Optical Methods for Solid and Fluid Mechanics

Prof. Koushik Viswanathan
Department of Mechanical Engineering
Indian Institute of Technology - Kharagpur

Module No # 09
Lecture No # 42
The Inverse Problem and Radon Transform to 2D Section

So far we have been looking at this radon transform and we started analyzing a method for inverting the radon transform and we established something called the Fourier slice theorem. I will just backtrack a little bit and recap to make sure that we are all on the same page so that our discussion today will carry on from where we left off in the last session.

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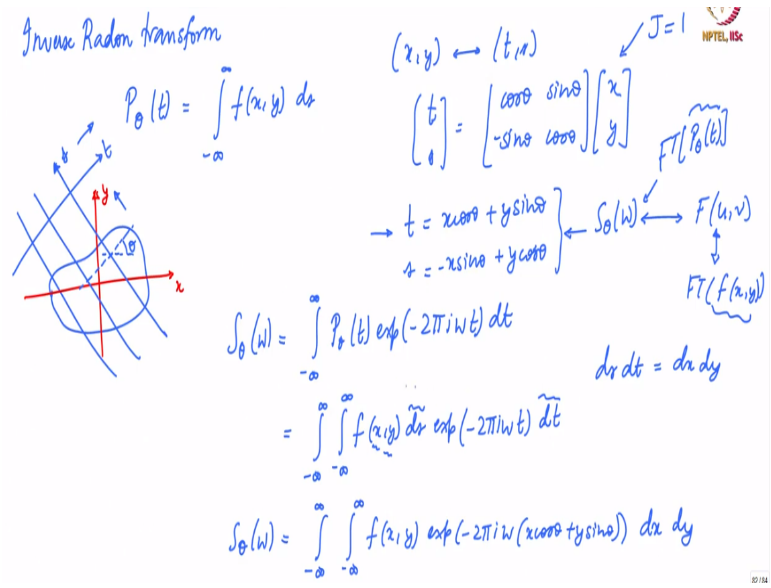

So just to recollect the, inverse radon transforms the radon transform itself as you can see here on the screen is given by this integral of this function. And remember the function f of x y is what we want to find this is your mu in the Beer Lambert law we discussed this at the very beginning when we introduced this idea of tomography. So f of x y you can think of that as like a density for example right and our aim is, to get these projections and then invert these projections p theta of t that a function of 2 variables theta and t.

And then get the actual density function f of x y to do this we made note of the fact that there is a there is a coordinate transformation that relates the t and s coordinates t being along the direction of the detector and as being along the direction of the ray. They are both, perpendicular to each other if you have a parallel beam of incoming rays. And the ts are related to the x y coordinates of the object by a simple rotation matrix right like we have written down here.

And this is just another way this particular relation for t is just another way to look at the equation of one particular line t being the offset from the center right. So we said that if you take us, 1D Fourier transform of p that is shown over here. Then you get this function called s theta of w is a spatial frequency because now t is getting removed t is like a spatial variable in this axis.

And this s theta of w is a multiplayer transform of p theta of t and the s theta of w we said was related to the Fourier transform 2D Fourier transform of, f of x y right. And we worked out this, relation precisely so for instance if you take the expression for s theta of w you will see that there is this integral basically the Fourier integral the 1D t variable. And then p theta itself is of course also an integral right.

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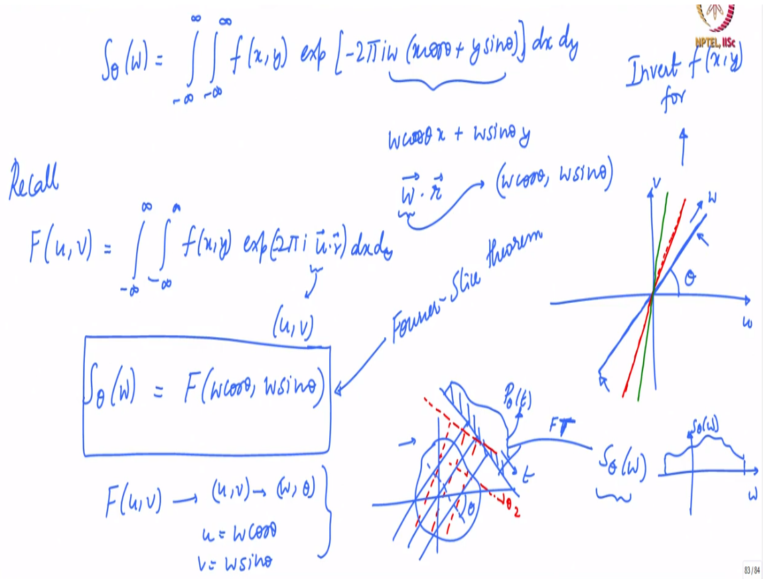

And so we use this to work out in the general case what we would need to relate a set of w to f of u, v right. And we found that by doing some simple manipulation we basically got this expression in the box which we call the Fourier slice theorem. And this relates the Fourier transform of the 1D projection to the 2D Fourier transform in polar coordinates right. And we looked at this in the frequency domain if you take a single value p theta of t you take its one different transform you are basically going to get one sliver along this line a particular line at, an angle theta in the Fourier space right.

And you can imagine doing this at various different angles so you take different projections and then you look at them stack them up one by one you are stacking them radially in 2D space. And the hope is that if you have enough theta slices then you have enough information to cover the entire 2D plane 2D frequency plane u v plane. And once you have that you, can then use the inverse 2D inverse Fourier transform to completely recover the original function f of x, y.

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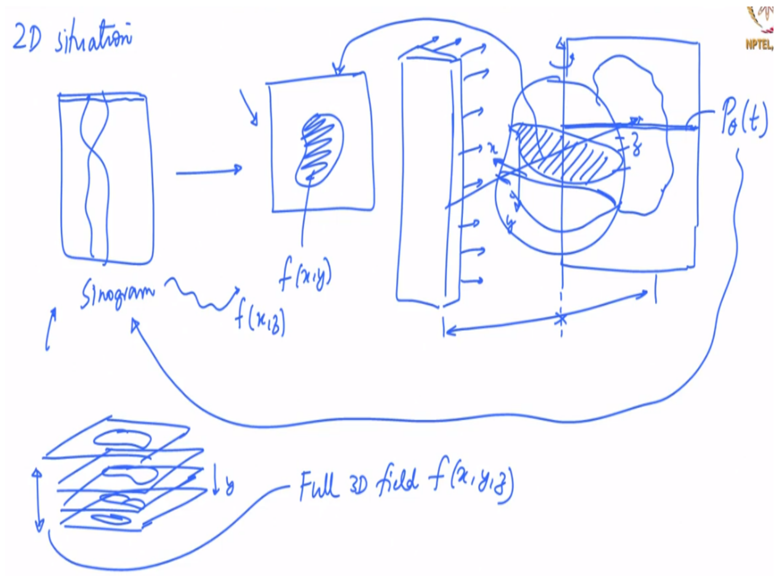

So we said that the 2D situation was like this and I showed you a sequence of images also where you have the sinogram this is generated by taking so each row in this image. For example if you look at the intensity of that image then in each row the image intensity corresponds to p, theta of t right. So you take one of them you take a 1D transform and then you stack it up and then you will get this you get the 2D field right.

We also made a brief mention about how you would do a 3D reconstruction we will talk a little bit more about this perhaps in the next session. But then the idea is simple you have the same configuration but now it is only done layer by layer. So every single, stack is lined up and you do the same thing for every stack you have a huge you know parallel beam coming from here that is the assumption.

Of course in practice it never works like it never works like that you have a point source and so we will talk about how to deal with point sources a little later on. But in principle if you had something like this you can then reconstruct stack by stack for, each stack you have a p theta being the rotation about the vertical axis for the object. So that will give you the set of stacks and you will get a full 3D field in principle again theoretically. So today we look at a slightly different interpretation of the scheme we developed the Fourier slice theorem.

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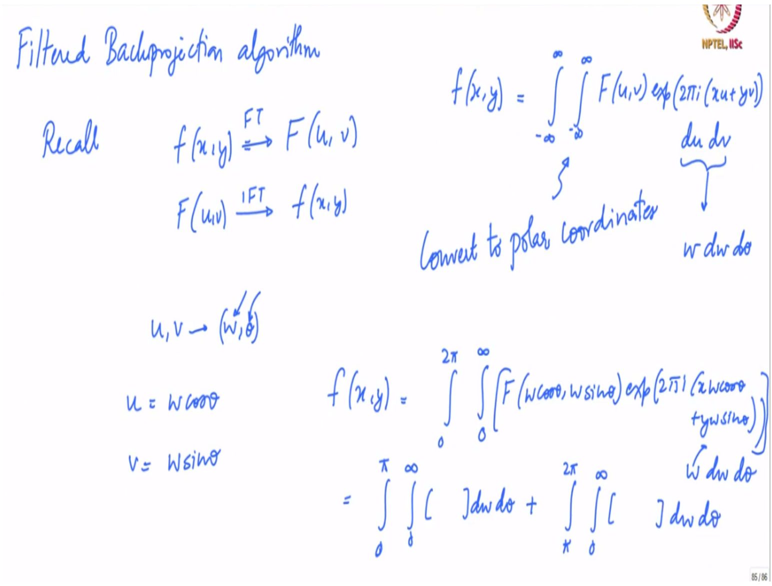

And this is called the filtered back projection algorithm it is an interesting algorithm. Because it tells you it tells you just by looking at the expression from the Fourier slicing is slightly different point of view. You can actually get a completely different interpretation and use that for an actual efficient implementation. Now in the interest of time you will not be able to go through an implementation from start to finish like we did for the case with DIC for example.

But, nonetheless this discussion of the filter track projection should give you an idea of how to get started? So let us say if you recall the 2D inverse Fourier transform of a function let me write down so you have a function f of x y and then you do a Fourier transform you will get F of x y of u v. And to get the inverse transform this is the forward transform, and to get the inverse transform you do the IFT you will get f of x y right.

Now let us start by writing down an expression for the inverse transform if you recall you have something goes minus infinity plus infinity minus infinity plus infinity f of u, v exponential 2 pi i x u + y b times du dv right. For the Fourier transform you are integrating over x and y now we are integrating over u, and v. So we now manipulate this expression and then use the Fourier slice theorem to come up with a more signal processing heavy interpretation of the exact same expression like I mentioned.

So let us start by converting this to polar coordinates so you basically convert u, v to w, theta just like we had before w is like a radial coordinate and theta is like an angular coordinate, but this is in frequency space. So your u will be w Cos theta and v will be w sin theta so F of x, y will be integral from 0 to 2 pi for the theta variable integral 0 to infinity for the w variable f of w Cos theta, w sin theta exponential to pi i x w Cos theta + y w sin theta.

And du dv if you convert to polar coordinates will give you w d w d theta right like r d r d theta only, now when the frequency domain. So this is nothing fancy so far we have only written down the inverse Fourier transform in polar coordinates. Let me just now maybe I will just use this so just use this instead of rewriting it again let me just split this into 2 parts right. So I have 0 to pi 0 to infinity of this entire thing dw d theta this entire thing with the w inside right plus, integral pi to 2 pi integral 0 to infinity this entire thing again dw d theta I am only splitting the integral.

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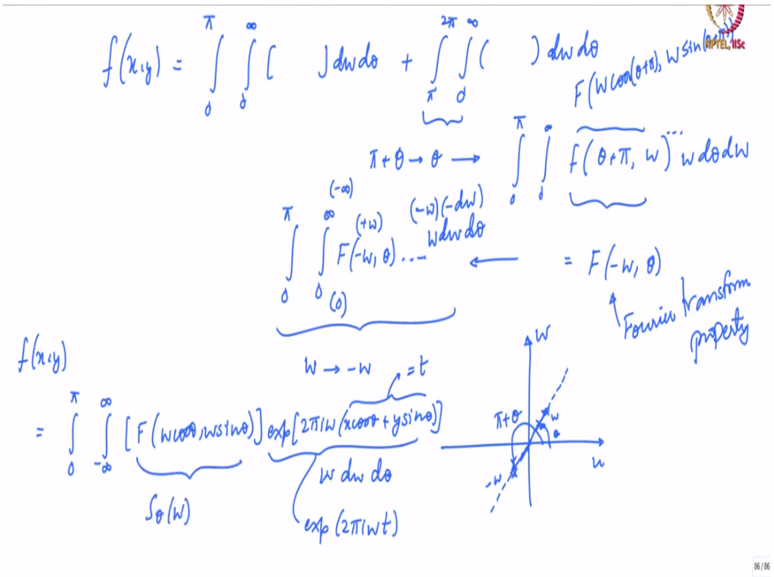

So this is easier to rewrite because I am not writing the full expression so you have integral 0 to pi integral 0 to infinity the entire thing dw d theta + integral pi to 2 pi 0 to infinity w d theta. Now in the second integral I can change variables so I change pi plus, theta to theta so pi + theta I am going to call theta right which means I have to subtract pi from the actual theta that is in the integral.

So the second integral if I do this will be 0 to pi 0 to infinity everything else will remain unchanged but you will get f of theta + pi, w right this is the same as f of w Cos theta + pi and so on right. Now there is a property of, the 2D Fourier transform that this is equal to F of minus w, theta this is a property of the Fourier transform which basically says if you go in the frequency domain and you take some you use w, as an angle as a directional variable.

So w in this direction is positive w the direction is negative then w at theta and w at pi + theta can be related you know F at this particular point. For, example at f at this point is the same as F of minus w at theta that is what this says this is u this is v. So if you do that then this integral gets converted to 0 to pi integral 0 to infinity F of minus w, theta with all the exponential stuff that is all here I have not written that down in the previous integral to you have w dw d theta.

Now in this I will change w, to minus w change of variable if you do that you will get a minus w here you will get a -dw here and you will get a + w here and your limits will go from minus infinity to 0. So if you do that and you combine that with the first integral you can basically write f of x, y is integral 0 to pi integral minus infinity to plus infinity of this entire thing which I am, going to write down now which is F of w Cos theta w sine theta exponential 2 pi i w x Cos theta+ y sine theta w dw d theta.

Just some simple manipulation nothing very fancy right and it is just still the expression for the inverse transforms. Now remember the equation for a line was x Cos theta + y sin theta = t that was the equation for the line that we used at the start, right. So this fellow inside is exponential of 2 pi i w t and remember from the Fourier slice theorem this guy was s theta of w right this was the 1D transform of the single projection.

Remember if you go back we have this right this is the Fourier, slice theorem this was s theta of w right.

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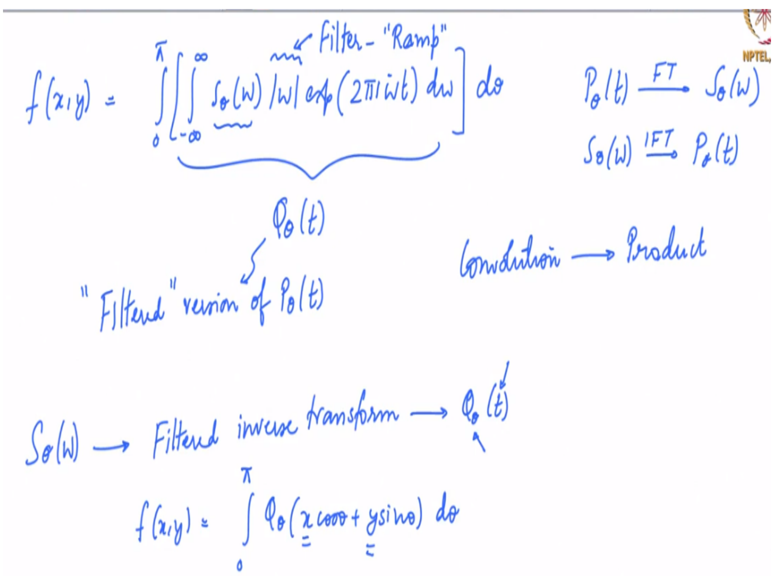

So now if you put this stuff together you will get f of x, y = integral 0 to pi, integral minus infinity to plus infinity s theta of w times w with a magnitude exponential 2 pi i w t dw d theta. This is a function of t because w is going to be integrated out the only remaining variable is t. So we will call this function q theta of t if you did not have this mod w sitting in the middle then this is basically just p theta of t because it will be the inverse 1D Fourier, transform of s remember p theta of t if you do a Fourier transform you get s theta of w.

So if you do an inverse transform of s theta of w you should get p theta of t right. So if the w was not here this mod magnitude w then you will only have p theta of t. But now you have a different function which I am calling q theta of t and this q theta of t is a filtered version of p theta of t. There is a property of Fourier transforms which we have not really touched on here which is that if you have a convolution integral again I am just going to mention it might not be as critical now because it needs a little bit of discussion.

If you have a convolution of two functions in real space then in the Fourier space a convolution becomes a product right. So this is like a filter that is, being applied to this before you take the inverse transform this filter is called a ram filter because it is proportional to the frequency. So you basically take s theta of w you take a filtered inverse transform and you will get q theta of t the function.

So you can do this filtering very easily in the Fourier domain now once you have q theta of t for a particular theta you can get f of x, square n for a particular t of course. You can get f x y f of x y by just integrating over all theta so now you have 0 to pi q theta of remember t was t was x Cos theta + y sine theta d theta right. So for every single theta you just add up the contributions from q naught at any particular x and y and you will get the actual function.

Now this is a much better implementation much better way of, implementing this entire scheme as opposed to just naively taking slices taking 1D transforms and then taking the inverse 2D Fourier transform like we discussed with the Fourier slice theorem. And the reason is because it has a nice geometric interpretation so we look at that now. So what do we have?

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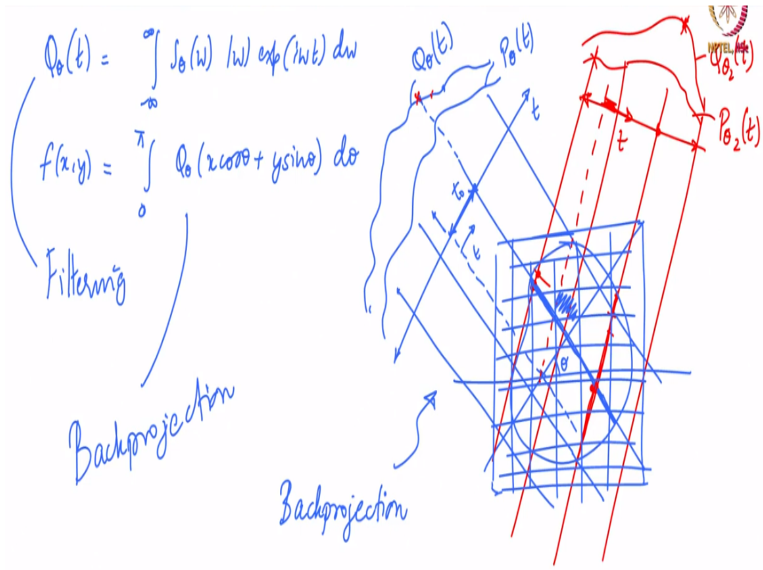

We have q theta of t is this fellow and we have f of x y is 0 to pi q theta of t let me write, this explicitly right x Cos theta + y sin theta d theta so let us look at what this what these 2 expressions are telling us geometrically? So here is our f of x y we have some unknown function and we take a projection let us say this is our projection so our rays are coming like this let me erase this we know what f of x y is and we need some space to draw this carefully.

And now you, have you have this guy so you have a p theta this is your p theta of t this is your t variable of course then you take a Fourier transform of this you will get an s theta of w. Then you do a filtered inverse and sum of that you will get a q theta of t it is a filtered version of p theta of t. Now at a particular t location remember this is all at a particular, theta right so this is your normal and this is your theta right we define theta with respect to the normal so we will use the same convention.

Assume that these 2 lines are parallel right up try to indicate that now at this particular theta this single projection the single q theta at a particular location t so let us take this location t right. So our origin is here let us say so t, is, measured from here so this value let us say at some t naught at this point you have a value of the function q right that has some value right. Now you take that value and assign it to this entire line initially you initially do not know anything about f of x y everything is 0.

Let us say this is the first projection you are starting with you go back and you assign this to every single point, along this line. Now if you have another projection let us say like this you have another p like this whatever you have another q let us say this is your q theta 2 of t p theta 2 of t let us say. Now you go back for some value of again t = 0 is here this is the origin so t is measured from here.

Now for this value of t you go back for this entire slice you update the value of f at this, slice as the value that it had before plus the value of this q go back to this point value third plus the value of q this point value at that value. So initially if this is all 0 you have only updated this blue line first when you come here you take this value of q theta of t and you update that by adding to it this value of q theta to 50.

So one point now has become a little bit more focused than, the others which is a point of intersection of these 2 likewise if you go to some other t let us say you go to this t well this t is outside let us take this t then this point which at this value has now been more refined. Likewise this point which had a value coming from this q theta now is more refined because it has a value coming, this q theta.

So this process is called back, projection so you are projecting back the q thetas into the actual spatial domain one projection at a time. So you have a filtering step this is called a filtering step like we discussed and this is called a back projection step. Now you can easily imagine setting this up if you have a 2D array again the interest of time you will not be able to do this for you but I think they should give, you enough information to get started right.

So let us say you have 2D array and you want to reconstruct the value of f of x y on the 2D array. So at each location each pixel location you can back project this value of the line that is passing through this pixel location on this projection at this theta to that you add up the projection coming from this theta. And to that you add up the, position coming from another theta and so only keep going till you cover all the projections and finally you will have the value of x f of x y at that point.

So this is a very clean way of implementing this reconstruction scheme right as opposed to doing an inverse Fourier transform we only have radial slices at periodic intervals. The other problem with doing an inverse Fourier transform I will just mention that now is.

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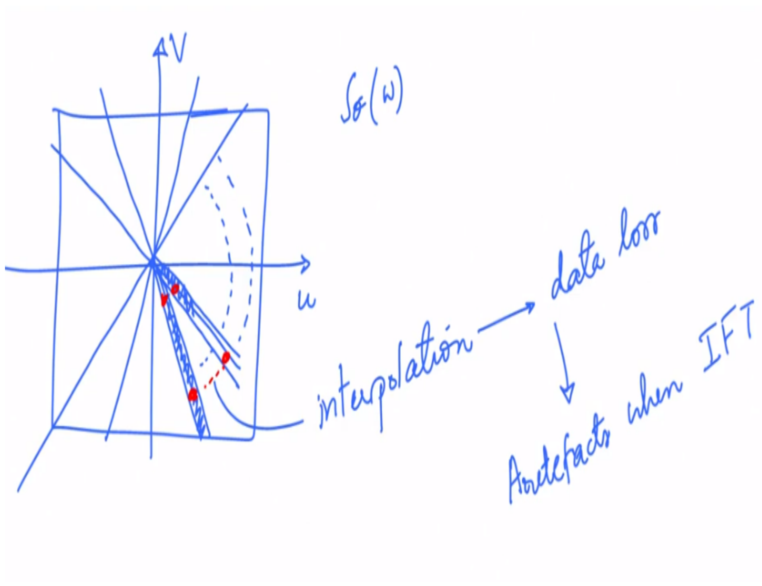

Remember we said that if this is your u v domain u v sorry then every time you take an s theta of w you are looking at one radial slice right. So you have one radial slice here you have one radial slice here one like this and so on. Now the problem is you have data along this slice but the further you go from the origin you will start seeing gaps in the data right. Because obviously here you know this data point and this data point are next to each other this and this.

But this data point if these are the only projections you have this data point and this data point have a lot of missing information in the middle so you have to do some interpolation. If you do the filtering back projection you are not you do not have this issue right so this issue becomes, amplified because you have to do filtering for these points sorry you have to do interpolation for this point.

So if you did the naive 2D Fourier slice business you have to do interpolation for everything in between that is a problem everywhere right. So you have to do more interpolation at larger distances from the origin which really does not make sense. Because you have a parallel beam you cannot, have a dependence on the radial distance right. Because the parallel beam is seemingly sampling everything equally so you cannot have this loss of data as you go further away right.

So the interpolation leads to results from a rather results from data loss so you will have artifacts when you do the inverse 2D inverse Fourier transform because the data is not dense. So that is a basic problem, with just directly implementing the Fourier slice theorem but now with the filtered back projection scheme you should not have you should not have this issue.

So I will just outline the sequence of steps so that if you want to implement your own scheme it should be fairly simple to do.

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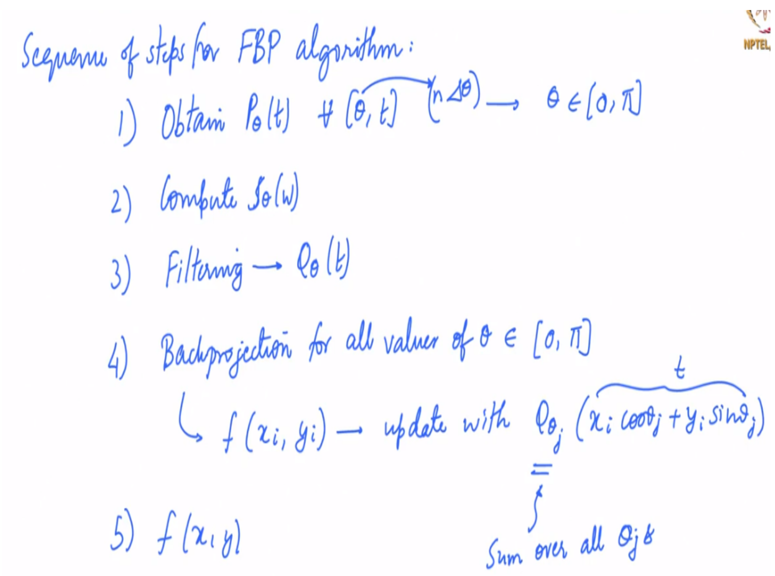

So first step so here let me summarize so sequence of steps for FBP stands for filtered back projection. The first step is you obtain of course p theta of t for all values of theta and t whatever within some domain you have discretized rate is obtained at some angle you know at some angular you know some discretized angular distribution right. So you have some delta theta n delta theta you have and so you have those many slices.

The second is for each of them you compute s theta of w take, the 1D Fourier transform which is easy enough to do because you are not messing with the slices you are messing only with the t dependence. Now you then do the filtering step so you take s theta of w and then you compute q theta of t again. So you know filtering the data and remember the ramp is doing the ramp is sort of weighting these distributions back.

Because the further you are the, larger weight you assign to the further you are from the origin the larger weight is assigned to that particular data point. And so it is actually filtering and giving you a new filtered value q theta of t which presumably does not have this artifact of data loss for larger radial distances in the frequency plane. Once you get q theta of t you do the back projection and the back position is done for, all values of theta.

And notice one more subtle thing it just perhaps slipped under the radar a little bit we had this discussion about having the integral go only from 0 to pi right. Because if you have one ray and you rotate the ray by 180 degrees you will get the same information so this automatically is taken care of in our limit so the integral. So you only have to do when you do the, discretization you only have to do 0 to pi because you have a parallel beam of course if you have a fan beam the situation is slightly different.

And so you back project for all of the theta projections you have and when you do the back projection you update at some i-th pixel location you update the value with q theta from the corresponding you know whatever theta j of x i Cos theta j this is your, t right. So you updated with this which means you add this value to what you already have and you sum over all theta j's.

So for every single projection you go back add it and then you get it and then at the end of the step you have your f of x y another right. Now you can imagine like we discussed for the 3 D case you can imagine doing the same thing you can start with a single slice do this, entire operation for a single slice and then move to the next slice and then keep going from there. So that basically covers how one would go about implementing a filtered back projection scheme.

What we will do next in the next session is to look at more realistic imaging configuration we will talk about what is called a fan beam projection. And then I will discuss briefly this idea about a cone, beam projection and what is called the felt camp algorithm. And based on that you should be able to appreciate how actual practical implementations are done with more realistic sources and with that we will wrap up our discussion about optical tomography.