Optical Methods for Solid and Fluid Mechanics
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Lecture – 04
Lagrangian and Eulerian Perspectives

Welcome back everyone. So, last time we had left our notes right here, where we were looking at the deformation process.

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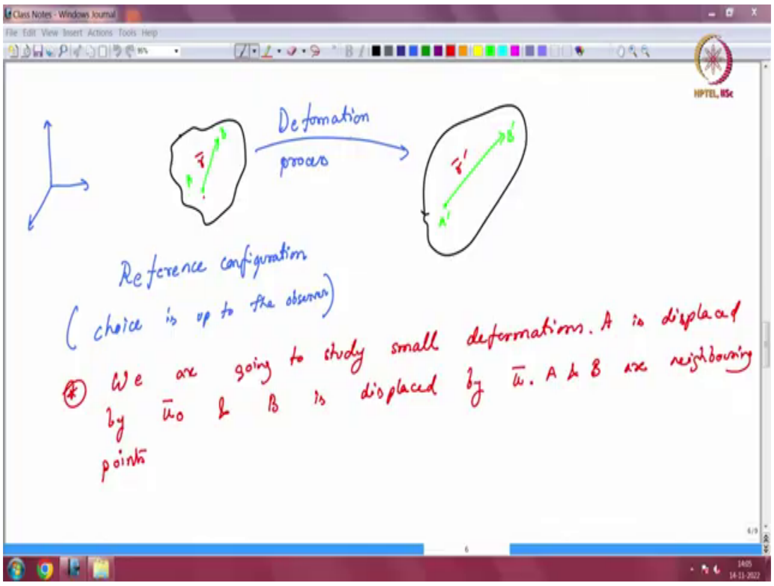

And in order to understand the tensor, how tensors come into the picture. We had talked about this issue of radius vector that is getting transformed, as the body undergoes deformation. And we had talked about a reference configuration with respect to which we are now comparing this new different deformed state. And then we had done a little bit of mathematics assuming small deformation we had used the Taylor series approximation for neighbouring points.

And we had seen that what we end up getting is a tensor which controls the transformation between two vectors here r ij r j and this is the ri. This is written in now, tensorial notation. So, you have to be careful with that and this here is a denotes, a differentiation. So, if you are not very familiar with index notations for tensors, you might want to go into it. We will not be using it a whole lot but it is very straightforward to get introduced to it.

Because it helps us write very complicated equations like this matrix on this right hand side. We can just express this as a simple notation on the left hand side. And then we had seen that we can break this tensor up into two tenses, one we will call the strain tensor and the other will call the rotation tensor. Now, the same thing we can also understand, through geometry here we just took advantage of a Taylor series approximation.

But there exists such geometrical interpolation to the strain tensor also. Again, I must note that this is in the case of small deformation, so that is a very important issue.

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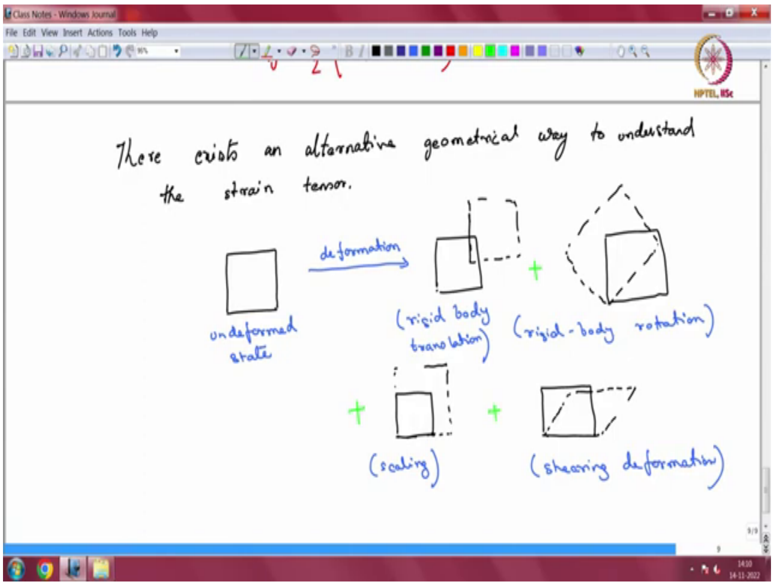

So, there exists an alternative geometrical way to understand the strain tensor. Now, in this geometrical way, what we do is? Let us say we have an undeformed state now this state is going to undergo some deformation. So, this is your on deformed state or rather basically, your reference configuration in a sense. And now what can happen to this small element? So, we assume that by the way I have drawn a box but this basically represents a very small infinite decimal area or volume depending on a 2-D or 3-D problem.

Now, in this case, it can undergo, let us say something called rigid body translation. So, this, this black box can translate to some other new location. So, it can undergo what we call the rigid body translation or it can undergo let us say you have this box it can now undergo rotation. This is the same box by the way, so, pardon my drawing things changing a little bit. So, rigid body rotation or it can scale.

So, it can scale on either of the two axes in this particular case. So, it can undergo a scaling transformation where it becomes bigger than the previous box or so, this is; so, scaling say. And you now have sharing component so, you have this box let us now the new box has undergone a shear deformation. So now, any small deformation like this for undeformed state can be looked upon as the sum of all these different types of deformation.

So, you have the rigid body translation or and or rigid body translation and or a scaling operation and this is a shearing deformation. So, this can be looked upon as this. So now, we can easily understand the rigid body translation is not going to affect the relative positions between any two points of the body by very the definition of a rigid body translation. Similarly, for rigid body rotation, although the vector will rotate it will but so that that will happen.

So, although rigid body rotation is looks apparently similar to the rigid body translation but it does end up changing the apparent position of the radius vector in that sense, it ends up rotating that. So, as we saw before we are going to look at deformed the deformation of an element and we can see that as the sum of these different type of deformations.

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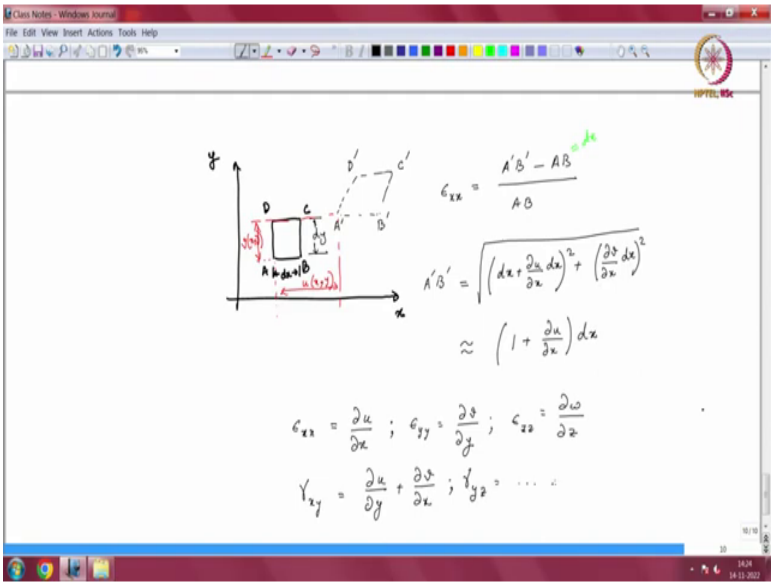

So now, we want to look at a geometrical explanation of the same strain values that we had discussed earlier. So, let us say you have a infinite decimal element now this is let us say these are the four vertices a, b, c, d and this is x and this is y and this is dx. So, this distance is just very small it is dx. Now, I want to be very clear that what I am drawing here is an infinite decimal volume or an area.

And I am the reason it is exaggerated here on the screen as the drawing is just because it is easier to understand. Otherwise, it becomes very difficult but basically the point B here is a neighbouring point of A in the calculus sense in the limit sense. So now, this element is now under going to undergo some sort of a deformation. And let us say this new element is now something like this.

So, this box A, B, C, D that we have drawn this now deforms to a new box A dash B dash, c dash and d dash. And we will say that the point a is now being displaced in the x direction by an amount u x, y and it is undergoing a displacement in the y direction by an amount called v x, y. So, my A has not moved to A dash. So, this is my u xy and v xy they help me locate A in the new reform state.

Now similarly, B has moved to B dash and I for me to be able to get the geometry accurately I also need to account for the movement in B. Now, this we will do just like we did before, in the previous example, where we had assumed. So, if I go back to this example, we had talked about the displacement of one point which is neighbouring another point and the other point is undergoing a displacement of u naught.

So, in that case, my displacement of B dash is related to the displacement of A simply through a Taylor series expansion. So now, in this particular case, what we will do is? We will say that there has been an extension in the length of my box and this extension is essentially if I need to calculate that I will define it as an increase in length in the new configuration minus the length in the previous configuration divided by the length in the previous configuration.

So, all I now need to do is to find out this new length. I know A B which is equal to dx. So, this portion is already known this is equal to dx. So, if I have to do A dash B dash then I again applied a Taylor series and what I find is this is d x + del u del x by dx whole square + my del v del x. It is a two dimensional problem. So, we are just going to take two dimensions here.

So now, when I do my when and this is if I now have to simplify this down this is nothing but 1 + del u by del x into dx and I have dropped out the second order terms here. So, I have only kept the first order terms of my equation over here. So, now, if I replace my value of a dash B dash into the previous equation here, I calculate epsilon xx. What I get is simply del u by del x because in the denominator, you have a del x.

So, this is nothing but del u by del x. So, this is something that you might be very familiar with if you have done a course in strength of materials or solid mechanics. And you would be very familiar with this similarly, I mean for a two dimensional problem. If you want to ever extend this, you can just go ahead and quickly show that the strains in the other directions are given as these and then there are shear strains.

So, for example, gamma x y which is can be written as del u by del y + del v by del x. And similarly, for the other strains, I am not going to write them down. I am just stating this anyway for completeness and to show you how this is related to the previous derivation.

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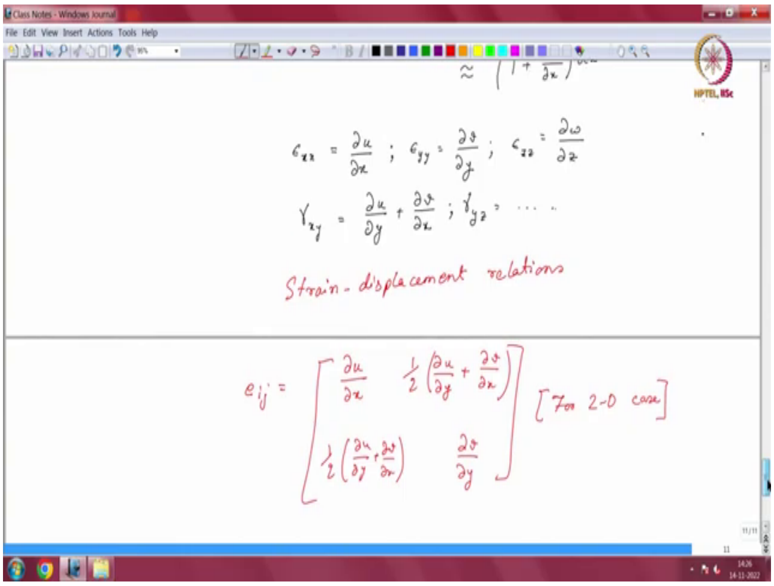

So, now these are also called the strain displacement relationships and in matrix form. When we write this down, we usually would say e ij is in two dimensions if I have to write this it is this matrix. So, we put a half here for 2D case. So, this shows you this the reason I did this geometrical method is simply because this is something that you might have come across in a textbook.

Again, as I said, in a solid mechanics textbook or in the case of strength of materials course and I just wanted to show you the similarity and the between the two methods and you end up the geometrical method and the other method they end up giving you the same tensor.

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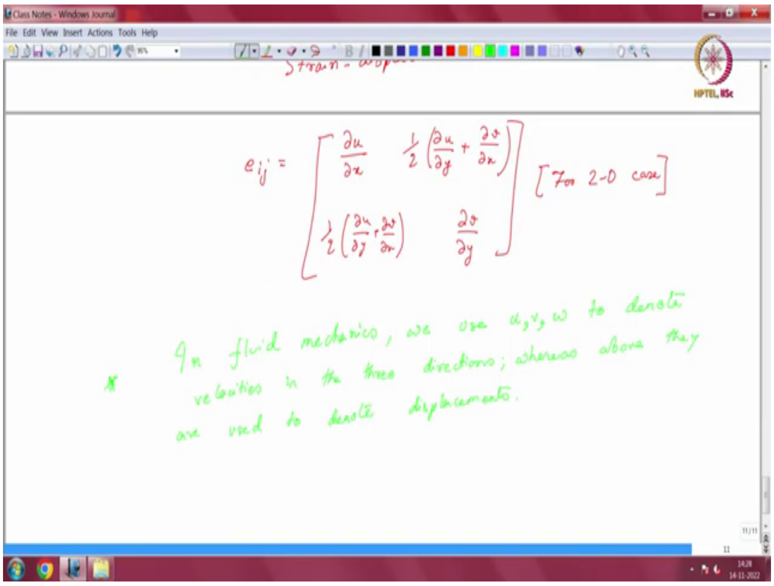

So, overall, we have been relating the idea of strain and we had introduced ourselves to this concept called tensors which is a slightly more important form that is related to vectors. It is a transformation essentially and here we showed you how they are relevant to our problems in solid and fluid mechanics. And I had also mentioned that in fluid mechanics, we are not going to be dealing with.

So, I am going to make a star mark here because the portion I will be dealing with is going to be fluid mechanics. So, I just want to in fluid mechanics, we use u, v, w to denote velocities in the three directions. So and whereas in solid mechanics, whereas here, whereas in above they are used to denote displacements. Now, my tensors look exactly the same, the idea is very, very similar.

And we are going to deal with the strain rate tensor in fluid mechanics where you will have these will now become velocities. So but the idea is essentially the same the reason we are using a tensor is the same. It is essentially to track deformation but because in fluid mechanics the strain rates are more important or the velocities are more important than velocities sorry displacements.

Velocities are more important than displacements we end up using the strain rate tensor rather than the strain tensor. So, my colleague Kaushik, when you will be talking about the solid mechanics aspect. He will probably look a lot into the strain tensor whereas we would more be interested in looking at the strain rate tensor when we discuss the experimental methods for fluid mechanics.

So, this was a very important aspect as to try and understand the relevance of tenses in our course. The other important portion that we want to look at is how to track velocities. And we will see that there is a very important difference between velocity measurements, in our system versus velocity measurements in the dynamics case.

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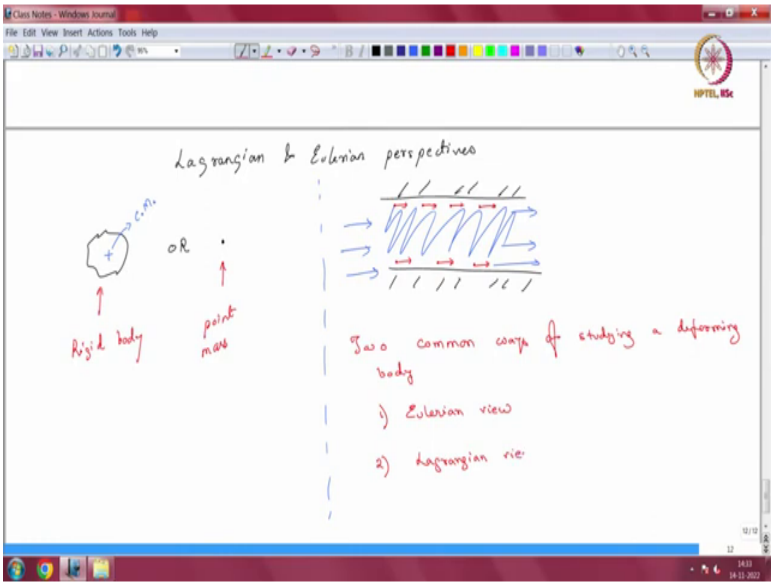

So, we go back now we want to look at what I will do here is so, if what now we look at something called Lagrangian and Eulerian perspectives. In Newtonian dynamics we were confronted with the idea of tracking a rigid body. So, if you have a rigid body and or a point mass, so, you have this was a rigid body. And this is our point mass and usually the problem we are looking at is understanding the velocity acceleration and position of either of these two cases.

In the case of rigid bodies what we do is? We track something called the centre of mass. So, let us say we have a body and this body will have a CM or a centre of mass and on the application of force is the centre of mass will move. And you will track that in the case of point mass this is an idealized point an infinite decimal point and you are just going to track that as time goes.

Now, all real bodies deform and the problem of tracking becomes far more complex and to give you a perspective into that. So, let us say you have a channel and you have fluid here that is now flowing. So, this fluid is now flowing outwards it is coming into the pipe and it is going outwards. So, obviously you are suffering some sort of displacement and some sort of velocities are generated in the system and now you have to track.

Now, we obviously cannot use the rigid body idea, so, we have to look at this fluid as a ensemble of point masses. So, every point here has some velocity that is it as a function of time and that is why the entire fluid is moving. So, every single point now in container mechanics we do not consider or we are sort of agnostic to the molecular nature of matter. So, this region of space is considered to be the same as the as I actually use the word space is considered the same as space.

So, this fluid is considered is supposed to be continuous just as space is continuous. So, essentially, this entire fluid now has an uncountably infinite number of points. So, as this deforms, you have to now keep track of the velocities of an uncountably infinite number of points which obviously is a very very difficult case. So, in now because this problem has when we are talking about fluids or even solids.

The problem has become so much more complex than the Newtonian case, where you are just looking at a point mass life was much easier. But here now that you are looking at infinite decimal infinite number of infinitesimal points. Life is far more complex, so, in order to simplify things or even make sure that we understand deformation properly. There are two common perspectives that are applied and we have to be careful in which one is being applied.

So, two common ways of studying a deforming body. One is called the Eulerian view and the second one is called the Lagrangian view. These are nothing but ways in which we keep track of the deformation.

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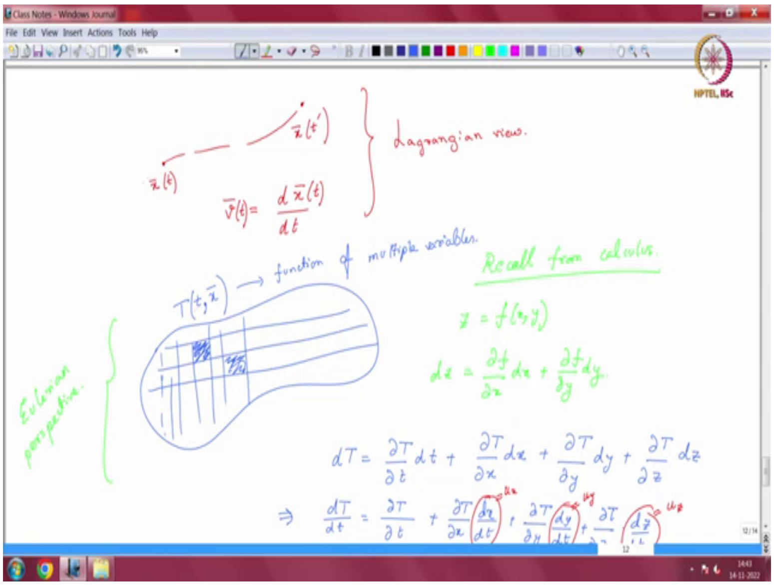

Now, we will discuss the Lagrangian view first because that is the more intuitive one and that is what we are accustomed to with respect to dynamics. So, let us say you have a point mass and this point mass is now moving and this is the locus it is going to track out. So, this was, it is position at some time t and now at some other point time it has a new position let us say this is t dash.

So, in order to so, we, if we if this path is known then the velocity of this particular body is simply t x by d t so, my x which is just a function of time. I have to have to take a derivative with respect to time and I get a velocity which is also now a function of time in general. This is just the way we do dynamics, so, you basically take a packet of material and you follow it as it goes from one location to the other.

So, this is also called the Lagrangian view. In a in a continuous body there is as it deforms so, let us say this is a continuous body. Now, let us say you want to look at the temperature of this body as a function of time and space. Another thing we could do is break this entire thing up into grids and we can say that will keep this grid fixed and we are going to monitor.

Let us say I want to monitor the temperature of this location capital T then I have to now just monitor the temperature at these locations over time. So, this now becomes a function of time, this temperature and if I go to a different window, it will have a different temperature. So, this is also now a function of my temperature is a function of which window I use. So, this is obviously exaggerated view you are now going to use a grid with infinite decimal points.

Essentially, so, basically, you are going to be looking at this variable temperature at each and every point in the system. So, my variable now also becomes a function of another variable. So, my the variable of interest temperature here now becomes a function is a multivariable function. So, it is now dependent on time it is also depending on the exact grid location we are using or the x vector.

So, this is now a function of multiple variables. You need not do this with something like temperature you could do it with other things also. You could associate velocity with each of the each of these grid points and say I am just going to look at the instantaneous velocity of this grid. And I am going to call I am going to say that this grid has a velocity v if the element or the mass that is occupying this grid point has a certain velocity at that time.

I am just going to say that this grid point has the same velocity like that so, you can associate other things, also just in the way we have done it here. I am just using temperature here to simplify something that I want to explain now. So, now that you have this, you see in the previous case this. So, this is your Eulerian perspective. And you can see that how these are different in the Eulerian perspective, you have grid points.

The grid points are stationary and you measure the variable at the grid points, whereas in this case you follow the particle, you do not care where it is? But you follow the particle and you just measure it is location with respect to time. So, now, what will happen is if I want to take the derivative of my variable temperature here. It is not going to be as simple as the previous case because now you have a function of multiple variables.

You have to apply multivariable calculus. Now, if I have to so, recall and I am going to use a result from my calculus. So, if I had a function z which was let us say a function of x, y my dz is given as del f del x into dx + del f del y into dy. So, this is a result that I mean, if you are not familiar with it, I urge you to look up multivariable calculus for this. So, now let us see if we have to measure dt.

So, I have to apply this multivariable calculus idea here and what I will end up having is simply del t, plus let us say del x. So, for three dimensions if I had to do this, I will end up with this. So, this says that my dt by total derivative here is nothing but so, what are you these quantities right here? Well, these are nothing but the local velocities. So, I can write this as ux. So, now that we have entered the realm of fluids, we are going to use u for velocity.

So, just to make sure that I am using a slightly different variable naught not a whole lot I am just going to use this subscript so, make our life easier. So, this is now these are velocities. So, you can now this applies not just to temperature it applies to anything, any other scalar or even a vector that you are measuring in the same manner.

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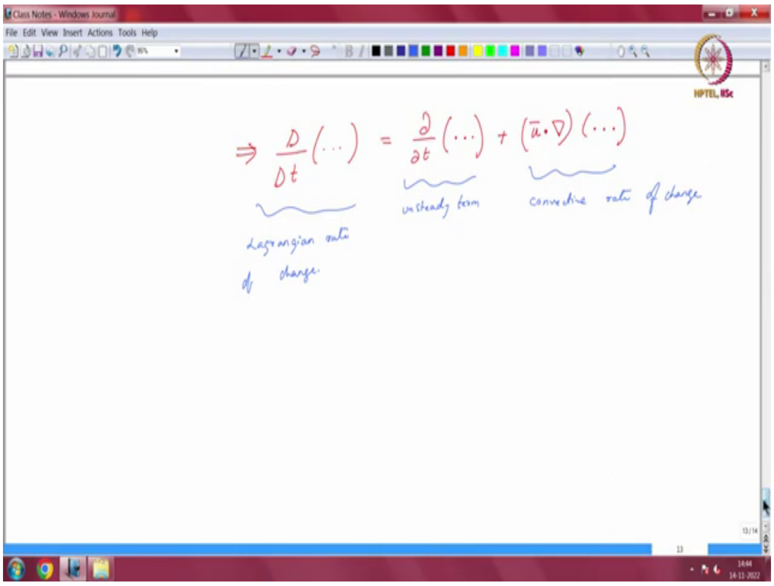

So, we go ahead and we usually say that to take the derivative and for an Eulerian variable, you must take to create to calculate the Lagrangian rate of change. You must evaluate it using this equation, so, basically it will consist of so, here I am writing dot dot dot for anything else that you might have here in the above equation it is temperature but it might as well be something else like velocity.

So, you will have a del t which is a partial derivative of the same quantity. And that reflects the local or the unsteady effect. And then you have a spatial component which if I have to, if you allow me to simplify, I can write in vector form as this. So, here u again is the velocity. This is now the velocity vector, so, I have used here I have written it down in the component form and just to make my life easier.

I have just written it here down in the vector notation. So, this is now my convective rate of change. This is my unsteady term and this people call by various names, for example, total derivative or Lagrangian derivative. So, this is this just shows my Lagrangian rate of change.

So, I want to emphasize upon why we had to come up with this different operator for time differentiation.

And the reason for that lies in the way we are looking at things which is we have changed from the Lagrangian view in which format F = MA is written. So, F = MA must be interpreted in the Lagrangian sense. Whereas if we are using the Eulerian perspective, the bookkeeping we are doing for the variables is slightly different because now we are associating the variables with grid points.

And we are keeping a tab on how this variable changes with respect to time but not just time but also as the spatial variable. So, because of this issue that this now my variable becomes a function of multiple variables. My the way I do my total derivative changes and hence in order to get the Lagrangian rate of change which we have to use in the Newtonian equation F = MA we have to use this derivative and it has to be calculated in this particular manner.

So, today I will stop here and I will start away start from here again in the next class. So, thank you.