on which slice to update. From 2 you know the t coordinate so you know where to come over here at what point to update right so this value will be equated to this value in the fan beam, projection. So now you run through the entire family projection calculate t calculate theta and then you start updating all the points with the data from the final position you will get an equivalent parallel beam projection.

And now from here you do the filter back protection and you will get the final reconstruction that is the simplest most conceptually clean way of doing this. Now there is a slightly more sophisticated way of doing this using the impulse response function I will not discuss that here for want of time. But if you go back and look at the reference I provided on the previous slide you will find details on how this implementation is done.

So that was about the fan beam so this is how you would do a reconstruction for the fan beam for the equispaced, fan beam now I will just talk. Just for completion sake I will talk about the 3 D cone beam case.

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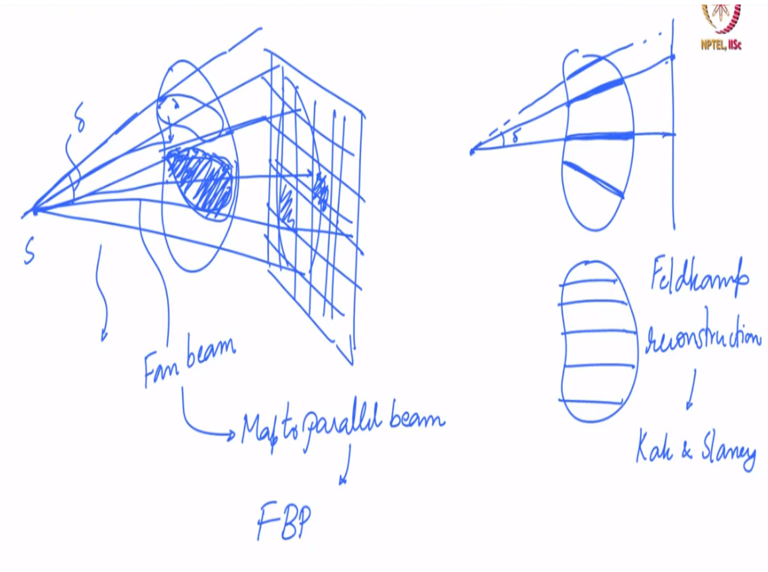

You know geometry is very the approach is very similar you have to just do the following. So let us say I have a 2 D equal space to the detector so each point here is 1 pixel let us say whatever equivalent of the CCD in your DSLR it is an x-ray detector just 1 pixel. And you have a source, that looks like this and then you have a beam it is going out like this and it is going to generate data on a certain set of pixels.

The idea here is exactly the same as the idea that we used for the fan beam case so you take each cross section in each cross section you will have a fan beam going like this in each cross section so this is a cross section you will have this will be a fan beam. So once you have the fan beam you map to a parallel beam and from there you do the filter back projection to get the data in this particular cross section.

And then you go back to the next section you have an equivalent fan beam but now if you look at this fan beam I have done it exactly in the same plane containing the detector and the source point. But if you go a little up then your fan beam, is basically tilted right so now you will have some additional geometric relations to basically project this back down to this parallel plane and then use this to map out an equivalent location for an equal and parallel beam and then do filter practitioner for that right.

So there will be some correction depending on this inclination angle which will call delta may be because if you look at the, side view this is a source the director this cross section is easy it is just straightforward it will be a fan beam in this plane. But if you look at this cross section at an angle delta this cross section. Now you will get information on an angled plane so you have to be careful in how you interpret it right.

And the reason you have to be careful is that a director now the distance will change so, you have to do something equivalent for a point here for a point here so you get cross sections like this. If you do the same slice by slice fan beam business that we discussed there is a slightly more geometric geometrically cleverer way of doing this which is basically to take each of these in a cone beam and map it back to parallel sections like you did for the 2 decays.

But now for the 3 D, case and this method is called the felt camp reconstruction method it is analogous to what we have discussed so far of back mapping using simple trigonometry. And you will find details of this in the book that I have discussed and slightly more tricky to implement but it is based on essentially the same ideas that we have discussed so far. And this is how typically cone beam, reconstructions are done right.

Again it is just a map slightly more sophisticated map but it is a map to come back to a parallel beam type case and then use this Fourier slice in the form of a for you in the form of the filtered back projection scheme.

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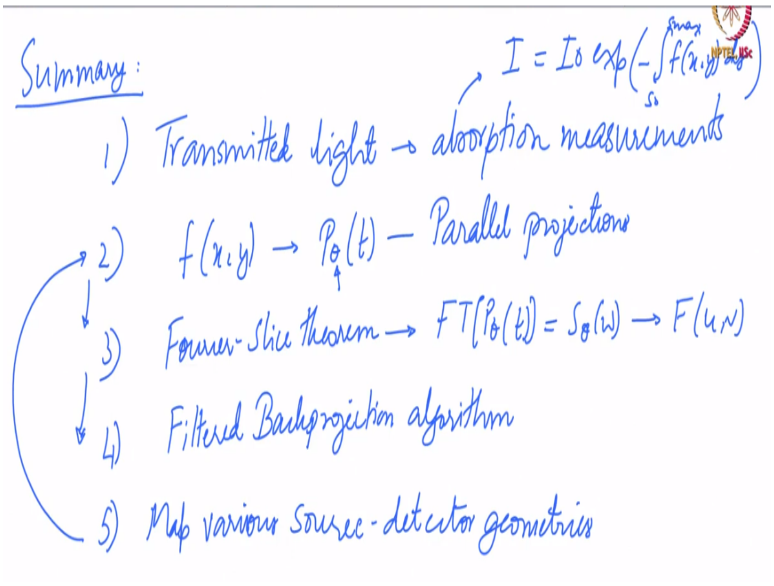

So with that I will just wrap up our discussion about tomography so the basic idea is you have first of course transmitted light and you are, looking at basically absorption measurements. Unlike the other cases where you are looking at changes in polarization you know changes in refractive index changes in speed. Now you are looking at absorption we set up this using this Beer Lambert law right.

And then we said that you have to find f of x y from a series of projections which we call p theta of t for various values of theta these were, all parallel projections. And then we derive this Fourier slice theorem to relate Fourier transforms of p theta of t which we called s theta of w to the 2 D Fourier transform f of u v along a radial line. And then we reinterpreted this Fourier exercise theorem to come up with a filtered back projection.

Algorithm which is a very clever and efficient way of implementing this inversion scheme based, on the Fourier flash theorem and then we saw how to map various geometries various source detector geometries. So if you have a fan beam as you have a cone beam equispace angular and so on right and this only use simple trigonometry and then it basically fed back into this. And then from there you could again go through the same sequence of steps.

Now the whole idea of x-ray, tomography is an area of active interest there is a lot of research that are still being done in this particular domain what we have done is barely scratch the surface. There are lots of interesting topics that you could pursue from here I hope this will give you a background for getting started. For instance this idea of using multi-energy x-rays to get spectral information is something that is very actively used both in you know difference applications airport security we mentioned some of that at the start.

There is also inversion with limited data so for example if you are trying to reconstruct the internals of a pipe where you do not have access to the entire 3 dimensional you know pi degrees of rotation you only have data that you have taken from a few angles. Let us say a, pipe is like this and you have only taken a few angles on the top how do you reconstruct from data that is you know restricted to a certain domain in the theta plane that is another area that is of active interest and there are also more sophisticated techniques.

So for example if your material not just is not just absorbing but also diffracts right if you have a crystalline sample and you have, a very high fidelity x-ray detector and you have diffraction effects how do your account for that? How; do you use that to interpret something about the system while also interpreting this f of x y type of function? So there are lots of possibilities and lots of possible directions to take this work in the book that I mentioned is a good starting point.

Although now it is a little bit dated it is about, 3, 4 decades old there are more recent monographs that you can get started on but hopefully this will give you some background on which you can build your future investigations in this particular area. So with that we come to the end of this entire module on optical methods for solids. As you can see the area is quite rich there are quite a few techniques some of them are new and they are, being readapted to more contemporary applications in solid mechanics.

Some are you know old and they are still being used and they are very profitable like let us say tomography for instance. And so it is an area that is very fundamental to our understanding of mechanics of deformation of internal structures and things like that. There is increased emphasis on some of these ideas more, recently with you know advances in manufacturing technologies in production, inspection, quality, control things like that and so.

I hope that this entire set of you know dozen or so 3 dozen modules has given you some idea of how to get started what some of the basic techniques are? And how you can imply how you can implement them in your own setting for your own application? And hopefully it will, spur you to investigate some of these techniques in a little bit more detail and try and understand them at a slightly more deeper level than we could cover here.

So with that we will sign off in this course and optical methods in solids and fluids and I hope you have learned significantly new information over the last 20 hours.