Optical Methods for Solid and Fluid Mechanics
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Module No # 08
Lecture No # 39
Signal Processing and Fourier Methods

In the previous session we were talking about what tomography is and why you need multiple views of a certain object to be able to reconstruct its internal structure which is essentially what tomography does. And we also discussed briefly the ideas behind radiography and what type of techniques you will need and we work through a couple of very simple examples that discuss this problem of, unique reconstruction. So today we will start discussing some of the more technical details the first step in this discussion is to look at what are called Fourier transform techniques.

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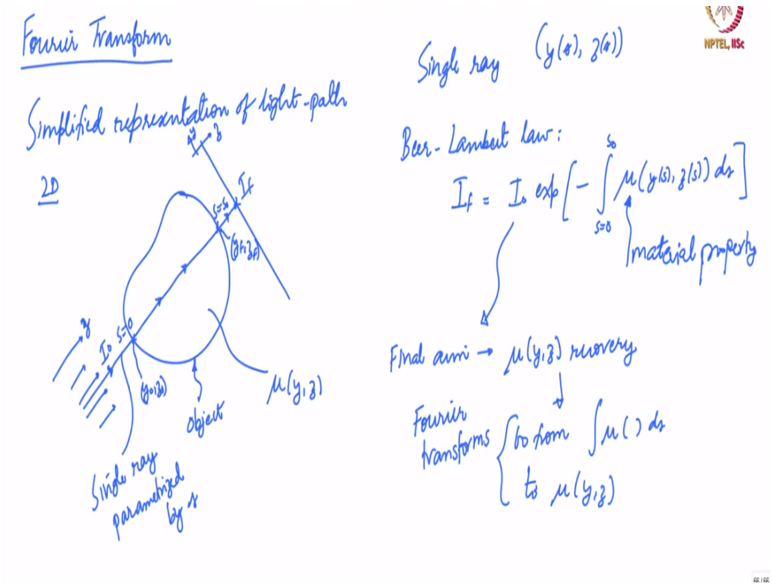

Now I understand most people have probably seen or even heard of this term before but just to make sure that all of us are on the same page. I will introduce the Fourier transform step by step and then we will see you, know why it is central to our discussion of what is eventually going to be called the radon transform? And how the fully transform can be related to the radon transform but we will get there in due course.

The first step before we start on this sort of journey is to look at the simplified representation of a light path right. So what happen when light passes through a material so for example we will take a 2D case this is 2 dimensional this is your incoming light. So I am passing x rays like this is my object and my screen is here remember the screen is perpendicular to the light source right.

So if you remember our coordinates from the previous discussion it was z and this was x y so this is let us say y does not matter right and this is again side but we look at one cross section of a, material or a 2D situation. So let us take one light ray if you start from here is going to enter here and it is going to come and reach the final screen. So this is a single ray the let us take the point at which it enters the object s = 0.

We are going to parameterize this ray by s ray parameterized by s so it enters the object at s = 0 and then it leaves the, object that s = s naught. And we are basically interested in how this intensity of the light ray changes as it goes from s = 0 to s = s naught that is basically what we are interested in right. Now let us assume that the material has some absorptivity called mu now this is a general property right.

Now we do not have to specify what physically mu corresponds, to it could correspond to x ray absorptivity it could correspond to density correspond to various things depending on the context we will talk about that in a minute but let us assume that the variable is mu and it is a function of x and y and z sorry it is a function of x, y and z. But in this case because it is 2 dimensional we are only looking at the y, z plane right remember y is like, this and z is like this a detector which is originally a screen.

Now looks like a line the 2 dimensional versions so we will drop the x dependence and look only at y and z. So our single light ray let us say has an equation y of s, z of s and the coordinates of the point of entry here y 0, z 0 let us say and the coordinates here are y f, z f let us see. So you can easily write the equation of, this line if you knew these 2 points right you can get the equation of any point on the line but what happens to the intensity?

So it turns out that a simple way to represent what happens to light intensity as it passes through materials is given by something called the Beer-Lambert law this is an approximation. Of course it is possible to derive some variant of this starting from matrix equations, in a systematic way we will obviously not do that. But it is physically based right and it says that the intensity that you see finally is the initial intensity times an exponential function of minus times the integral mu of y of s z of s t s is going from 0 to s naught.

So it tells you that the intensity is exponentially falling off as the light ray passes through the material. And this sort of, makes some sense because you know the farther you go the more absorption is happening and so you would expect this exponential falloff or a strong fall off to be present and the differential is exactly that. So I naught of course is the initial intensity you had when the light wave was entering and I f is what you pick up on the screen.

And we will assume that there is no absorption you know in the, material surrounding the object of course there is a light x-ray going through air or whatever it is practically 0. So the material property is really encoded here in mu this is your material property. So a single ray measures the integral of the material property along the line so you can imagine if you have a sequence of rays that go like this parallel to this ray which is what our, incoming x-ray source does.

Then you have a sequence of these integrals that are present on the final screen right so that is what you are going to pick up. So now we will talk a little bit more about this mu the final aim like we said is to be able to determine mu of y, z. So that is in this plane and if you look at subsequent planes below you have a x dependence also coming in so you want to be, able to recover this right.

And the way to do this is remember what you are measuring is an integral of mu is to go from integral of mu something along a line to actually mu that is basically the mathematical problem we have to solve. And the way to do this is using Fourier transforms as you will see maybe towards the end of today's session.

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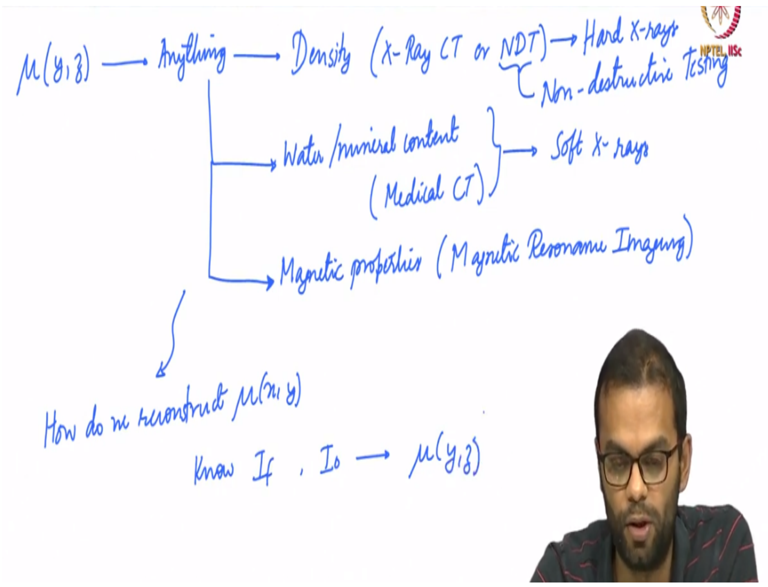

Now this mu itself mu of y z or whatever x, depending on the direction this can be anything it can be any physical property that is contributing to absorption. So for instance it could be density so more matter means more electrons which means more light is absorbed. So it could be proportional to the density in some way and so if that is the case then you are looking at x-ray what is typical with x-ray microtomography or in, non-destructive testing.

If you want to look for flaws you want to size the flaws shape the flaws inside the material that you cannot access then you do something called non-destructive testing. It is a very common way of figuring out and quantifying defects inside material specially, metallic materials. It could be for instance proportional to the water content or some mineral content different minerals absorb different x-rays to different degrees.

This is what happens in biomedical or medical city where you are actually measuring mu that is proportional to this particular content is again sort of similar to the density but not quite the same. There is a distinction between these 2 incidentally so usually when you are working with metals the intensity and the energy of the x-rays that are incoming should be somewhat larger because their absorptivity is larger.

So you use typically what are called hard x-rays over here so these are higher energy smaller wavelength extras and over here you usually use what are called soft text rays. So these are lower energy larger wavelength lower frequency x-rays that is just a detail you want to keep in mind. Or it could actually even be this mu, could actually even be proportional to the magnetic properties of the material.

So sometimes there is a magnetic moment or a distribution of magnetic moments or magnetization you want to measure and that it turns out there is also described by the same Beer- Lambert law. In which case the reconstruction scheme that you do to get mu is actually measuring the magnetic properties and this is exactly, what happens in magnetic resonance imaging or an MRI scan if you have had the misfortune of going through one.

And the techniques are somewhat similar right there are obviously differences from one to the other there are some things you have to change but the broad ideas are more or less the same. So let us start looking at what we have to do to reconstruct mu of x, y in general right, irrespective of what it represents we will just treat it as a mathematical quantity right now. We know that we are measuring If we know I naught we want to use this to invert to get mu that is are basically our problem y z x y whatever some 2 coordinates in the plane.

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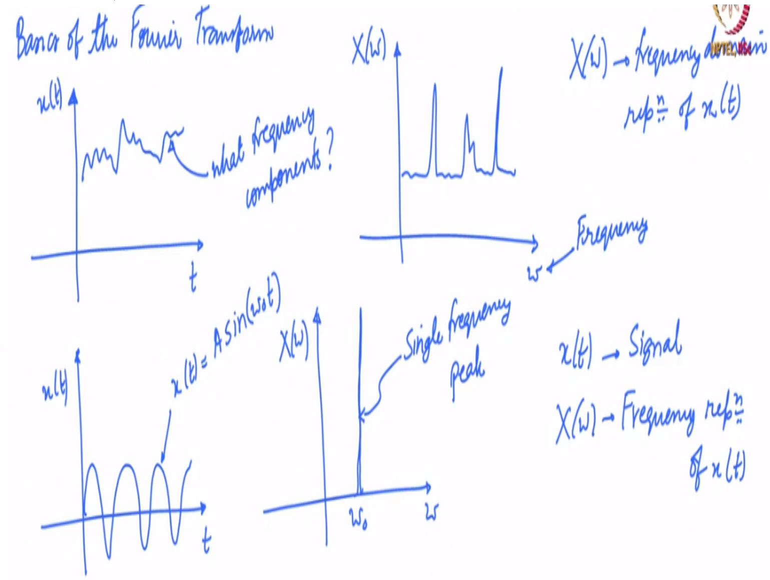

To do this we have to like I said use the Fourier transform so we will just basically look at some background material on the, Fourier transform before we can start using it to make sure that everybody is on the same page. So we will go over some of the basics of the Fourier transform so if you have a signal we have already mentioned this at some point when discussing DIC but nonetheless.

So let us say you have a signal as a function of t is time or length or distance or whatever it does not matter some variable we, will for now assume it is time because it is intuitively easier to make sense of and the term frequency also comes from assuming t is time. So let us say you have a signal like this some shady looking signal. If you want to figure out what frequency components are there or what frequency components make up this signal.

Then you have to convert it into a frequency domain of course and the, frequency domain will represent the signal by X and that will tell you what frequencies are contained in the signal right. So this omega's frequency so if t has units of second then omega's units of one by second or hertz and x is the frequency domain representation of x. The simplest way to look at this of course is if you have a simple sine wave.

So this is x of t is say a sine, omega t omega naught t let us say this is x of t this is t this is a simple sine wave. The frequency representation of this if you ask the question what frequency components are in the signal then the frequency representation is basically 0 everywhere and it has a large peak at omega = omega naught right. So there is a single frequency peak so Fourier transform basically is giving you the, information that is contained this frequency information is contained in the signal and this type of information becomes very useful.

One of the reasons is very useful is because it allows you to evaluate what are called convolutions again we discussed this in the context of digital image correlation. But I will briefly mention it again where convolutions are basically time integrals of a, signal and or integrals along t whatever t is could be a special variable also. And the corresponding representation in the Fourier space becomes a multiplication just a product right so it is easier to evaluate.

Now before we go there let us look at the definition of a Fourier transform so if you have x of t is your signal and x of omega is the frequency or Fourier representation. Then you can get X, from x by doing a transform and the transform is defined like this.

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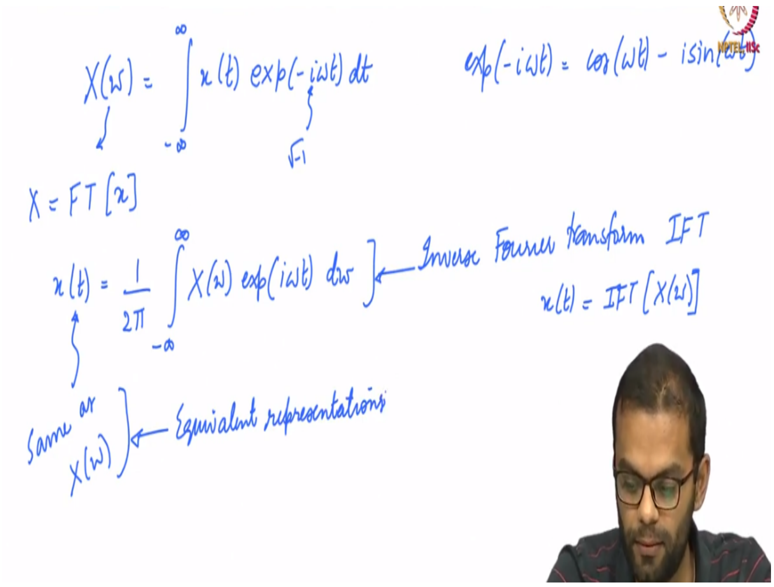

Where you have an exponential I of course is square root -1 omega is a frequency that is the variable that comes out and t is integrated over. Notice I put a limit going from minus infinity to plus infinity so t does not really have to be from 0 time = 0 t can be any variable like we mentioned, it can be a spatial variable also, in which case it can in principle go from minus infinity to plus infinity so this gives you x of omega.

The exponential of course is obtained from this Euler relation you have probably seen this in a complex variables course so it really is 2 parts a real partner imaginary part that you have to take into account good. Now this is called the f t or this, is x = f t of x f t being Fourier transform you can recon reconstruct x if you knew X by doing what is called the inverse transform. And it is almost analogous looking with the plus sign replacing the minus that was here there over here and you have a 1 by 2 pi factor which comes up in the 1D case.

So this is what the inverse Fourier transform looks like we will call this IFT so your x is IFT is, X. So if your knew x of t then it is as good as knowing X of omega or if you knew X of omega you know everything about the signal if you knew a x of t you know everything about the signal. So they are both equivalent representations good.

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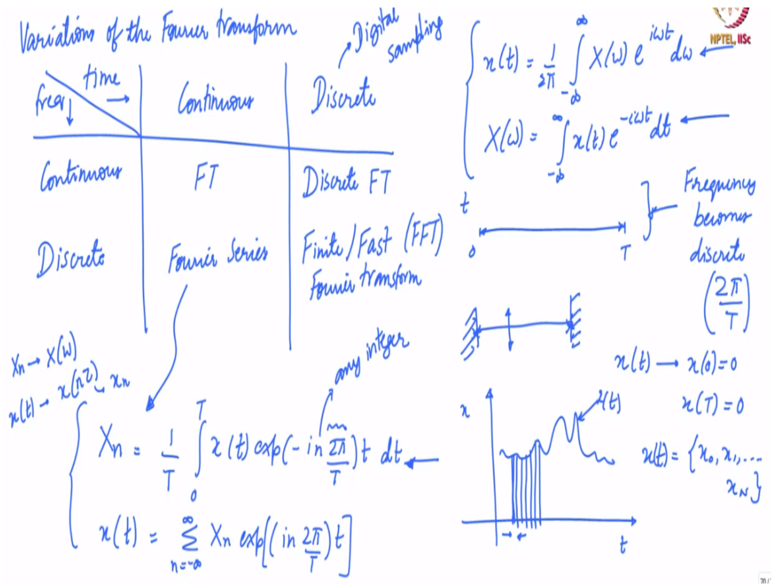

So let us look at a few different variations and the reason these, variations are important are because they come up in practical implementations. So we look at the variations of the Fourier transform so I am going to draw a table here so we will have just time and we have frequency here going down. So when time is a continuous variable and the frequency also is a continuous variable you get the Fourier transform that we just discussed.

So you have x of t, is 1 by 2 pi integral minus infinity + infinity x of omega e to the power of I omega t d omega and the corresponding expression for X right. So these are 2 integrals the integrals are done one over, t 1 over omega and you get the corresponding representation and this is what the Fourier transform is. So you assume that the time variable is continuous and you assume the, frequency variables also continuous.

If the time is not continuous but time goes or t being t right if t goes from being 0 to T so it is not going from minus infinity to plus infinity it is not an infinite domain but it is a finite domain. Then in cases like this the frequency becomes discrete and they become multiples of what is called a fundamental frequency which is 2, pi by t. So this is a very general phenomenon it happens in all sorts of situations the most common situation is if you take a string and you tie it between 2 you know fixed points.

And then you try to oscillate the string you will only get certain modes right you only get certain modes corresponding to certain frequencies. And so these frequencies are now discrete but if you have a free string that is infinite string then you can do all sorts of oscillations because it does not have to satisfy boundary conditions. So if you have a function x of t that has to satisfy let us say x of 0 = 0 and x of T = 0.

I am just using 0 as a number it could be x naught and x 1 or x1 and x 2 whatever it basically satisfies 2 boundary conditions. Then the frequency, spectrum corresponding representation in the frequency domain will become discrete. So whenever you have a discrete frequency representation you do not have a Fourier transform but you have something called a Fourier series.

So if you have taken a course in engineering maths you have probably looked at solutions of differential equations or you know heat equation and the solution often comes up, in the form of a series the Fourier series and that is because the differential equation has a finite domain. And the finiteness of the domain means that the frequency becomes discrete right. So the thing that changes when you do this is in the discrete case when you have Fourier series.

You have not any frequency but you have a discrete set of frequencies and they are given by this. There is, no omega now there is only x r the n becomes like your multiple of this fundamental frequency which is 2 pi by t. And the representation for x changes goes from minus infinity to plus infinity n is of course any integer and this now becomes x n times e to the power of + in 2 pi by t times t.

So Fourier series representation becomes like this your time is continuous there is no remember this, is an integral now. So there is no constraint on this but the frequency is discrete because you only have this index n you only have multiples of the basic frequency which is 2 pi by t. So this is again the moment you know if you want a physical picture this oscillating string is a good picture to keep in mind.

So this is kind of obvious right we have seen this in some form or the other now if, you look at digital signals you are always sampling the signal at a particular frequency. So you do not have in principle x of t with infinite resolution in t. So what you are actually measuring is if this is your x of t if this is your actual signal what you are actually measuring is at some intervals right whatever device it is for using for recording stuff is actually measuring at some intervals.

And so you really do not have an infinite or a completely continuous signal but you have a discrete set of points that represent the signal. So now we have to talk also about situation where this time is discrete so this happens when you have digital sampling. And your time now is a discrete variable so signal x of t is represented by a collection of values x 0 x 1 up to x N depending, on how many samples you have taken and so you have a discrete time signal.

If you have an infinite domain discrete time signal which means you do not have boundary conditions like this then your frequency will be continuous because the frequency has no reason to become discrete. But your time is now discrete when you have a situation like this you have a variation of the Fourier transform that is, called the discrete Fourier transform. So you still have a continuous frequency spectrum but you have a discrete set of points in time.

And now this the relationship will slip so I am not going to write this down because we do not need it for now you only need it for an implementation so we will talk about a later stage but the implementation will the representation will flip. So now x n will be, represented as an integral as a summation X n. so X n will actually become x of omega it will be an integral so it will be a sum sorry and x of t will now become x of n times tau or x n and this will now be represented as native.

So you flip the 2 equations here and you change the signs accordingly that is what will happen now it is not pragmatic to have a, signal defined over an infinite domain and so you have typically it signals defined over a finite domain. So you really have to look at this last you know cell in this table which is you have a discrete time which means you have a digital signal sampled. And you also have a discrete frequency because you have a finite domain.

And in this region the Fourier transform that you know corresponding, transform that you have to do is called a finite or a fast Fourier transform or sometimes called an FFT. So if you open up Mat lab or python or something you will see package to do FFT and not FT. And that is because you have to work with discrete signals you have to work with arrays and over a finite domain and so you will get something that looks like you know sum in both cases.

So what you, will do here is you will approximate this as a sum and this is a sum over a discrete set of time points and a discrete set of frequencies. So these distinctions are important to keep in mind because they come in handy when you look at practical implementations. But for now we will skip over some of the details and move on to the actual description of how this Fourier transform comes in handy.

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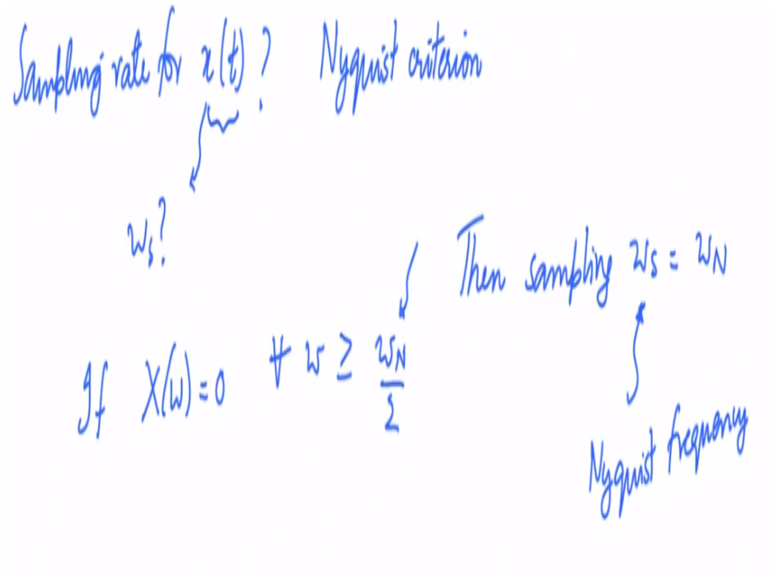

But before we do that I just wanted to mention one last thing which is this idea that you have to sample. So if you have a continuous variability you have to sample at a certain frequency right. And so the natural question comes up what should be the you know ideal rate for sampling if you have a frequency sampling frequency let us say omega s for you know want to have a better subscript s for, sampling then what should the value of omega s be?

And this is given by something called the Nyquist criterion and this criterion is important to keep in mind. Because when we look at signals for the case of you know tomography red words called the Releon transform you will see that we have a t dependent signal like this. And we are trying to reconstruct it from its Fourier transform that, is what we will be doing? And so you naturally need to know what the smallest resolution is that you can get when you do this type of thing right so the Nyquist criterion basically tells you the following.

So if x of omega = 0 for all omega greater than equal to sum omega n by 2 so in your frequency representation of the actual function the values are all zero after some particular, frequency which we are calling omega n by 2. Then the sampling frequency omega s must be equal to omega n and this is called a Nyquist frequency. And this tells you really that you need at least 2 samples per cycle of the large of you know fastest changing frequency in your signal that is basically what the Nyquist is.

If you do not sample at this frequency or higher, you can obviously sample higher however high you want right. But if you do not sample if you sample lower than this particular frequency then you will it will lead to data loss you will not be able to reconstruct the signal entirely is there something to keep in mind. Now I will end this today's discussion with a more generalized version of the Fourier transform called the 2d Fourier, transform.

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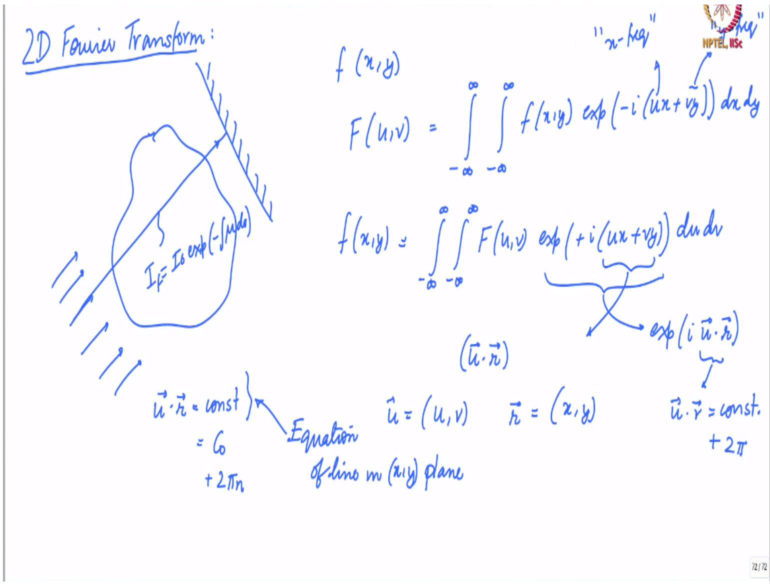

Now so far you have not seen perhaps it is not obvious how this Fourier transform relates to the situation we described. Remember the situation we described was this we had a light ray coming like this it is got a bunch of lighters coming like this and then you have a screen here. And then you have you know whatever intensity along one particular line is given by I f = I naught, exponential of minus integral mu d s going from s naught to s f or whatever right or s 0 or 0 to s naught.

This was our problem now to see where we you know where this comes from we want you know work out the entire detail entire bunch of details now we will do that in the next session. But to get started let me introduce a more generalized form or a higher dimensional form of the free transform, called the 2 d Fourier transform. So let us say instead of x of t I have a function mu of x, y some function or for generality let us use f of x, y d 2 D transform of, f is given by f of u, v now you have 2 variables.

So you transform into 2 variables and it is given by you know the analogous expression so you take f of x, y you do a 2D integral for now you do,- i times u x + v y times dx dy. So it is just you know if you do not know anything about Fourier transforms. Somebody said you know generalize the 1D case or 2D case this is the expression you would write down right without thinking and it is correct.

So u becomes like an x frequency and v becomes like a y frequency so you have 2 frequencies now to deal with one in the x direction on the y, direction the inverse transform is analogous you have to now integrate over all frequencies with this kernel that has a positive sign replacing the negative sign and exactly like the 1D case it looks like this is the inverse 2D Fourier transform.

Now to see where we are going with this I want you to focus on this so remember u is like an x frequency v is like a y frequency so, it has dimensions if x is dimension of length then u has dimensions of one by length or wave number likewise v has dimensions of one by length. This expression can be written as u bar dot r bar u bar is a vector 2 component vector it has components u and v and r bar is also a vector which has components x and y.

So r bar is the composition vector u bar is like a frequency vector let us see now this e to the power of i u bar dot r bar. This function will have the same value when u bar dot r bar is constant. So when u by dot bar is constant e to the power of i u bar del r bar is constant. So anywhere where u bar dot r bar is constant this function will have the same value not only that I can add 2 pi to this constant and the exponential will get the same value and that is, really the key for starting off on the 2D problem for tomography.

This value of u bar dot r bar equal to constant this fellow represents the equation of a line in the plane in the x y plane in this case right. And the line is distant that says constant is equal to some c naught the distance of the line the offset or the intercept on the y axis is given by c naught right. Not only that it represents, for this exponential to have the same value it represents a family of lines all separated by 2 pi n.

So if you have c naught + 2 pi n e to the power of i u dot r will give you the same value because e to the power of i 2 pi = 1 so you now have a family of parallel lines all separated by 2 pi n.

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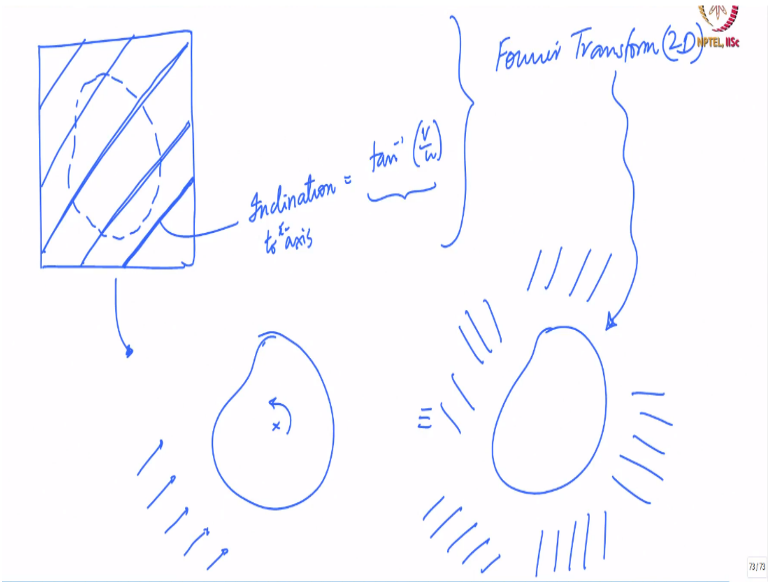

So what this means is that if you take a plane your e to the power of i times 2, pi n is going to have the same value on a family of parallel lines that go like this. The inclination of the line is of course given by tan inverse of v by u in relation to x axis of course it related y axis is tan mass of u by v but you have a family of line. So on this line exactly on this line e to the power of i times u dot r has the same value as it has exactly on this line as it has exactly on, this line and so on.

And if u dot r = 0 let us say for example then the corresponding orientation will change and so you will have a different value of the exponential but nonetheless it will still give you a family of lines. So now you see already there is some relation now you have a family of lines that are passing through the material. And if your cross section were like this you, could think of this family of lines as sampling the function you know f of x y along a particular direction characterized by a pair of constants u, v by changing this u, v.

By changing the orientation of the family of lines you are sampling a different location in the frequency domain. I hope this sort of thing is loosely at least intuitively clear this I would like to think about a, little bit more this is somewhat analogous to having the same sample and rotating the sample. If you have arrays like this having the sample and rotating it about you know 180 degrees because this is analogous to having the sample and looking at a family of lines like this and so on.

And this is exactly what the 2D, Fourier transform does so you now have some type of hopefully intuitive idea of where this Fourier transform fits in. So the 2D Fourier transform is related to what you are actually doing in a tomography problem right. So you are sampling these integrals along those lines and in some sense you are actually probing the frequency domain directions given by this.

So we will just think about this idea, we will start from here in the next session we will make this a little bit more concrete there is something called the Fourier slice theorem we will talk about that. And once we know exactly what we are sampling and how it is related to the Fourier transform we can then look at how reconstructions can be done and how the actual function f of x i which in our case will be mu of x y can be, recovered. So we will start with that in the next session.