**Optical Methods for Solid and Fluid Mechanics**
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**Module No # 08**
**Lecture No # 40**
**Rays and the Radon Transforms**

So far we have been looking at we have set up the basic problem in tomography we have discussed what the differences are where tomography comes from? And what you know basic ideas underlying single projections multiple projections when you need multiple projections and so on. And we also discussed this idea of a Fourier transform which as we will see today will be central in determining the, actual cross section and inverting the actual measurements to get cross-section information.

So I think we have a good background to get started with the actual formulation where we are looking at how to reconstruct a cross section from a set of projections. So that is the basic problem like we set up we will do this for a 2 dimensional case first and then I will touch briefly on the, 3 dimensional case towards the end it is a little bit more complicated algebraically but the idea is more or less the same.

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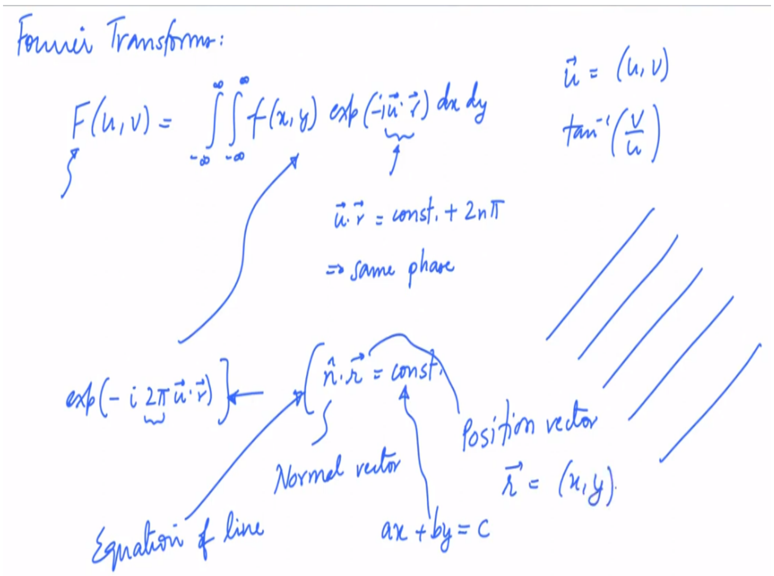


So let us get back to a discussion on Fourier transforms so I will just make a couple of notes first I discussed that the 2D Fourier transform was basically something that looked like this. If you have a function f of x y then you take’s this double integral and, you will get its Fourier representation F of u v like u is like a x direction frequency v is like a y direction frequency.

And we saw at the end that this particular form with an, I here sorry this particular form is sort of representative of a line in 2 dimensions.

And this is the equation of a line of course depending on the constant you get families of parallel lines. So whenever u bar, dot r bar is constant or a constant plus sum to n pi you will get the same phase right because e to the power of 2 i pi = 1. And so you get a family of parallel lines and the direction of the line is basically tan inverse of the components of this which is this of course this were the vector u and the direction of these lines the standard was a v by u.

So I will make a couple, of changes to this notation so this notation is intuitive this u dot r equals constant is intuitive but it turns out if you instead use slightly different definition of the frequency which has an explicit factor of 2 pi outside. So from now on we will start using something like this instead of just u dot r and the reason being that if you do not do this then there'll be factors of 2 pi that keep, coming out and we do not want to keep track of any factors of 2 pi.

So we will just put the 2 pi inside and u will now be not like an angular frequency but it will be like an actual physical frequency for a physical inverse of wavelength right not a wave number but an inverse of wavelength that type of thing. So that is one piece of information now towards the end I mentioned that there, seems to be some relation between this family of parallel lines that we have corresponding to this tan inversely by u for a given pair of values u, b or for a given vector u.

And this is not a coincidence right so if you think about the equation of a line this is what the equation of a line in 2 D looks like n is a normal vector normal to the line and r is the position vector and depending on the constant this gives you the equation of a line. The constant of course determines how far from the origin the line actually is if the constant is equal to 0 the line is passing through the origin if the constant is positive it is in the supposing through the top right quadrant and so on.

So this is just a slightly different way of looking at the equation of a line of course you, know that ax +b y = c or y = mx + c or whatever this is the standard equation this is the same as this right so because r bar is x, v x y sorry.

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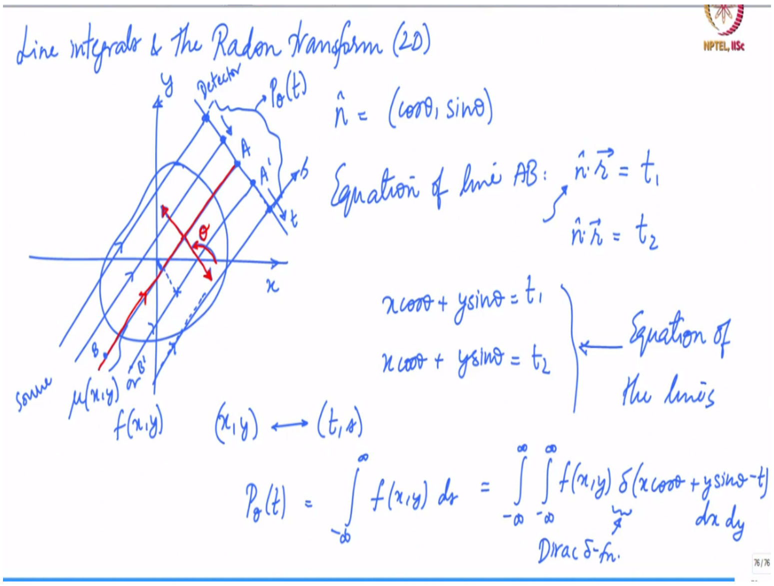


So with this in mind the second notation I also wanted to mention is you remember the Beer-Lambert law we are going back to that now. So this had I f is I 0 exponential of minus integral mu of x y along some line with, some you know limits does not matter for now. We would not look at I f but we will instead look at minus log of I f by I naught. So we will not actually look directly at the intensity that we measure but instead we look at the logarithm of the ratio of the measure intensity to the incident intensity.

The reason being that this is simply this integral right and it is easier to work with this integral, than an exponential of the integral just a notation. So this is what we are interested in since we know I f and we know I naught we know log of I f pi naught so we have the value of this integral along various lines. And our aim of course is to try and recover the value of me the value of mu as a function of x and y right that is the problem we set up initially.

So we will now, start doing some algebra for this with these 2 notations in mind so we will talk about what is called line integral and excuse me what is called the radon transform. So let us say this is our cross section so this is let us say we have a square and inside the square this is our actual object we are talking only about 2D for now 3D case will come later. So this is our bounds and the actual object, is here there is some variation in f or mu and mu of x y is non-zero inside the shape and it is 0 everywhere else right.

So we want to be able to reconstruct the shape from a series of projections so I will leave out this bounding area because it is all 0 there. So this is our field mu of x y we will actually refer to it as f of x y just so that we can relate back to the Fourier transform, good. So when you take a projection let us now cross some axis for concreteness so this is our x axis and this is our y axis and when you take a projection.

So let us say my light ray is coming like this my x-ray is also light it is coming like this so the source is here and there is a detector here and at each point on the detector there is some type of charge couple device that is, sensitive to x-rays let us say and it picks up an intensity right. So let us take one of these lines so if you were to draw just one of these we will just take this fellow the normal to this line goes in this direction right this is the normal vector normal to any of these lines they are all parallels.

So normally to all of them is the same but this is the normal direction or this it can, be up or down does not matter right whichever normal. So let us now take assume that this normal makes an angle theta with the x axis. So for normal probably use the same color again so this normal line makes an angle theta with the x axis. So the n hat is basically Cos theta comma sine theta it is a unit vector so Cos square + sine square is equal to 1.

Now the equation of the line, the red line let us call it as given name this is B and this is A equation of line AB is what it is n hat dot r or x does not matter dot r bar is equal to some constant and that constant we are going to call t 1 some constant t 1. And what is called what is t 1 measuring is basically measuring the perpendicular distance from the origin to the particular array AB that is.

What t 1 is? So this distance this is at a point p 1 at point p 1 if you take the next one the equation of this line b prime a prime is n hat dot r bar is equal to some t 2 and remember there is an n here so the theta is implicit inside here right. So your actual equation is x Cos theta + y sine theta = t 1 and x Cos theta + y sine theta = t 2 right. So this is the, equation of the lines the line integral measuring so the detector picks up something that looks like this we are going to call this function p theta of t.

For a given theta is already parameterized the normal vector or the perpendicular direction for the wave for the light rays or the x rays and t is the distance along this axis right this is our t axis. Perpendicular to the t axis, of course is the direction of propagation we will call that the s axis. So we have an x y coordinates and we also have a t s coordinates for a given theta you have to axis perpendicular to each other that are basically rotated version of the x y coordinates.

So let us keep that in mind so p theta of t basically measures the f of x, y along the line whichever line at a particular, t which is parameterized now by d s. So d s is along the line direction like this t is telling you which line you are picking right and the integral is over s. So in other words this is basically can be written down like this I will explain what this notation means.

In case you have not seen it before it can be written as a double integral with this function called a delta function or, a direct delta function.

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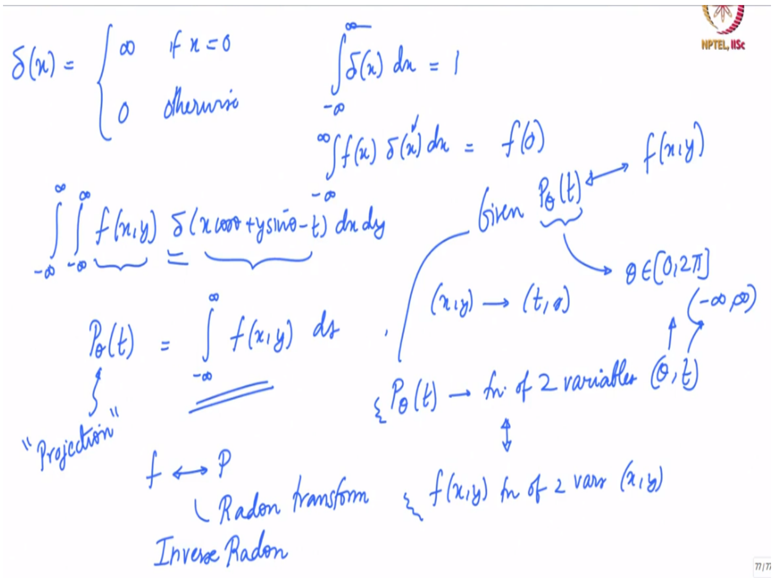


The direct delta function has the following properties it is infinite if I click equals 0 it is 0 otherwise and integral delta of x dx so minus infinity to plus infinity is equal to 1. So just a way of sampling one particular value out of infinite of them right so if you do for example integral f of x delta of x d x from minus infinity to plus infinity everywhere, else except at x = 0 delta is 0. So this integral will basically vanish except at x = 0 where the integral is bounded so you will get f of 0.

That is an important property of the delta function it is basically a manipulative tool it is not really a function in the conventional sense of the word it is a manipulative tool that you can use to represent integrals in some form, right. So when we wrote this double integral it basically means that when you had f of x y delta of x Cos theta - t x d y. It is only picking out this delta guy is only picking out the values of f along this line right just like x = 0 was picked out here this is picking out this whole argument equal to 0.

The whole argument is equal to 0 is the equation of the line right so it is, going to pick out the value of f along that line and you will get an integral in the direction of that line that is a representation of t same integral. But nonetheless we will just assume that we have this fellow going where the integral is taken along s and x, y are transformed to t, s and they are taken along the line direction.

So this is what is what we will call a, projection this particular integral does not have t it does not have s in it because s is integrated out but it is a function of t right. Because depending on which line you take you will get different value of the projection and just like if you look at the earlier schematic the value here at t 1 is different from the value here at t 2 and so on right. And so the function the you know p is, obviously a function of t it is also a function of theta because if you rotate this in a different direction and you do the same thing again you will get a different functional dependence on t right naturally because the shape is not symmetric.

So this p theta of t is a function of 2 variables theta, t and its exactly analogous to the original function it contains the same amount of, information as the original function which is also a function of 2 variables x, y right. So if you had this p theta of t for all values of theta from 0 to 2 pi let us say we do not need all this. But let us for now assume that 0 to 2 pi is needed and t let us say from minus infinity to plus infinity this is equivalent this is a information contained in p theta is the same as information, contained in f of x.

Just like if you took a Fourier transform the information contained in f of u v is the same as the information contained in f of x y right. So this conversion from f to, p theta p or whatever this is called the radon transform just like you had a Fourier transform you have a radon transform which is represented by this. At some angle theta some distance t you find this line, integral along a particular line.

So just like we have an inverse Fourier transform there is also an inverse radon transform and that basically means that if you are given p theta of t you invert it and get f of x y. And given p theta of t by that I mean that you are given this functional form for all pairs of combinations of theta and t right just like you would need that for f of x y.

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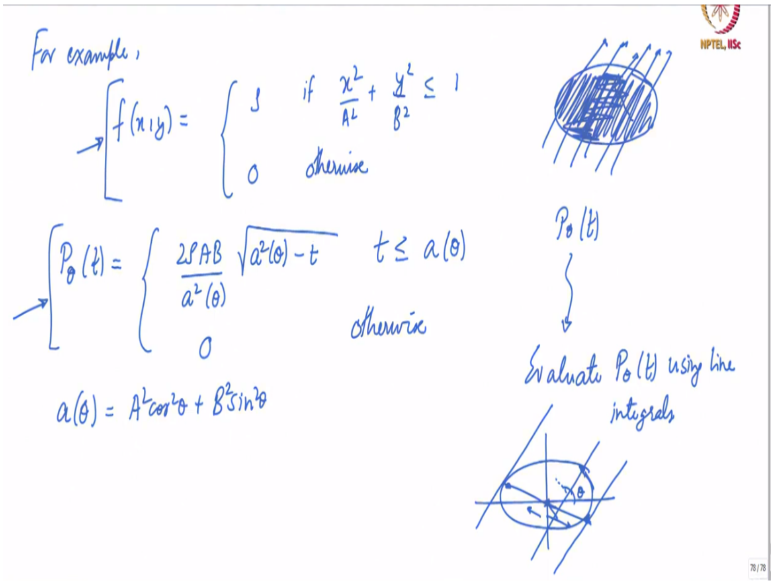


So, it is easy enough let us just look at one example just to make sure that you know everything is clear for us. So let us say for example f of x y = rho if x squared by a square + y square by b square is less than equal to 1 and a 0 otherwise. So this is our function right this is our cross section function it is basically an ellipse it looks like this or like this depending on, which of a or b is larger.

Inside this it is constant right and outside this is 0 and you want to reconstruct this and so the representation you will get if you pass light rays like this through this parallely of course will be p theta t right. And so you can evaluate p theta of t using line integrals I am not going to work this out because it is a little bit algebraic but it is very, straightforward to do you basically take an ellipse like this the center of in the origin of the center of coordinates I will center the ellipse sorry.

And then you take an arbitrary line like this and then you parameterize this line and then you find for what values do you get extreme lines right so you basically have to get the coordinates of this point and the coordinates of this point. And, for all values inside you will get a non-zero value of p for everything outside you will get 0 right. And once you do that you parameterize by distance along this particular line that will be your t and this angle or the perpendicular pi by 2 plus this angle if you take the normal that will be theta.

And so you will get p of theta comma if you do that it turns out you will get an, expression that looks like this it is probably easier to work this out for a circle where a and b are equal and the expression become much simpler as well but nonetheless. So you will get if you do this transform you will get something that looks like this for p of theta t and a, of theta of course is a square Cos square theta + b square sine square theta.

So the reason I put this, example out is because this representation of what is inside here is intuitive right f of x y is rho inside the 0 otherwise that is our actual object. So it is representation in this form is intuitive but is absolutely equivalent to representation of this form right. So if somebody came and gave you this expression and said this is the actual object it is giving you it is just as good as, giving you the first piece of information.

Just like if somebody told you enter Fourier transform you know the actual value by inverting so the radon transform is identical.

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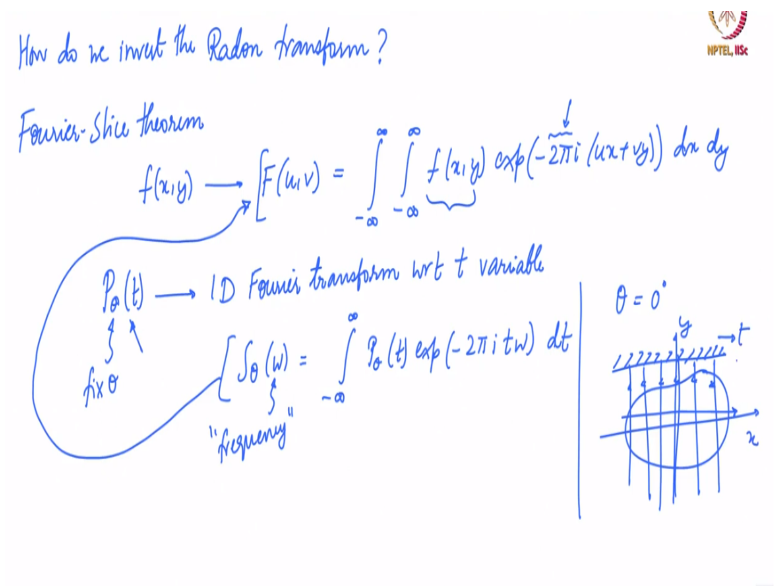


So the question now finally that we get to after this you know sort of background setting is how do you invert the radon transform? Now to do this there is something called, the Fourier slice theorem and this is what will work out now. So let us say we have our function f and I take a Fourier transform of it f of u v of course this is given by remember we had this extra factor of 2 pi so I am going to include that here.

So this factor of 2 pi is now here and the f is just obtained by inverting this transform right you have a plus sign here and you integrate, over u and v as a product x y. Now if you take a single projection so a given value of theta so you fix theta first so now it is a function of one variable along the line direction t. And you take a 1D Fourier transform of this with respect to the t variable because that is the only thing remaining right.

So you take a 1D transform of the radar transform along a certain theta, direction what do you get? We get something that will call s theta of w this is minus infinity to plus infinity p theta of t exponential -2 pi i t w d t right just like the usual 1D Fourier transform that we discussed. Now the t dependence is gone and you get a w dependent so w is like your frequency right in the frequency domain.

Now this Fourier slice theorem basically relates, this see 1D Fourier transform to this. So the Fourier transform of the actual radon transform along a certain direction is directly related to the 2 D Fourier transform of the actual function right. Remember our aim is to recover f that is what we want to find out that is the unknown. So this is how you relate the 1D Fourier transform of the projection p theta of t at a given value of, theta to a 2 D Fourier transform of the function return so that is basically what the Fourier slice theorem tells us.

Now how this is related is what will work out now to do that let us take the simpler case of theta equal to 0 now theta = 0 degrees then basically this is your object. So let us say x axis is your y axis you are basically measuring along this because, remember the normal direction was what we used to set the value of theta right the normal vector. So now all the actual light rays or x-rays are going parallel to the y axis this is x this is y and t is basically nothing but x right your t coordinate is horizontal because theta = 0 degrees good.

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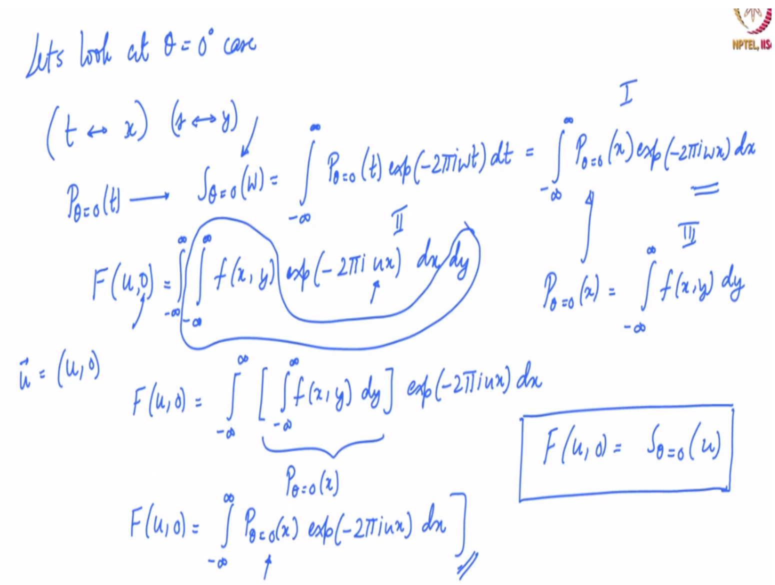


So let us look at either theta = 0 degree case what happens here is of course t, is the same as x coordinate and so p theta = 0. Remember we said this was for a fixed value of theta right you are taking a Fourier transform 1 D Fourier transform for a fixed value of theta so we will transform this. So you will get s theta = 0 of w is integral minus infinity to plus infinity p theta = 0 of t exponential of -2 pi i w t d t.

So now t is the same, as x right so we will just write this as minus infinity to plus infinity p theta = 0 of x exponential -2 pi i w x d x. Now what is the 2 different transform look like of the original function right p theta is a projection we are now looking at F. So F of u v for theta = 0 let us take a look at f of u v, 0 u, 0 what does this look like so its minus, infinity + infinity F of x, y double integral because you are integrating over x and y exponential -2 pi I u x because v is 0 dx dy.

There is no v here because v = 0 so we are looking at the frequency vector u bar is basically u, 0 right just to see if you can relate the 2 of these 2 integrals. Remember since t was the same as x and if you look, back at that figure we drew over here on the right bottom s is in the ray direction and t is in the directed direction right. So s is basically y t is x so s is basically y correct so what is p theta of p theta = 0 of x is what?

It is integral along y because s is along y of, F of x y correct because s is along the y direction so now we can put this and this together. And you will immediately see that this whole thing becomes quite obvious. So f of u, 0 is we are going to split this integral so we will do this part first so we will keep the x integral out this is p of theta = 0 of x because of this. So F of u, 0 is minus infinity to plus infinity let me erase this line since you know what it is because of p theta = 0 of x exponential -2 pi I u x, d x.

But this is nothing but the Fourier transform of p 1D for return form of p right which is nothing but s that is what 1 is right one is the same this guy is the same as this fellow. So F of u, 0 is nothing but the 1D Fourier transform a theta = 0 this s that we have here but with w being replaced by u. So if you take a single projection for a fixed value of theta and you, take its Fourier transform somehow it is related to the full 2 D Fourier transform of the original function.

Now this is for a specific case theta = 0 but the idea is that that is related we have to get the exact relationship and that is what we will do next but I hope this special case situation is clear.

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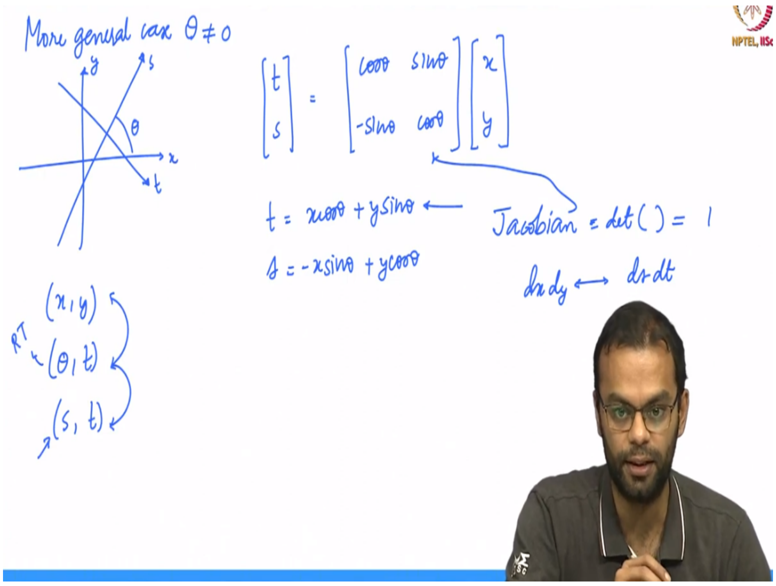


Now we look at the more general case for theta not equal to 0 so in general you have this pair of coordinates x and y and you have this pair of coordinates this is your s direction this is your t direction and you have some theta between s and t. So you can in general convert the t, s coordinates using a rotation matrix in terms of the x y coordinates. Now to establish the Fourier stress theorem we will work it out as a simple algebra.

But you have to remember this fact right that the fact that you have an orthonormal coordinates system t, s which is only a rotated version of the original x y coordinate system. So your t is basically x Cos theta + y sine theta and s - x sine theta + y Cos theta right this is just a coordinate change under rotation right. Remember this was the equation of a line that is where we started from, so it is exactly the same equation we are only looking at it from a different point of view.

Now we are looking at it as a coordinate change one of the coordinates in the new pair of coordinates that we are using the t, s system in terms of the original coordinates as a function of the rotation angle theta. So if you do this then the Jacobian of this transformation which is basically the, determinant of this matrix will be 1 because you will have Cos square + sine square that is 1.

And so the area element d x d y will be the same as the area element d x d t so what we will do next is we will see how we can establish this for a more general case for theta naught equal to zero we will do the algebra it will take a little bit of rewriting. And you will see immediately that, this Fourier transform becomes absolutely critical in determining what the projections Fourier transform looks like right.

So you can use the 2d Fourier transform to invert the radon transform and the Fourier slice theorem basically relates that and we will work that out in the next session in more detail. So in the meanwhile please remember that this is not you know something new right we are, just looking at the same thing from a slightly different point of view. We are treating the t and the line direction as new coordinates and theta is something that parameterizes the new coordinates right in this room.

So there are 3 things going on there is this x y coordinates that you have by default there is the theta, t which gives us the radon transform right p theta of t but there is, also s, t which are new coordinates. So in the radon transform or on the line in the projection you do not have, s because it is integrated over right you integrate f over, s. But now we are going to sort of work with all three of them this being the radon transform and this being used for this transformation here to relate the radon transform to the original Fourier transform of the 2D function, so we will work out some of these details in the next session.