Dynamics and Control of Mechanical Systems
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Lecture-09
Inertia, Momentum and External Forces

Welcome to this NPTEL lectures on dynamics and control of mechanical systems. My name is Ashitava Ghosal. I am a professor in the department of mechanical engineering and in the center for product design and manufacturing, and also in the Robert Bosch center for cyber physical systems at the Indian Institute of Science, Bangalore. In this week, we will look at the main topics are inertia, momentum and external forces.

In the last week, we have looked at kinematics. So basically we are looked at linear and angular velocities and accelerations.

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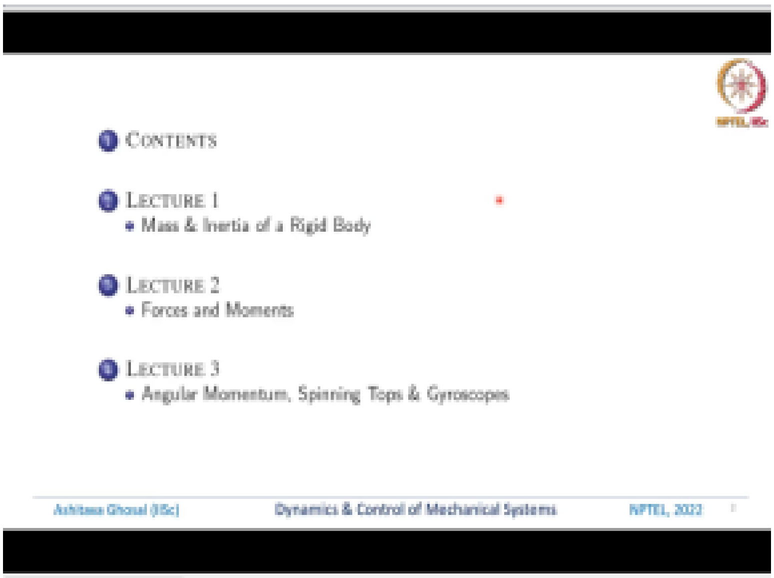

In the first lecture, we will look at mass and inertia of a rigid body in 3D space. In the second lecture, we will look at external forces and moments which could be acting on a rigid body or a system of rigid bodies. And in the last lecture, of in this week, we will look at angular momentum. And we will see how angular momentum is involved in spinning tops and gyroscopes.

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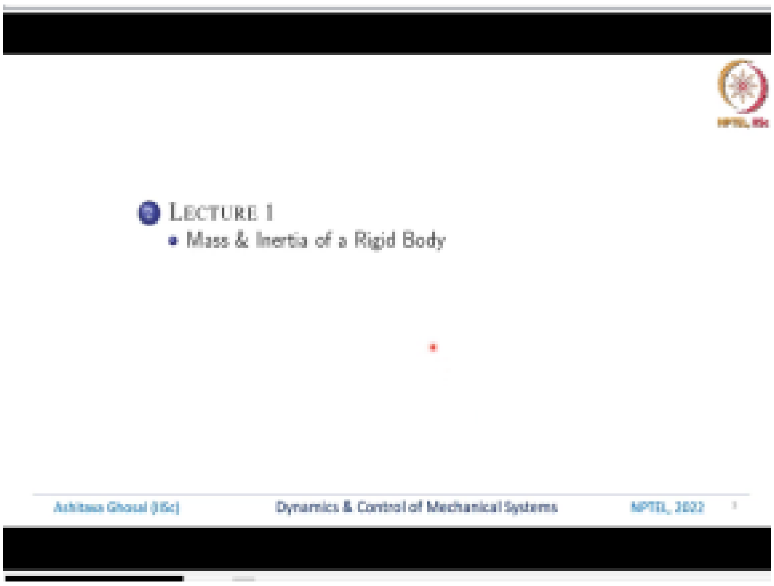

So, first lecture mass and inertia of a rigid body.

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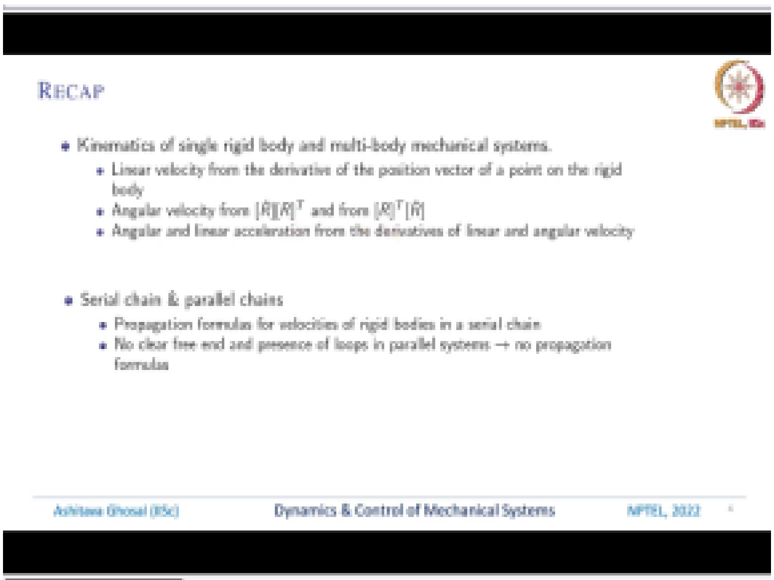

A quick recap: In kinematics of a single rigid body and multi-body mechanical systems, we looked at the linear velocity from the derivative of the position vector of a point on the rigid body. We also looked at the angular velocity and it was derived from boost cue symmetric matrices, one was $[\dot{R}]\ [R]^T$ and from $[R]^T\ [\dot{R}]$, where R is a rotation matrix which describes rigid body be in a reference coordinate system A.

And from this angular velocity, we could also look at angular and linear accelerations and these were basically derivatives of the linear and angular velocity. In serial and parallel chains with derived propagation formulas for velocities of rigid bodies in a serial chain. For parallel chains, I showed you that there are no clear free ends and there are loops in a parallel system.

So hence, it is not possible to obtain propagation formulas. So, propagation formulas, if you recall or nothing, but we give the velocity of one link and we go to the next link and find the linear and angular velocity and linear and angular acceleration with respect to the previous link and so on. So, we start from a fixed base purely the linear and angular velocity are zero. For the fixed base and then we have joints connecting to the first link.

And then we can find the linear and angular velocity of the first link, then based on the propagation formula, we can go to the second link and so on. And all the way to the free end in a serial chain. Whereas in a parallel system, there is no such clear, fixed base and a free end, because there are loops, you can start from one fixed body, go through some chains, go through some rigid bodies and joints and come back to the fixed ways.

So in such cases, we go back to our basic definition of linear and angular velocity. So, we pick a point on one of the output links and then we find the derivative of that position vector which gives the linear velocity. For angular velocity we find the rotation matrix of that moving coordinate system or the moving object, which is our chosen output and we do $[\dot{R}]\ [R]^T$ or $[R]^T\ [\dot{R}]$.

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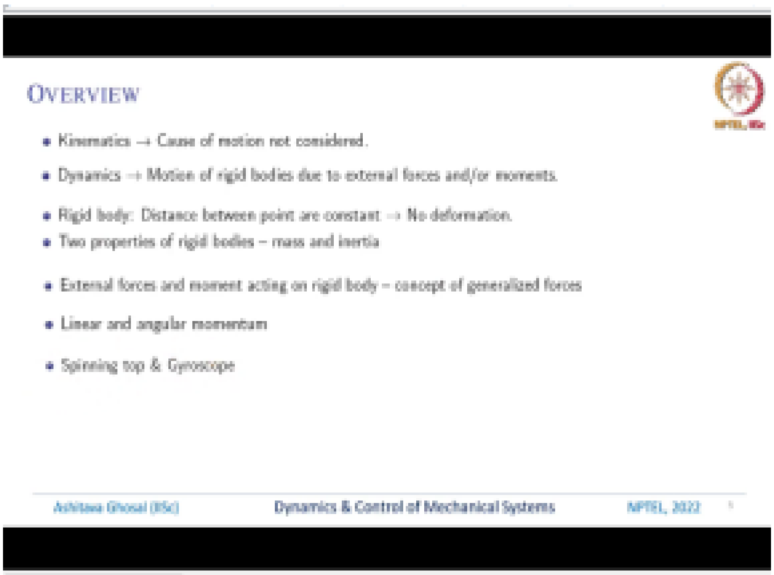

In kinematics the cause of motion was not considered. We are slowly going towards this topic of dynamics of mechanical systems. And in dynamics, we will look at the motion of rigid body due to external forces and or moments. So, as I said earlier, in a rigid body the distance between two points are constant, there is no deformation. So, the two properties of a rigid body are mass and inertia.

And we will also look at the external forces and moments acting on a rigid body, which brings us to this interesting and new concept called generalized forces. And then we will look at linear and angular momentum. And as an example, as an application of angular momentum, we will look at a spinning top and gyroscope.

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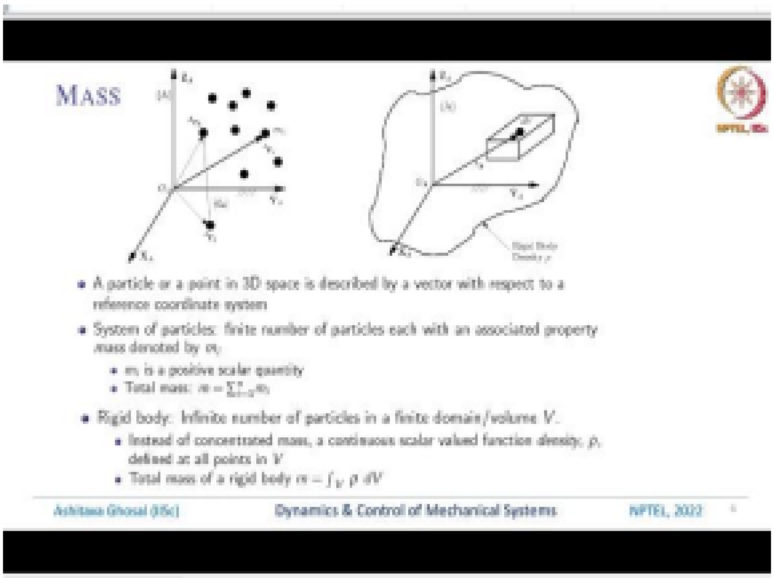

So, let us look at mass of a rigid body. So, this figure shows a reference coordinate system as usual it is unit vector $\hat{X}_A$, $\hat{Y}_A$ and $\hat{Z}_A$ with an origin O_A and what I have shown here are particles, each particle has a mass m_i and it is located from this origin of this reference coordinate system by a vector Ar_i. So, in a rigid body; so, the distance between two particles let us say this Kth and the Lth particle.

And the distance is d_{kl}, it is nothing but the Euclidean distance. So, $\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2$ and $\left(z_2 - z_1\right)^2$ and the square root of that, that is the distance between these two particles, they remain same, they do not change. So, a particle or a point in 3D space is described by a vector with respect to a reference coordinate system. So, if you have a system of particles, basically, we have a system of finite number of particles.

And let us assume each of these particles has a property called mass which is m_i. So, m_i is a positive scalar quantity. So, the total mass of the system is summation over all the individual masses from i equals 1 through n and that is given by m. Whereas, if you do not have a system of particles, but you have a rigid body, then you have this rigid body. And then we look at a volume element which is dV.

And we look at say the center of this volume element which is A r and again this vector Ar is with respect to a reference coordinate system A which has $\hat{X}_A$, $\hat{Y}_A$ $\hat{Z}_A$ and an origin O_A. So, in the case of a rigid body, we have infinite number of particles in a finite domain finite volume. And instead of concentrated masses like here, we have a continuous scalar valued function which is called the density denoted by ρ.

And this is defined that all the points in this finite volume. So, the total mass of this rigid body is nothing but the volume integral of ρ dV and that is given by m. So, in the case of a set of particles, that total mass is summation of all the individual particles, masses of all the individual particles. So, m is summation i = 1 through n m_i, whereas, in the case of a rigid body it is the volume integral of ρ times dV, dV is a volume element of this rigid body.

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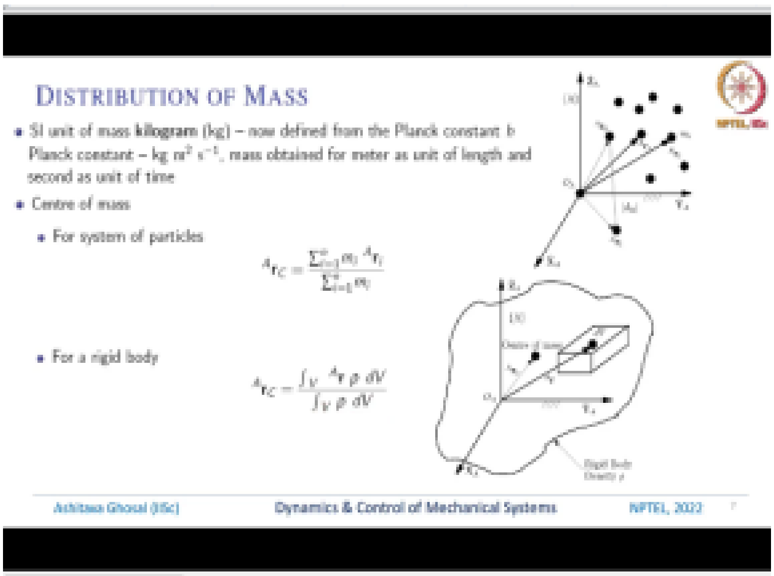

So, the unit of mass is kilograms and for a long time this kilogram was defined as a platinum iridium alloy, one small cylinder, but nowadays it is defined in terms of Planck's constant h. So, the unit of Planck's constant is kg meter square second inverse. So, if you can define what is meter and second, again meter and second are defined in terms of some atomic length and frequencies of certain oscillating atoms.

Then, if you assume that the Planck's constant is a fixed number that automatically defines what is the unit of mass or what is a kg. Now, if you have a distribution of particles or a rigid body there is an important notion called the centre of mass of this rigid body? So, in the case of a set of particles and again we have various particles with respect to this reference coordinate system.

And each of these particles are located by some vector Ar_i locates the ith particle of mass m i. So, for this set of particles, we can find a point which is a Ar_c which is basically the vector locating the center of mass of this set of particles and it is defined as in this way. It does not thing but i = 1 through n summation m_i Ar_i . So, basically some vector and then you associate a particular mass m_i.

And then we sum over all of these particles and divide it by the total number of particles i = 1 through m and some of that. In the case of a rigid body, again we instead of summation, we have to do some integration. So, for the rigid body, the location of the center of mass is given by the volume integral of this vector Ar into ρ dV divided by volume integral of ρ dV.

So, now, the denominator volume integral ρ dV is nothing but the mass of the rigid body.

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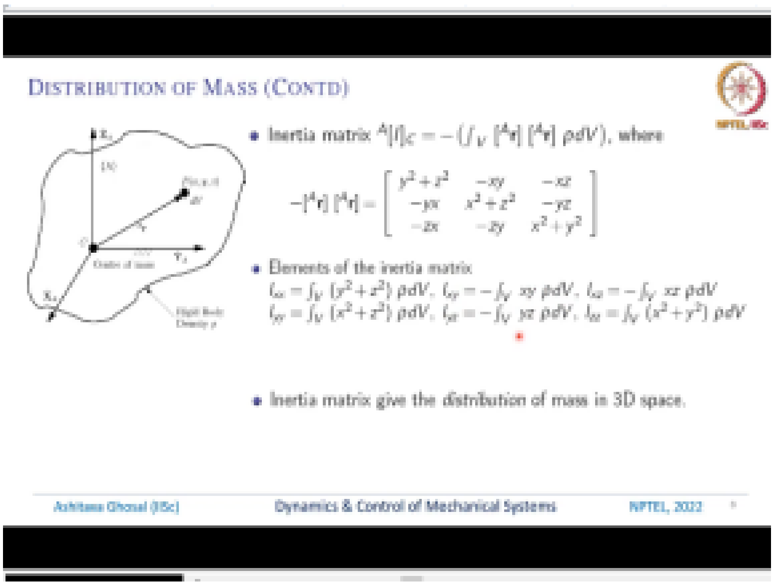

So, just the mass itself is not enough we also have this notion of distribution of a mass of a rigid body and that is what is called as inertia. So, the inertia is given in this form. So, I have a rigid body and let us pick a point on this rigid body which is that X, Y, Z. There is a small volume element dV at that point. So, what we want to do is we want to integrate this Ar and ρ dV over this volume with the minus sign.

Now, this Ar is the skew symmetric matrix which we have discussed earlier. So, this Ar is nothing Ar into Ar is nothing but $y^2 + z^2 - xy - xz - yx x^2 + x^2 - yz - zx - zy$ and $x^2 + y^2$. So, Ar is a skew symmetric matrix with diagonals as 0 and then $-z, -y$ and so on just like what we had discussed earlier. So, from the skew symmetric matrix Ar if you multiply both of these two matrices with a minus sign you will get this matrix in terms of x, y, z.

So, the elements of this matrix in i around the center of mass and again with respect to a reference coordinate system A is given by these elements. So, we have I_{xx} which is nothing but $\int_V y^2 \rho\, dV$, I_{xy} is $\int_V - xy \rho\, dV$, I_{xz} is $\int_V - xz \rho\, dV$, I_{yy} $\int_V x^2 + z^2 \rho\, dV$ again volume integral and I_{yz} is $\int_V - yz \rho\, dV$ and I_{zz} is $\int_V x^2 + y^2 \rho\, dV$.

So, this inertia matrix tells you how the mass is distributed in 3D space. So, in certain directions if the mass is more then some of the elements of the inertia matrix will be more than the others. So, intuitively what is I_{xx}? It is how the mass is distributed in some sense about this x axis. Likewise I_{xy} is tells you this element of this inertia matrix tells you how the mass is distributed in some other plane.

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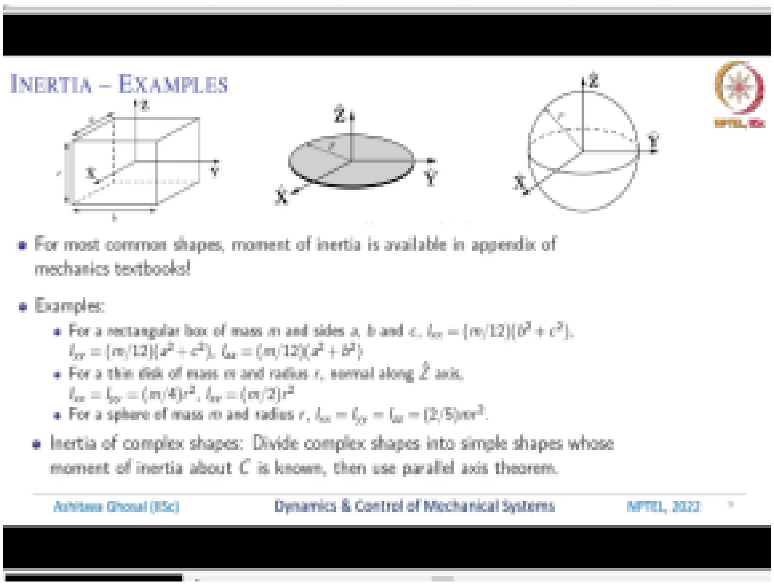

So, how do we calculate this inertia matrix? So, for simple examples like as cuboid or a disc or a sphere, these are available in all textbooks. So, you do not have to actually do all the integrations, these are well known, well studied and they are available in textbooks. So, for most common shapes a moment of inertia is available in the appendix of some mechanics textbooks.

So, for example, if you have a box of sites A, B and C, I_{xx} is nothing but $\frac{m}{12}(b^2 + c^2)$ I_{yy} is $\frac{m}{12}(a^2 + c^2)$, I_{zz} is $\frac{m}{12}\left(a^2 + b^2\right)$. What is m? M is the mass of this box, so, for the thin disc like this where the disc is in the xy plane and the normal to the disc is z. So, I_{xx} and I_{yy} or $(m/4)\, r^2$, where r is the radius of this disc and I_{zz} is $\left(\frac{m}{2}\right)r^2$.

Similarly for the sphere of mass m and radius r, all the three moments of inertia are exactly the same which is $(2/5)\, mr^2$. Inertia of complex shapes on the other hand is not obvious how to find. There are two main techniques which are used to obtain the inertia of complex shapes, which we will see a little later, but basic idea is you divide this complex shape into simple shapes whose moment of inertia about the center of mass is known.

So, I have a very complicated shape I can always discretize that complicated shapes into small cuboids, often a tetrahedron is also used and then we will use what is called as the parallel axis theorem.

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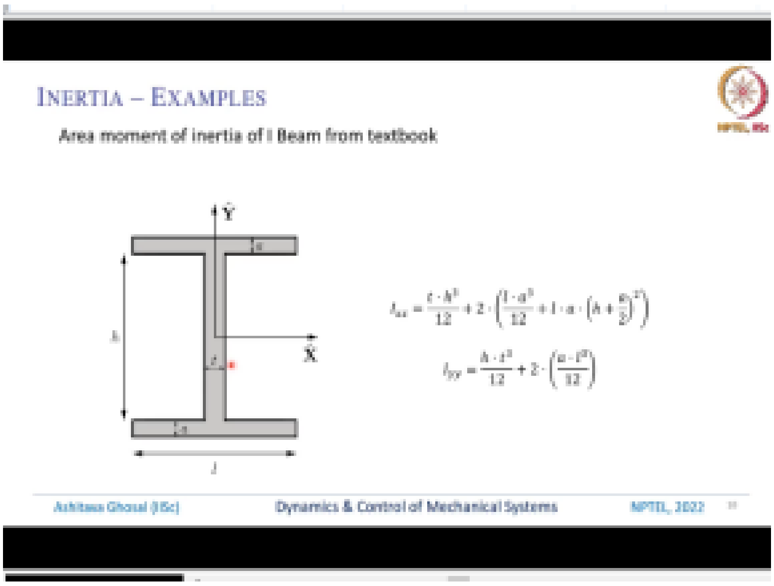

So, as an example of another inertia of a object from textbook, this is a very well known engineering cross section, this is called the I beam. So, there is one thicken portion here, there is a thin portion here and then there is another flat portion here. So, if this dimension is a, this is t and this is a and this height is h. Then I_{xx} is given by $\frac{t\,h^3}{12} + 2$ into something similarly, I_{yy} is given by this.

So,

I_{zz} depends on how much is the length of this I beam into the page. So, this is a very, very well known cross section which is used in many mechanics and in structures, because, there are some nice properties of this beam that the deflection and bending and stresses are in some desired way.

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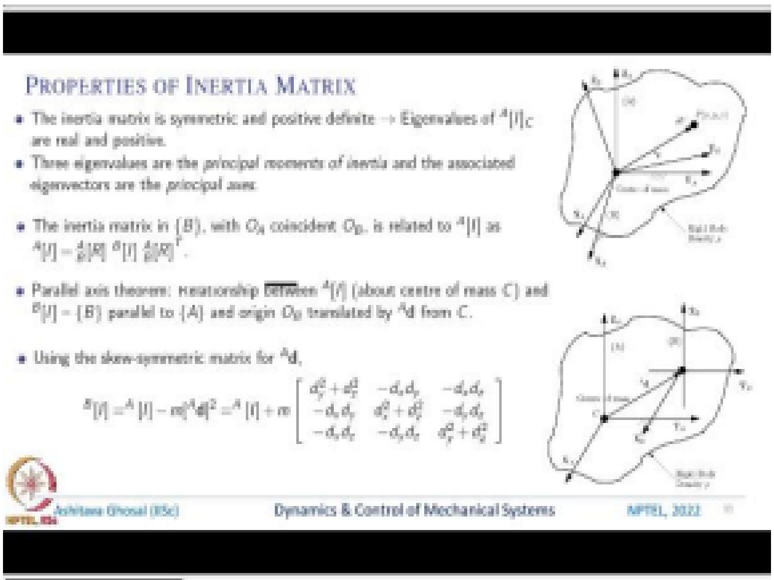


So, let us look at this inertia matrix in a little bit more detail. First thing to remember is this inertia matrix is symmetric and positive definite basically the Eigen values of that inertia matrix with I_{xx}, I_{yy}, I_{zz} and all the cross moments which is I_{xy} and so on. The eigenvalues of this matrix are always real and positive. And if you have a matrix was eigenvalues are real and positive you can get three positive eigenvalues.

So, these are called the principal moments of inertia and for each of these eigenvalues I can find an Eigen vector and these are called the principal axis. So, if you have this rigid body and if you have a reference coordinate system $\hat{X}_A$, $\hat{Y}_A$ $\hat{Z}_A$ and if you have another coordinate system $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$, which is rotated with respect to the A coordinate system and both of them are located at the center of mass.

Then I can show that the inertia matrix in the A coordinate system is related to the inertia matrix in the B coordinate system by this formula. So, A I is nothing but $BA[R]\ B[I]\ BA[R]^T$. So, this is a typical form when matrices are converted from one coordinate system to another rotated coordinate system. So, this tells you that if I have an inertia matrix given in one coordinate system and if I rotate the coordinate system, so, I have a rotation matrix $BA[R]$.

So, B with respect to A, then the inertia matrix in the A and B coordinate system are related to this matrix multiplication. The other important result for inertia matrix is this something called parallel axis theorem. So, what it says is it is the relationship between this matrix in the

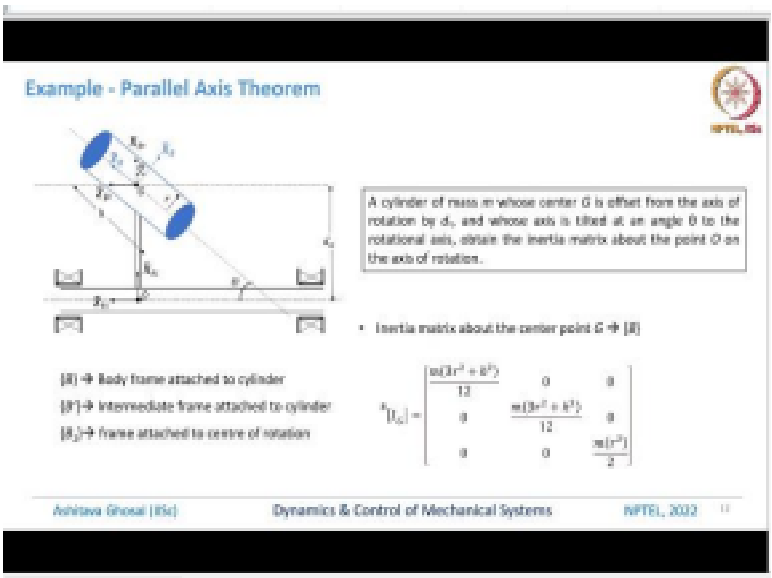
A coordinate system about the center of mass and another coordinate system B which is parallel to A but the origin is translated by a vector Ad .

So, I have a rigid body, I have a reference coordinate system or a coordinate system A which is fixed at the center of mass I have another coordinate system which is translated by a vector A d and this is the B coordinate system and translated with respect to the center of mass. This is important. This B coordinate system is not with respect to some other coordinate system which is not at the center of mass.

So, if I write this vector A d as a skew symmetric matrix again skew symmetric matrices have very useful tool. So, zeros in the diagonal and enough diagonals you have A ij is minus A ji. So, the inertia matrix in the B coordinate system can be written as the inertia matrix in the A coordinate system minus m into this $[Ad]\ square$ where Ad is the skew symmetric matrix and what is $-\ [Ad]\ square$? It is this quantity here.

So, you can see the terms in the diagonals are $d_y^{\ 2}+ d_z^{\ 2}$and $d_x^{\ 2}+ d_z^{\ 2}$ and $d_y^{\ 2} + d_z^{\ 2}$. So, basically if I have inertia matrix of a rigid body given in some coordinate system and then if I translate it then I can find the inertia matrix of that body in this translated coordinate system and so, we are always going to go from the center of mass to some other point.

We can do it the other direction also. So, if I know what is A this I can translate it back to the center of mass. Basically, I can use this expression to find the elements of inertia matrix between two coordinate systems which are translated from the center of mass. **(Refer Slide Time: 21:40)**



Let us look at an example. So, what we have is a cylinder of mass m and its center of mass is translated from some axis of rotation by this distance B 0. In addition the axis of the cylinder is inclined to this rotation axis by theta and we want to find the inertia of this cylinder with respect to this point O and in a coordinate system $\hat{X}_{B1}$ $\hat{Z}_{B1}$. So, remember Y is somewhere forms a right handed coordinate system.

So, how do we start? So, first thing is we will assign a coordinate system $\hat{X}_B$ and $\hat{Z}_B$ again Y is perpendicular to both $\hat{X}_B$ and $\hat{Z}_B$ and this B coordinate system is at the center of mass and we know for a cylinder what is the inertia I_{xx}, I_{yy}, I_{zz} and so on when reference coordinate system is at this center of mass and the principal axis or along the axis of the cylinders and two other perpendicular in this central plane.

So, that we know the elements of the inertia matrix in the $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ coordinate system.

First thing is we rotate about theta because we want the inertia matrix in the $\hat{X}_{B'}$, $\hat{Z}_{B'}$ coordinate system. So, how do I do this rotation? I use the formula which I showed you earlier. That A i is nothing but $[R]$. $[R]^T$. So, we use this formula of pre multiplying and post multiplying by rotation matrices.

So, do we know what is the rotation matrix of $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ with an $\hat{X}_{B'}$, $\hat{Z}_{B'}$, $\hat{Y}_{B'}$, yes because it is just a rotation of about the y axis by some angle theta. Then we translate this $\hat{X}_{B'}$, $\hat{Y}_{B'}$, $\hat{Z}_{B'}$ to this $\hat{X}_{B1}$, $\hat{Z}_{B1}$. So, this is the parallel axis theorem. So, if you do these two steps, one is rotation and then one by parallel axis then we can find the inertia matrix with respect to the rotation axis.

So, these are the steps which we need to do. So, the inertia matrix in the B coordinate system is available in textbooks. So, it is some $\frac{m}{12}\left(3r^2 + h^2\right)$ that h is the length of the cylinder and r is the radius. So, this is well known, this can be obtained from any textbook I_{yy} is same as I_{xx}, but I_{zz} is $\frac{m}{2}r^2$.

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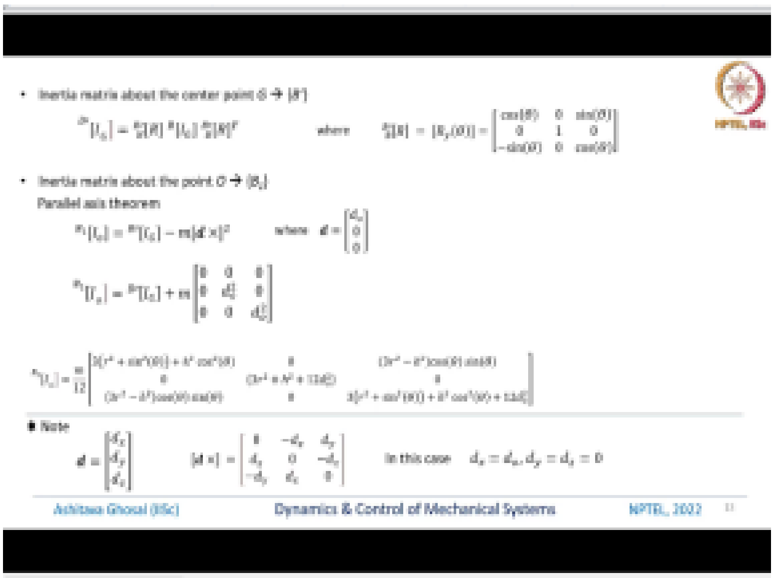

So, once we have found the inertia matrix so, as I had indicated I want to find it in the B prime coordinates system. So, basically we have to do the inertia matrix in the B prime coordinate system is nothing but some rotation matrix into this given rotation matrix into the rotation matrix transpose and as I said this rotation is nothing but rotation about the y axis. So, hence the rotation matrix is $cos\theta$ 0 $sin\theta$ 0 1 0 - $sin\theta$ 0 $cos\theta$.

So, that is the standard rotation about the y axis. Then we go from O to B 1 by using this parallel axis theorem, which is that the inertia matrix is related by this inertia matrix into $B'[I_G] - m[d\times]^2$ where d is d 0 0. So, hence $B_1[I_o]$ inertia matrix at the point O is given by $B'[I_G] +$ m into this $d_o^2\, d_o^2$. So, there is no x component because it just moving in this vertical direction.

Hence, we can do all this work and then we can show that the $B_1[I_o]$ basically the inertia matrix at O in the B_1 coordinate system B_1 remember the coordinate system is aligned with the rotation axis is nothing but m by 12. And we can find out one term which is $3(r^2 + \theta) + h^2\theta$. And then similarly, the I_{yy} is $3r^2 + h^2 + 12\, d_o^2$, I_{zz} is this. And now we have an I_{xz} component also, which is $(3r^2 - h^2) cos\theta\ sin\theta$, it is a symmetric matrix. So, the 3, 1 element is also the same as 1, 3 element. So, hence, d is dx dy dz, the skew symmetric matrix corresponding to d is 0 and the diagonals this is – dz, this is dy - dx and then this is + dz - dy dx. So, this is the standard skew symmetric matrix which we have used earlier, we have used it for omega, we have used it for position vector and various other places.

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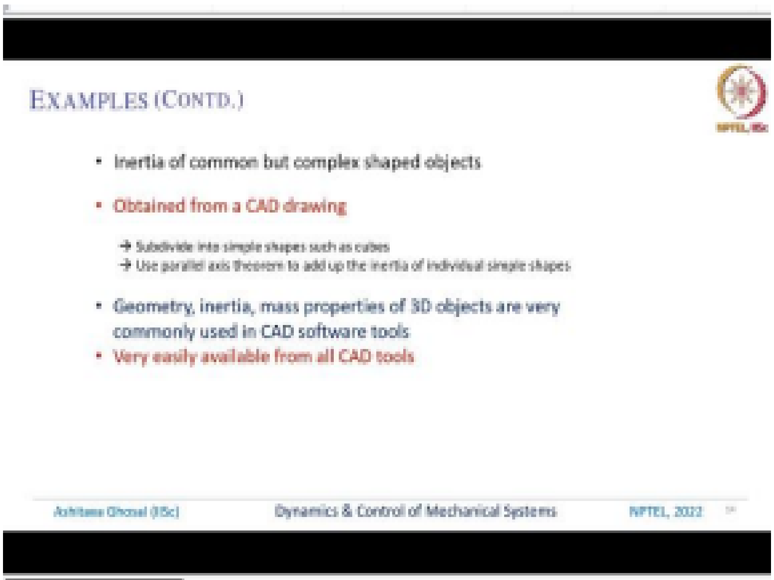

So, as I said by using this rotation and parallel axis we can find simple inertia of some cylinders or some other quantities with respect to another point and rotation also can be taken into account, but there are many other complex shapes. So, for example, we can have a very complicated shape obtained from a CAD drawing and I will show you some examples. So, how do I find the inertia matrix for a complicated object which I draw using some CAD software?

The basic idea is again is subdivided into simple shapes such as cubes, then we either use parallel axis theorem or we can also rotate this about the CG. So, combination of these two will give me the inertia of each of these small, small simple shapes which are cubes, sometimes tetrahedrons are used, sometimes triangles are used, nevertheless these are very simple standard shapes for which the elements of the inertia matrix are known.

And then we use these two techniques of rotation and parallel axis theorem to find the complete object. This geometry properties inertia, mass and all these properties are very commonly used in CAD software tools. So, for example, if you are going to do finite element analysis of a complicated shape you would like to know what is the mass, what is the surface area, what is the volume all these things and that can be all obtained in all CAD software tools, it is very easily available.

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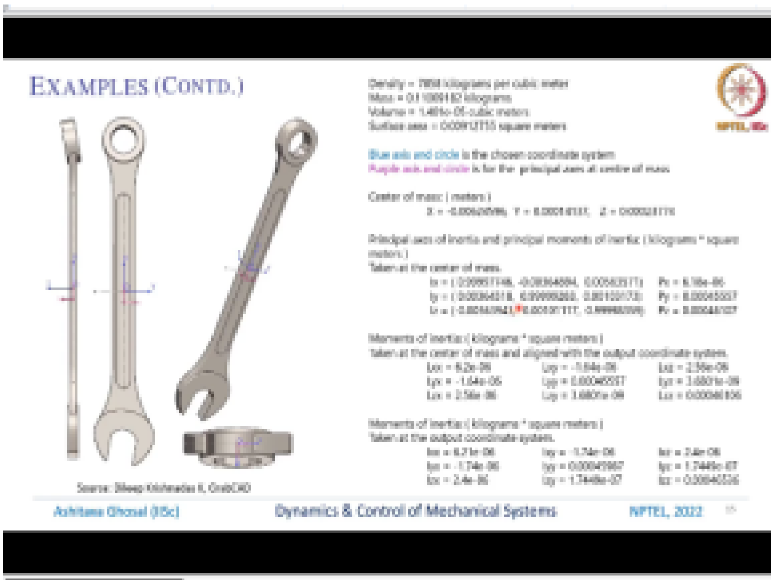

So, here is an example of a wrench. So, these are the two views of this wrench and we have chosen this blue point and this x and blue z axis at some arbitrarily chosen reference coordinate system. And then we make this drawing in these CAD software tools, so it looks like a wrench and then you can give it to the software and tell that this is material whose density is 7858 kilograms per cubic meter.

So, it is basically some kind of steel and it will tell you what is the mass of this? So, how did it find out the mass? It made small, small volume elements and it integrated over this or actually it made small, small mass elements and it did ρ m i ρ into dV integral volume and then it found the mass. The volume is some 1.4 – 5 cubic meters and it can also find the surface area.

So, once you have this, then it can ask you to obtain what is the center of mass and what are the inertia matrix elements? So, once you say what is it can find the center of mass by again using the formula of whatever we did. So, it is integral volume into some r into dV divided by total mass. So, the center of mass is x = -0062, y is 00014, z is 00023. These are all in meters.

So, with respect to the blue chosen coordinate system, the center of mass is slightly large. You can see this purple axis and the circle denotes the center of mass and the principal axis. So, we can find the principal axis and the principal moments of inertia at the center of mass again by just clicking in some software tool, some elements and we can find again I_{xx}, I_{yy}, I_{zz}. So, I_{xx} is 0.99. So, what it is telling you is that we can find out all these masses the direction of the principal axis and all the inertia I_{xx}, I_{yx}, and I_{zz}; everything by using this software tool.

So, I_{xx} means this is the axis corresponding to one of the principal moments of inertia. So, this is along the x axis. So, roughly speaking this is 100, the y axis 010 and z axis is this 001. That is small errors are there because it is a numerical tool after all. And this position of this center of mass can also be obtained little bit of the x direction; y and z are also very small. The moment of inertia taken at the center of mass and aligned with the output coordinate system is this.

And similarly all the elements of the inertia matrix taken at the output coordinate system is this. So, whether it is aligned along blue x, y, z or aligned along blue x, y, z and at the purple point. These are the two sets of inertia elements. So, we do not have to do much. So, you give me any complicated CAD drawing of any object. So, in this case it is a wrench.

I can tell you what is the mass, I can tell you what is the volume, I can tell you what is the surface area, I can tell you where is the center of mass with respect to wherever you are starting or your reference point and then I can find out all the principal moments of inertia and the principal axis.

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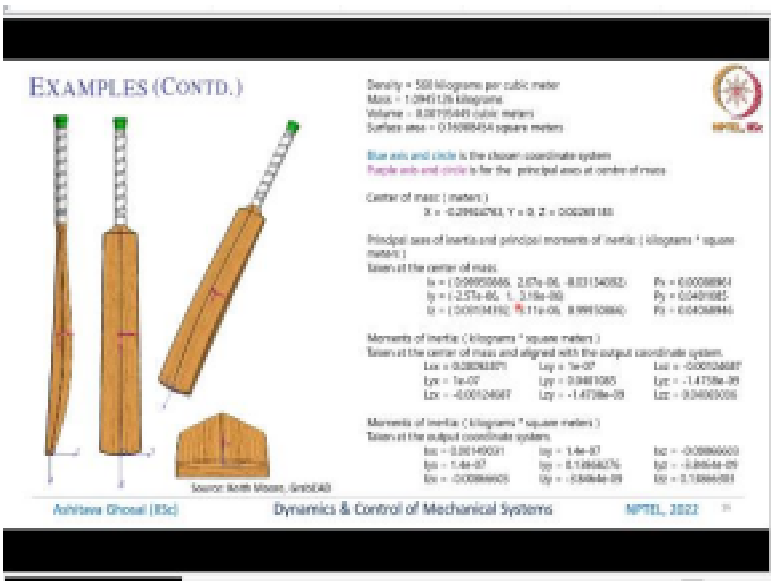

This is another example, this is a cricket bat. So, as you can see a cricket bat is a reasonably complex object, this is almost like a cylinder but then this is some flat area which is slightly bent and then in between it is very thickened. So, most cricket bats are nowadays such that you put lot of material somewhere in the middle, it is thinner here and again very thinner here but lot of material is here.

So, I would like to know what is the moments of inertia of this? So, maybe later on you want to design a different cricket bat and somebody tries it out and say that okay, if the moments of inertia are slightly less then you can hit better. So, again there is a reference coordinate system or we draw from this blue x, y, z and this point here and then you find that the density of wood is some 560 kilograms per cubic meter.

So, we find that the mass of this bat is 1.09 kilograms. So, this is a cooked up example, so maybe the mass of an actual bat might be more, the volume is such and such 0.00195 cubic meters and then surface area is 0.169 meter square, the center of mass we can find out is at point x, y and z. So, the y is still 0, but x, y is somewhere here. So, you can see this purple arrows and this purple's circle that is the center of mass of this cricket bat.

So, in this other view you can see this is the purple place. So, y is this way and x and z are this way and again we can find the principal axis at the center of mass and you can see it is a little bit off. So, here it is 1 roughly 0 but the z component is not really, really 0 some numerical problems might be there. The y is definitely 1 and then z is again also 0.999. So, again the x, y, z are calculated the principal axis and they can be calculated.

We can also find what are the moments of inertia taken at the center of mass and aligned with the output coordinate system which are this and then center of mass taken at the output coordinate system. So, again we can find out I_{xx}, I_{yy}, I_{zz}. So, all the various components of inertia of this complicated looking shape of a cricket bat or for that matter any object can be found using modern CAD software tools.

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So, let us switch topics a little bit. Once we know what is the mass and what is the moment of inertia matrix then we can find define a few things. One is again let us recapitulate I am going to call this vector or the skew symmetric matrix corresponding to this vector Ar in this form. So, $[Ar]$ is 0 -z y, z 0 –x, -y x 0. So, we can find the linear momentum of a body in this form. So, we pick a point p on this rigid body.

Find the velocity of that point p again with respect to some reference coordinate system A. So, that is $\dot{Ap}$ and then $\dot{Ap}$ ρ dV, volume integral is the linear momentum. So, for a particle what is the linear momentum it is m into V, m into the velocity of that particle. Likewise we can define something called as the angular momentum of a rigid body. The angular momentum is nothing but the cross product of this $Ap\times$ the linear momentum.

So, it is the moment of the linear momentum vector, this is a vector $\dot{Ap}$ ρ dV. So, it is a vector along the direction of the velocity vector and the angular momentum is the moment of that linear momentum. So, it is $Ap\times$ $\dot{Ap}$ ρ dV. Again we can write this cross product as some skew symmetric matrix A p into $A\dot{p}$ ρ dV, volume integral of this quantity.

If you want to find the angular momentum about the center of mass, so how do I find the center of mass? We can locate the center of mass which is by some vector Ar_c . So, the angular momentum about the center of mass is nothing but this Ar which is I want to again take the cross product and $A\dot{r}$ C and $A\dot{r}$. So, the velocity vector is nothing but the velocity this plus velocity this.

So, derivative of these two position vectors and again we have to take the cross product because we want to find the moment of this linear velocity vector. So, let us take a step. So, this Ar into $A\dot{r}_c$ ρ dV can be written as, so this is like r cross p. So, I can take out this $A\dot{r}_c$. So, r cross velocity can be written as velocity minus velocity cross r. So A cross B is -B cross A.

So, I can take this $A\dot{r}_c$ derivative of Ar_c outside with a minus sign and then inside the integral we have Ar ρ dV. Now this has to be 0 because this is the definition of center of mass, the center of mass is such that sum of this vector around that center of mass is 0. So, what this tells you is that I can simplify this expression; we need one more result before we simplify.

So, first thing is we want to write what is the space fixed angular velocity vector? So, remember the space fixed angular velocity vector was obtained by this $BA[\dot{R}]\, BA[R]^T$. So, this is starting from $[R]$. $[R]^T$ equals identity and then we took the derivative. So, this is the right multiplication. So, we have a skew symmetric matrix $B\,A[\Omega]_R$ and from that we obtain the space fixed angular velocity vector.

So, now what is $A\dot{r}$? So, $A\dot{r}$ which is basically this term is nothing but $A\omega^s_B \times r$. Again we make sure that we write it in the same coordinate system, this is $A\omega^s_B \times r$ or we can write it as $B\,A[\Omega]_R \times\ Ar$. So, matrix into Ar and similarly this into this is - Ar cross A omega B which is again we can write it as minus a skew symmetric matrix into $A\omega_B^s$ s.

So, what is the final result? That the derivative of this position vector Ar is nothing but minus Ar into $A\omega_B^s$ some skew symmetric matrix into angular velocity vector. So, hence the angular momentum of a rigid body with respect to the centre of mass; remember previously we had this obtain the angular momentum with respect to a point O_A in the coordinate system.

With respect to the center of mass is nothing but minus volume integral Ar Ar ρ dV into $A\omega_B^s$. So, how did we get that? So, see there is one term which is Ar into $A\dot{r}$ ρ dV. But $A\dot{r}$ I showed you is nothing but –[Ar] skew symmetric matrix corresponding to r into $A\omega_B^s$. This is the proof. So, if I substitute $A\dot{r}$ which is - Ar $A\omega_B^s$ here. So, I have one Ar, there is another Ar.

So, this is two Ar 's skew symmetric matrix, there is a minus sign which is coming and then this ρ dV is here and then we have this angular velocity vector. So, this is a very useful formula that the angular velocity of a rigid body about the center of mass is nothing but you pick a point on the rigid body from the center of mass. Let that vector be Ar and then you write it as a skew symmetric matrix.

And you multiply these two skew symmetric matrices okay and then you integrate over the volume and then you multiply by this angular velocity vector and that took the space fixed angular velocity vector.

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So, in summary I showed you what is the mass of a rigid body? So, this is again some kind of a volume integral ρ dV or if it is a set of particles or a system of particle that is sum of all the masses. The inertia on the other hand gives the distribution of the mass about this principal directions or some other directions and it is described by 3 by 3 symmetric inertia matrix which is symmetric and positive definite.

So, basically this matrix has always positive and real eigenvalues. So, from the eigenvalues I can find the principal moments of inertia and the corresponding eigenvectors gives the principal axis. The inertia matrix transforms like a tensor. So, what do we mean? If I want to write the inertia matrix in another coordinate system such that the rotation matrix between A and B coordinate system is given by $BA[R]$.

Then we have to do $BA[R]$ into the inertia matrix into $BA[R]^T$. We can also use this useful idea of parallel axis theorem where we can find the inertia about another coordinate system which is translated from the center of mass not rotated only translated and so if I know what is the inertia matrix in one of this coordinate system I can find it in the translated coordinate system.

And as I showed, this will be very useful, later on we can define the center of mass and then we can find the velocity of the center of mass; let us call that V_c and if you multiply by m that is called as the linear momentum and the moment of this linear momentum is called the angular momentum and that the angular momentum can be written in terms of a matrix into some angular velocity and the matrix is at the C.g or at the center of mass with respect to some coordinate system and $A\omega_B^s$ is the space fixed angular velocity vector.