Dynamics and Control of Mechanical Systems
Prof. Ashitava Ghosal
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 07
Joints, Degrees of Freedom and Constraints

In this lecture we will look$R_{33}R_{33}$ at set of rigid bodies bunch of rigid bodies connected by joints and then arising from these connections with joints we have this concept of degrees of freedom and constraints.

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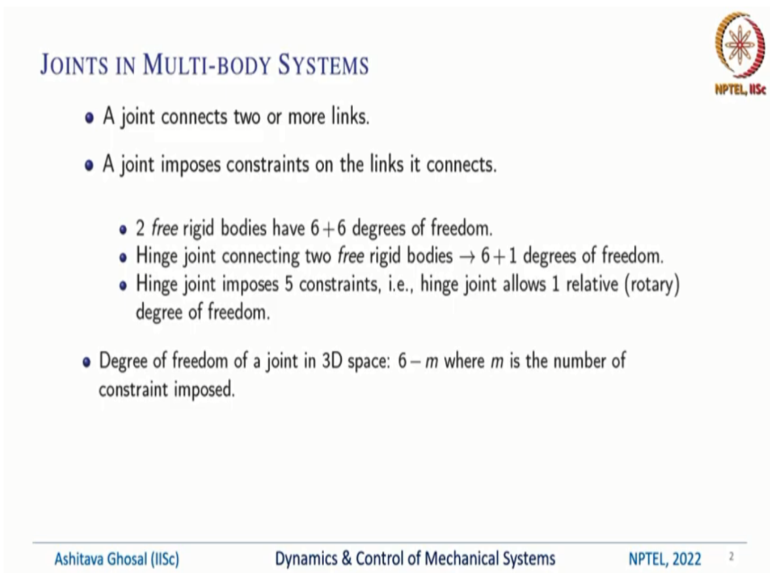

So, let us look at joints in a multi-body system a joint is something which connects two or more links and a joint basically imposes constraints on the links it connects a link, and a rigid body will be used interchangeably in these lectures. So, if you have two free rigid bodies in 3D space each of these rigid bodies have 6 degrees of freedom. So, we have seen earlier that it can be used to locate a point can be used to locate a rigid body in 3D space.

So, there are three coordinates x y z and then a rigid body also needs to be specified by its orientation, so then there are three parameters in orientation, so there are six parameters required to tell the configuration or position and orientation of a rigid body in 3D space. So, if you have two free rigid bodies then you have 6 + 6 degrees of freedom, you need 12 parameters. However, let us consider these two rigid bodies connected by a hinge joint.

A hinge joint is nothing but whatever connects a door to the wall that is like a hinge joint. So, now you can think of the wall as one rigid body and the door is another rigid body but now you can see that these two-digit bodies do not have 6 + 6 degrees of freedom they have only the wall or one of the rigid bodies has 6 degrees of freedom and the second rigid body which is connected to the first rigid body by this hinge joint has only one degree of freedom.

The hinge joint only allows one relative rotation between these two rigid bodies which it connects. So, another way of looking at this thing is that the hinge joint imposes 5 constraints. So, hinge joint allows one relative rotary degrees of freedom or it imposes 5 constraints between these two rigid bodies. So, the degree of freedom of a joint in 3D space is 6 - m where m is the number of constraints it imposes, and we will look at these m in more detail as we go along.

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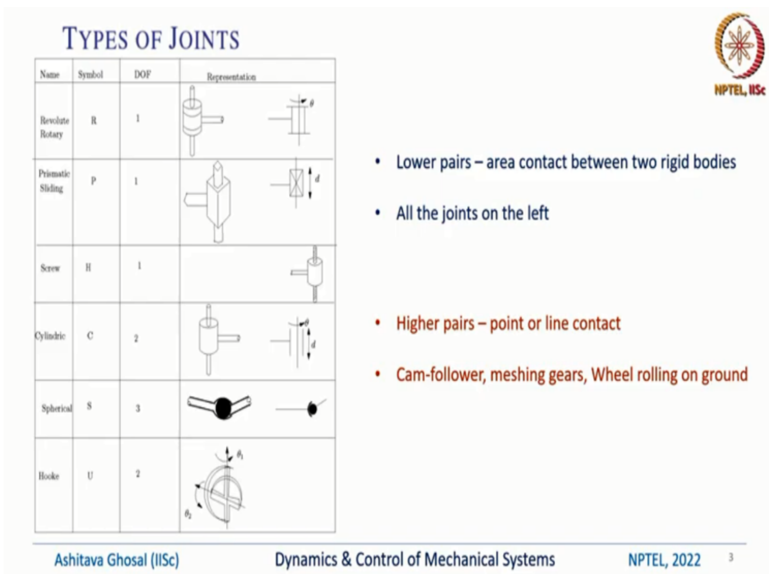

So, let us look at some of the common types of joints. So, one common type is this rotary joint the symbol used is R, it has 1 degree of freedom, and a rotary joint is same thing as a hinge joint. So, if you have one rigid body here and another rigid body here and they are connected by a hinge joint, the second rigid body can only rotate with respect to this first rigid body. This is also shown schematically here this is one rigid body, this is another rigid body.

This one can rotate with respect to the other one and this is the typical picture of a rotary joint or the hinge joint sometimes it is also called as a revolute joint. You can the other well-known or commonly used joint is a prismatic joint or a sliding joint, the symbol used is P, it has also 1 degree of freedom meaning what if I have one rigid body which is this one and another rigid body which is this one.

So, the second rigid body can only translate with respect to this first rigid body and intentionally I have shown it as a box here and this rigid body also has a like a square a sided thing. So, that it cannot rotate with respect to this first rigid body it can only slide up and down and this is a symbol which is used for prismatic joint. So, d denotes the degree of freedom in this case theta denotes the degree of freedom which is the rotation of the one rigid body with respect to the previous rigid body.

You can also have a screw joint which is not very commonly used, the symbol is H, it has also 1 degree of freedom. Basically, a screw joint is a combination of these two except that the relationship between d and theta the sliding and the rotation is related by the pitch of the screw. So, if most of you who are mechanical engineers will know that when you rotate the screw the nut moves up or down by the pitch.

So, d is p times theta, so it is still one degree of freedom there are two possible motions but the rotation and the translation are related by a constant. Then we have the cylindric joint, symbol is C, it has two degrees of freedom. So, in this case the d and the theta are not related. So, most of the time these kind of joints are used in closed loop mechanisms and in kinematics of some kind of mechanisms.

It is normally not used very often but it is possible to have what is called as a cylindric joint. Where one rigid body is this one, the other rigid body is this one schematically shown, so it can rotate with respect to the previous one and it can also slide. This is a well-known joint it is called the spherical joint; the symbol is S; it has 3 degrees of freedom. A spherical joint is very is the same thing as a ball and socket joint which is there in your arm.

The arm is connected to your body at the shoulder by a ball and socket joint. A ball and socket joint is shown this is one rigid body, this is another rigid body schematically shown. So, this rigid body can rotate about this direction it can rotate in the other direction out of the plane and it can also rotate about this axis. So, there are three possible rotations between one rigid body and the other rigid body connected by a ball and socket or a spherical joint.

The last sort of well-known joint is what is called as a Hooke joint it is also sometimes called as a universal joint and the symbol is U, it has 2 degrees of freedom. So, in this schematic drawing so, there are these two angles θ_1 and θ_2. So, one rigid body is connected to this ring, here the other rigid body is connected to this ring here and you can see that these two rings can rotate with two angles θ_1 and θ_2, so they have 2 degrees of freedom.

A hook joint is extensively used in many mechanical systems. So, for example in many cars and trucks if the engine is at the front and the wheels and it is a rear wheel drive then you have a transmission which is going from the front to the back and there are hook joints which connect the rear wheels to the front wheel. So, all these joints are lower pair joints, the word lower pair means that the contact between the two rigid bodies is through an area.

So, there is an area contact between the two rigid bodies and all these joints on the left are lower pair joints, all of them are lower paired because they all have area contact. We also have another kinds of joints which are called higher pair joints in which there is a point or a line contact. And a good example of a higher pair is a cam-follower mechanism or if you have a two gear teeth which are meshing or if you have a wheel which is rolling on the ground ideal wheel.

Because due to the load there is a area contact but assuming it is a ideal wheel then you have a line contact. Similarly, if you have two gear teeth which are meshing, they have a line contact and cam-follower could have a line contact or even a point contact.

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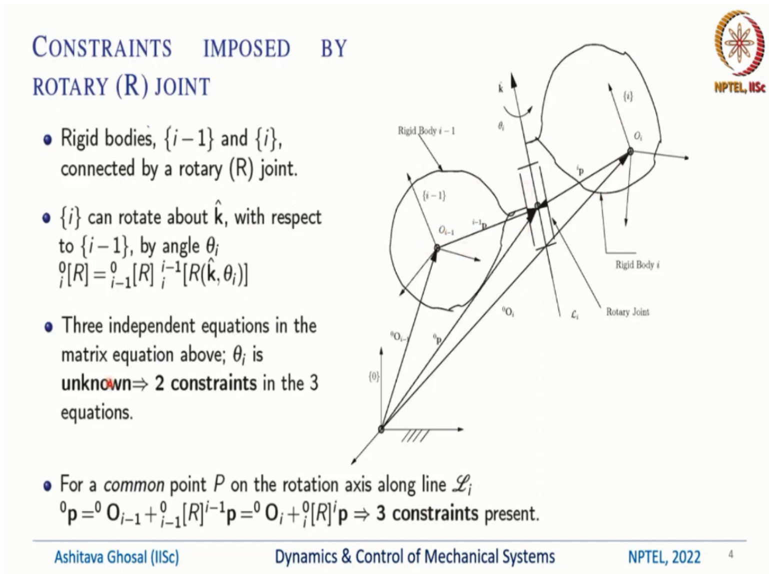

So, let us look at some of this notion of constraints imposed by joints. So, I will take an example of a rotary joint. So, I have two rigid bodies i - 1 and i, again coming from the last week or even the first week that we will denote a rigid body by means of these braces. So, i - 1 is a label for a rigid body which is labelled i - 1 which basically means that there is an X_{i-1}, Y_{i-1}, Z_{i-1} axis and then origin O_{i-1}. So, this is the rigid body i - 1.

Likewise, I have a rigid body i which also contains one x axis, y axis and z axis and an origin. So, we have these two rigid bodies i - 1 and i labelled i - 1 and i and they are connected by a rotary joint. So, this is the schematic view of a rotary joint that shows that there is a rotation possible between this rigid body and this rigid body and this rotation is always about a line and this line is labelled L_i and this axis about which it is rotating is called $\hat{k}$.

So, we have these two rigid bodies in summary connected by a rotary joint. The second rigid body can rotate with respect to the first rigid body about an $\hat{k}$ by an angle θ_i, so I can rotate about $\hat{k}$ with respect to $i - 1$ by angle θ_i. So, let us do a little bit of math we have done this before. So, what is the rotation matrix of i with respect to 0 is denoted this way and similarly i - 1 with respect to 0 is denoted by a rotation matrix.

And what you can see is the rotation matrix for this rigid body the ith rigid body will be related to this i - 1 is rigid body by 2 multiplication of 2 rotation matrices. The first one is $i - 10[R]$ and then between i - 1 and i which is nothing but a rotation of θ_i about the k axis. We had looked at this rotation matrix in terms of k and theta. So, remember it was some $k_x k_y k_z$ are the unit vectors about this k axis and θ is the rotation.

We could obtain the rotation matrix in this k, θ_i form. So, the orientation of this rigid body i is nothing but obtained from the rotation matrix$i - 10[R]$ $ii - 1[R]\left(k, \theta_i\right)$ and this R is in this $\left(k, \theta_i\right)$ form. And that is again because this revolute joint o r this rotary joint allows only one degree of freedom. So, let us just look at it a little bit more carefully. So, this equation which is a matrix equation contains 3 independent equations.

Why? Because a rotation matrix contains only 3 independent parameters. Orientation is described by only three independent parameters. So, if you equate the left-hand side to the right-hand side although there are 9 elements here, there will be 9 elements here but then there are this only 3 of them are independent. So, this equation contains 3 independent equations, however θ_i is an unknown.

So, hence out of those three independent equations there are only 2 constraints 2 are fixed and the third one is in functions related to θ_i. Let us also look at a point on this rotary joint. So, this point here is on the rotary joint and you can think of this point as a part of this rigid body i - 1 rigid body and also as a part of i th rigid body. So, the vector which is locating this point is $0p$ so what you can see is the $0p$ can be written as $0O_{i-1}$ to the origin of i - 1 rigid body.

And then a vector going from the origin to this point which is $i - 1p$ likewise the $0p$ can is also equal to 0 to O_i and then another vector ip . So, this plus this must be equal to this plus this, so this is a vector equation and we need to make sure that all the vectors are in the same coordinate system basically we need to pre multiply by certain rotation matrices. So, $0p$ is

$0O_{i-1}$ then $i - 10[R]\ i - 1p$, so this has to be pre multiplied by a rotation matrix.

Likewise, this has to be pre multiplied by a rotation matrix, so we have this vector equation. So, this is equal to this. Now this is a vector equation containing 3 coordinates, so maybe like x y and z 3 components. So, there are 3 constraints which are present here because in order to make this equal the 2 sides there are 3 constraints, there were 2 constraints here so hence we have 5 constraints.

So, mathematically I have showed you that if you have a rigid body, i connected to another rigid body i - 1 this rotary joint imposes 5 constraints, 2 from the orientation equation, 3 from the position equation.

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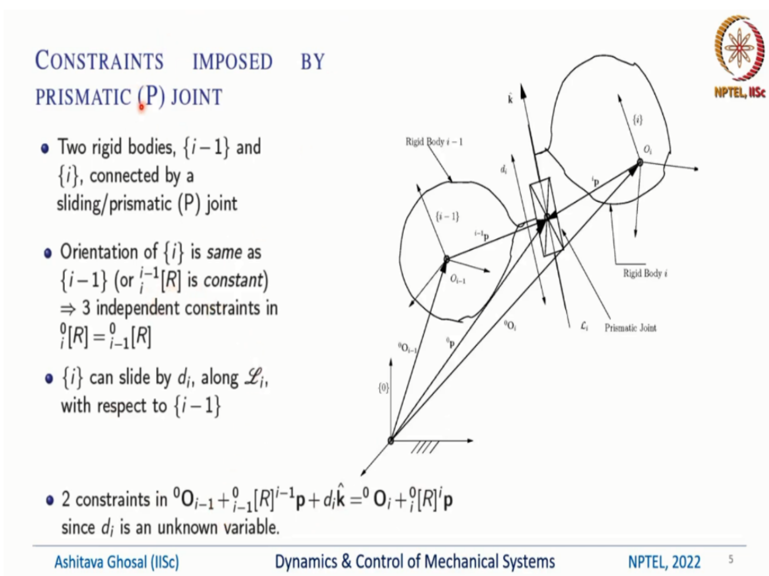

Now let us look at the constraints imposed by a prismatic joint. So, again we have two rigid bodies i - 1 and i which are connected by a sliding or a prismatic joint P joint. Schematically what we have here is a rigid body i - 1 a rigid body i they are connected with the prismatic joint. And as I said before instead of theta rotations by k in the case of the rotary joint, now we have a d vector a sliding vector d_i.

So, this rigid body can slide with or translate with respect to this rigid body along this line L_i or along this direction k. The same very similar to the idea as before except that now we have a d_i vector d_i along k. So, the orientation of this rigid body with this rigid body has to be same. Why? Because this rigid body can only slide or translate with respect to this rigid body along this $\hat{k}$ by a quantity d_i.

The rotations or the orientation of this rigid body does not change when we add it to each other using or we connect these two rigid bodies using a prismatic joint. So, hence what do we have? We have a equation which is $i0[R] = i - 10[R]$, so this contains 3 independent constraints, again rotation matrix has 9 parameters 3 by 3 rotation matrix but only 3 of them are independent which we have seen earlier.

This is the i rigid body can slide by d_i along this line or along this vector k with respect to i-1, I have mentioned this earlier. So, we can again write the vector equation which is pick a point on this prismatic joint, so this $0p$ will be equal to O to this origin of the i - 1 rigid body plus a vector here $i - 1p$. And that should be equal to this vector plus this vector but on the left-hand side we need to do this we need to go from here to here and then we need to core slide d_i k.

So, $O_{i-1} + i - 1p + d_i$ k should be equal to this vector plus this vector and again when whenever we are adding two vectors, we make sure that they are all in the correct coordinate system. So, in this equation it is a vector equation with 3 components but d_i is an unknown d_i is a variable. So, hence there are only 2 constraints so we have 2 constraints from here and there are 3 constraints from here, so the prismatic joint imposes 5 constraints.

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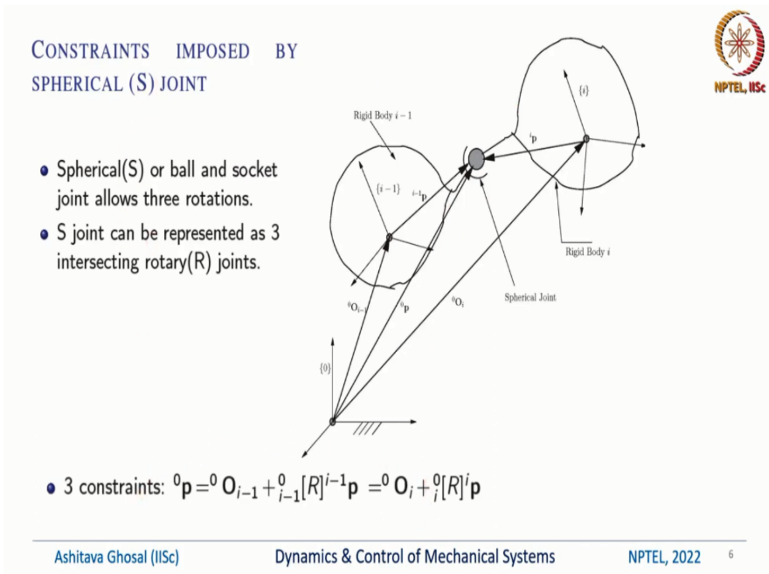

Then let us look at this spherical joint, so a spherical joint or a ball and socket joint allows three rotations. So, a spherical joint or an S joint can also be represented by three intersecting rotary joints, so remember it allows rotations in one direction perpendicular direction and about the third axis. So, let us look at this picture of again a rigid body i - 1 and another rigid body i and they are connected by this spherical joint.

So, what do we have here? We have that this rigid body can rotate with respect to this rigid body how it from about 3 axes. So, this can rotate about this line it can rotate in the plane or it can rotate out of the plane, so hence there are no constraints in terms of the rotations. This rotation matrix of this can be independent of the rotation matrix of this so there are no constraints. However, if you pick a point p on this spherical joint you we can see that this vector which is O to the origin of the rigid body i – 1.

And from the origin to the spherical joint must be equal to this vector which is to the origin of the ith coordinate system and from ith coordinate system to the centre of the spherical joint. So, basically, we have $0O_{i-1} + i - 10[R]\ i - 1p$ must be equal to $0O_{i} + i0[R]\ ip$, so this is again a vector equation it has 3 constraints. So, I can say that some quantity of this side the x component must be equal to the x component here so that is a constraint.

Similarly, the y part should be equal to the y component from the right-hand side. So, hence we get three constraints so this spherical joint imposes 3 constraints.

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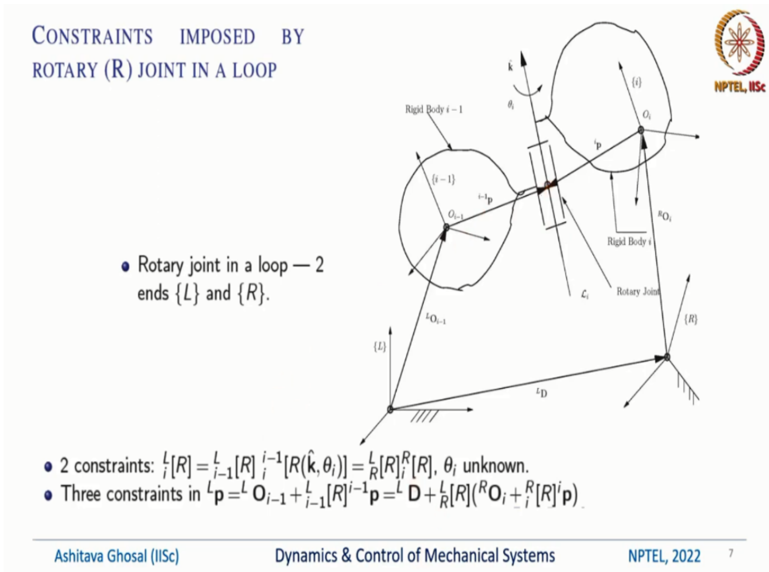

And finally let us look at the rotary joint in a loop. Remember we also will be looking at closed loop mechanisms or parallel mechanisms and as an example we look at a rotary joint in a loop. So, what is the loop here? So, I have a coordinate system I will label it as left, I have another coordinate system which is right both are fixed to the ground, this hash line shows that they are fixed to the ground.

Then we have a rigid body i - 1 and a rigid body i they are connected by a rotary joint. So, and these two coordinate systems which are both fixed to the ground are located with respect to each other by this vector LD , so this is a constant vector. So, what do we have? We have two constraints coming from this equation which is that $iL[R]$, so our rotation matrix will be same as $i - 1L[R]$ then to $ii - 1[R]\left(k, \theta_i\right)$.

So, the rotation of this is related to the rotation of this by multiplying by this rotation about k, so going from this side this $i - 1L[R]\, i - 1L[R\left(k, \theta_i\right)]$. So, $RL[R]\, iR[R]$ let us do it once more so the rotation matrix or the orientation of this rigid body can be obtained from first finding the rotation matrix of i - 1 rigid body post multiplied by the rotation at the rotary joint which is this part $i - 1L[R]\, i - 1L[R\left(k, \theta_i\right)]$.

This must be same as the rotation matrix going from here to here and then from here to here. So, $RL[R]\, iR[R]$, so as you can see many times, I have said this is i - 1 and i - 1 will sort of cancel out. And you will be left with L and i which is the rotation matrix for ith rigid body with respect to the left coordinate system. This is the same as if you cancel out this R and R if you go from this side then you have again the rotation matrix of the ith rigid body with respect to the left coordinate system again.

So, this is a matrix equation, there are three independent equations in this matrix equation, each of the matrices are 3 by 3 but again there are only three independent parameters and θ_i is an unknown whatever is happening at the rotary joint is a variable. So, hence there are two constraints in this then similar to what we had shown for the rotary joint I will get three constraints by considering the position vector.

So, this vector plus this vector should be equal to this vector plus this vector plus this vector and that is what exactly is shown here. So, the vector from L to p is this plus this and that should be equal to L to R then R to O_i and then O_i to this ip and again we have to pre multiply properly with rotation matrices so that all the vectors are in the same coordinate system. So, this is a vector equation so there are three components.

So, let us say the x component of the left-hand side will be equal to the x component on the right-hand side and so on and hence we will get three constraint equations. So, a rotary joint in a loop also has five constraints except why are we doing this because for a four-bar mechanism or some other mechanism in a loop and we want to analyse what is happening to this rigid body when it is connected in a loop to another rigid body with a rotary joint in between we can actually write down all the constraint equations.

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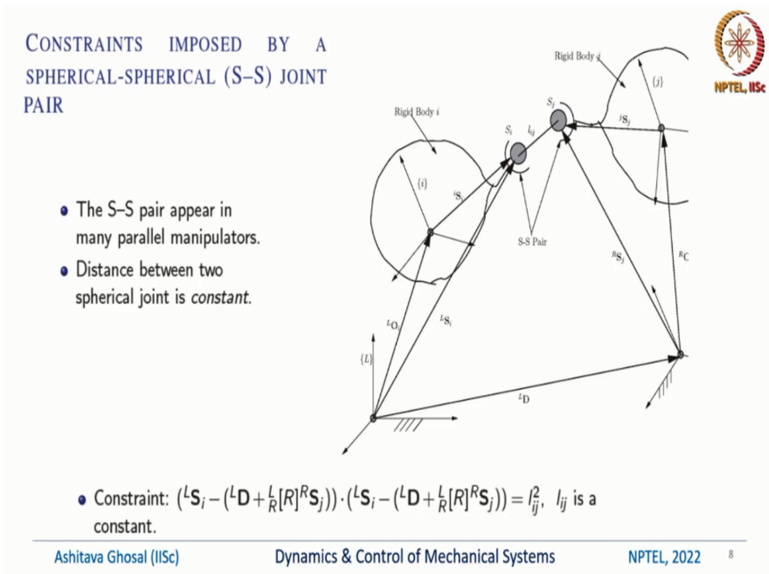

Finally, the last constraint set equation which is very useful and it will be used in many closed loop mechanisms and parallel robots this is thing called the spherical sphere spherical S S joint pair. So, I have a rigid body i, I have a rigid body j in this case, you know we could have written i - 1 and I but let us consider i and j. So, I have one spherical joint which is connected to this rigid body j and then there is another spherical joint which is connected to this rigid body i and in between there is a link.

So, that this S S pair appears in many parallel robots and what is the constraint between these two spherical pairs S_i and S_j, that this link L_{ij} is constant this will not change. So, how do I find what are the constraints? So, I can find this vector from left coordinate system to this centre of the spherical joint and I can also find the centre of this second spherical joint by going in this way to the right coordinate system and then this plus this.

And hence what we have is this vector LS_i which is this vector minus LD and this vector this will be equal to the magnitude of this subtraction should be equal to L_{ij}. So, that is what is shown here LS_i - $LD +$ RS_j the magnitude of this which is nothing but the dot product of the vector with itself will be equal to L_{ij}^2. So, what is this S S pair posing? It is imposing a single constraint which is that the distance between the two S joints is a constant.

And we will use this notion of the distance between two S joints in a loop or in a parallel robot is a constant quite often, it is extensively used.

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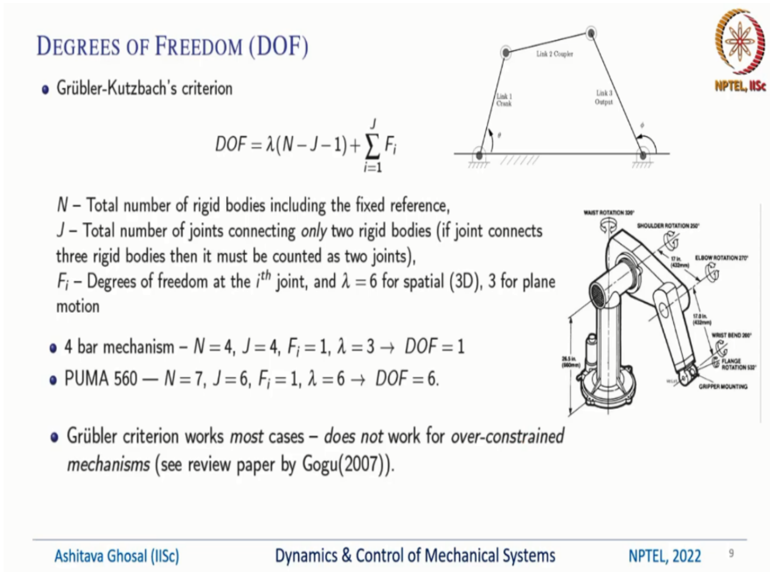

So, let us continue once we know that this rotary joint or a prismatic joint or the spherical joint either on its own or in a loop, they possess some constraints. We can look at the degree of freedom of a set of rigid bodies connected by various kinds of joints. So, this is the very well-known famous Grubler Kutzbach's criteria. What it tells you is that we can calculate the degree of freedom of a multi-body system any mechanical system connected by joints rigid mechanical systems as by this formula.

So, DOF is the degree of freedom, lambda is a parameter which is either 6 or 3, so if the mechanism moves in 3D space lambda has to be taken as 6. If the mechanism is moving in a plane the lambda has to be taken as 3 and then we have N - J - 1, N is the total number of rigid bodies including the fixed reference frame or the fixed rigid body, J is the total number of joints connecting only two rigid bodies this is important.

So, if a joint connects three rigid bodies, then it must be counted as two joints. And then we have this F_i which is the degree of freedom of the ith joint and so if there are J joints you sum these degrees of freedom from, i = 1 through J. So, the quantity which you can obtain is called the degree of freedom of the mechanical system. So, let us take an example we have a four bar mechanism, four bar mechanism by its name inherent name means N is 4.

Remember there was one joint and another link, another link and then there was a fixed link, so N is 4 the number of joints is 4. Each of those joints is a rotary joint it has one degree of freedom lambda was 3 and hence the degree of freedom is 1, so you can substitute lambda is 3, 4 - 1. So, this is - 3 plus there are 4 joints 1 + 1 + 1 + 1 so it is the total degree of freedom becomes 1. So, this is the picture, so I have link one which is the crank this is one link.

This is another link this is the third link and this as I said the fixed reference is also counted. So, N is 4, J is 4 you can see the joints are marked like this and these are rotary joints and hence we get degree of freedom is 1. In the 3D example of a robot which is this puma 560 robot which is well known there are 7 links, so the base is one, this is second, this is third, this is fourth and there are two more here. So, there are 7 links, N is 7 including the base there are 6 joints.

So, one joint is here, this is the second joint, this is the third joint and there are three joints here. So, the and each of these joints has rotary joints so each of them have degree of freedom as 1, so lambda is 6. So, let us substitute 6 into 7 - 6 - 1, so this is 0 and then sum of each joint degrees of freedom so it is 1 + 1 + 1, 6 so that is 6 degrees of freedom. So, this is a very powerful formula and very famous formula and it works for many cases but it does not work always unfortunately.

So, it does not work on what are called as over constrained mechanisms and there is a huge body of work research work which goes into why this formula does not work for a certain particular mechanism. Basically, this formula is just a counting formula you do not in any way look at the geometry of this mechanism. So, N is just a number J is just a number F_i are some numbers. There is no way which tells you that this link has some length or this is look rotated by some angle there are some special link lengths and angles.

So, whenever you have special link lengths and angles it fails this degree of freedom formula will fail and these are called over constraint mechanism and there are these very well-known pieces of work which deals with over constraint mechanisms. We will not go into over constraint mechanisms in this course.

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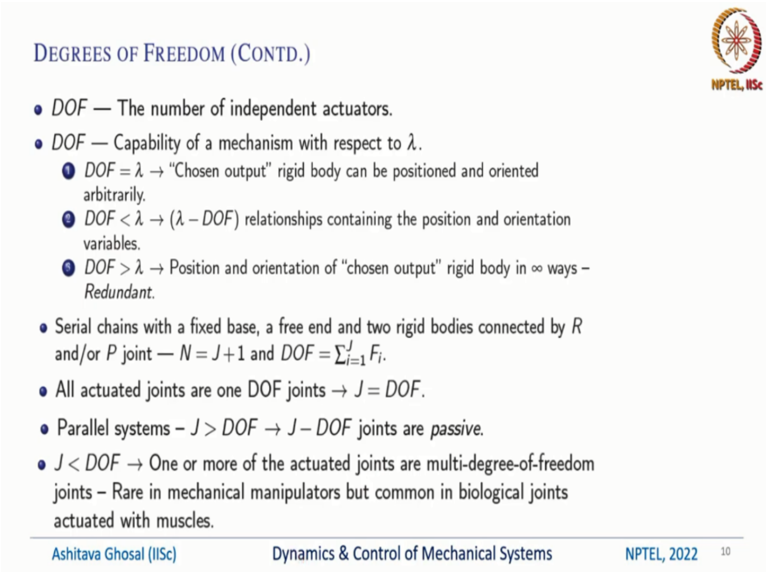

So, what does this degree of freedom mean? Basically, the degree of freedom means the number of independent actuators which I can put in the mechanism. So, in the case of a robot if it has six degrees of freedom what it means is I can have six motors at those joints and I can move the free end, a robot is a serial chain, I can move the free end in with six degrees of freedom. So, what does it mean by six degrees of freedom?

I can move the free end to any x y z and also attain any orientation. So, that is what the degree of freedom means that I can have 6 independent actuators the degree of freedom also shows you the capability with respect to lambda. So, lambda in some sense stands for the ambient space in which this mechanism is operating. So, if lambda is 3 then it is a planar mechanism if lambda is 6 then the mechanism is operating in 3D space, so if degree of freedom is same as lambda.

So, for example the puma the degree of freedom was 6 lambda over 6. What it means is the chosen output rigid body in which case it is the free end can be positioned and oriented arbitrarily I can achieve any x y z and any orientation if degree of freedom is less than lambda then lambda - DOF relationships containing the position and orientation variables are present. So for example if I have a robot which has 5 joints so lambda is 6 but the degree of freedom is 5.

So, then what will happen is I cannot pull position and orient the last free and arbitrarily in 3D space so one of them is constrained. So, basically lambda which is 6 minus degree of freedom 5, so there is one relationship between the position and orientation variables because I cannot achieve all arbitrary position and orientation. If lambda is less than degree of freedom or degree of freedom is greater than lambda.

So, then we can position and orient the free and in infinitely many ways. So, this is a very special class of mechanism these are called redundant mechanism, so our arm is a redundant mechanism. Many biological systems are redundant that robot which I showed you which was trying to trace a circle is redundant because it is moving in a plane lambda is 3 but there were 8 joints. So, I could have moved these eight joints in infinitely many ways to trace the circle by the free end.

So, the serial chains with fixed base one free end and only two rigid bodies connected by R and or P joints we have the special relationship, you can prove it that N must be equal to J + 1 and the degree of freedom is nothing but the sum of the degrees of freedom at each joints. If you have all actuated joints as one degree of freedom then J will also be equal to DOF. So, in the case of the puma there were six joints each joint was one degree of freedom.

Hence the total degree of freedom of the puma robot was 6 which is also the same as the number of joints. In parallel systems the number of joints is much more than the degree of freedom, so in the case of the four-bar mechanism you saw that there was only one degree of freedom but there were 4 joints in each of connecting the links to the fixed base and to each other. So, if you have four joints but it is one degree of freedom then three of those joints are passive.

You can only actuate one of them because remember the degree of freedom is same as the number of independent actuators. So, in a four-bar mechanism I can only put one motor I can only have one input and this is again well known. If you have J which is less than degree of freedom one or more of the actuated joints at multi degree of freedom joints. So, suppose I have degree of freedom of 6 but the number of joints is J is only three.

So, the only way that is possible is I have a multi degree of freedom joint. Maybe I have a spherical joint, so this is not very easily seen in mechanical systems and robots and so on but it is very common in biological joints which are actuated with muscles. So, your ball and socket joint is three degree of freedom joint at you know at the shoulder. It is not actuated by motors it there are some very complicated arrangements of muscles which can actuate this ball and circuit joint.

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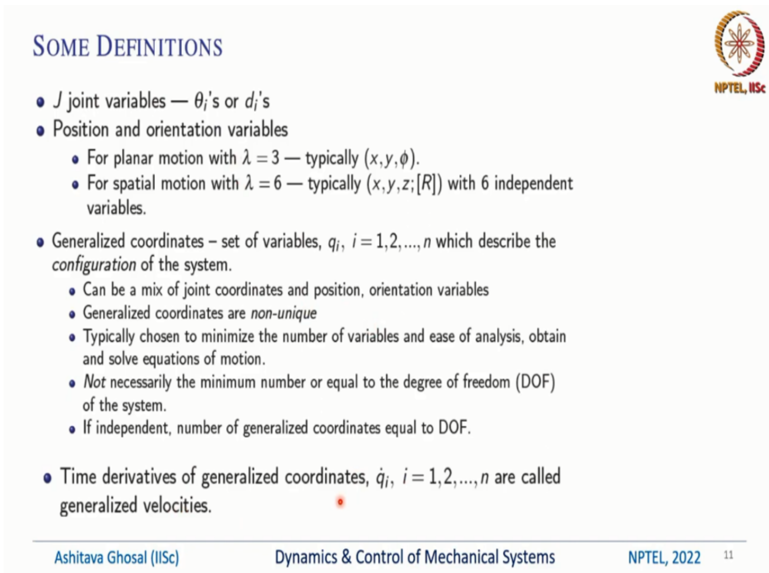

Let us look at a few definitions. So, the J joint variables which are either thetas or d_i's so we can these are look denoted by rotations θ_i's and d_i's which are the translation the position and orientation variables depends on what is lambda. So, if you have a planar motion with lambda = 3 then the position and orientation is determined by three variables, so we will use x, y and phi, just like in any you know mechanics.

So, if the body is moving on a plane, it can have an x coordinate and a y coordinate and it can orient about the z axis which is this angle phi. If for spatial motion with lambda = 6 there are typically six parameters you know rigid body has six degrees of freedom and there are six independent variables, so you can have x, y, z and a rotation matrix. So, this rotation matrix contains only 3 independent parameters.

We can also describe any mechanical system or a multi-body system by something called as generalized coordinates. So, this is a very useful concept let us just go over it little carefully and slowly so what is the generalized coordinate. It is basically a set of variables often denoted with q, so q_i that there could be n of these and this q_i is n of them describe the configuration of the system.

So, what do you mean by configuration of the system it I can given this q_i's I can completely describe the mechanism to you, so for example I can draw this mechanism on a sheet of paper or I can draw it in a computer system if I give you these q's. These q's could be a mix of joint coordinates which is theta and d and x, y and z also. So, these generalized coordinates could be a mix of joint variables and position and orientation variables the generalized coordinates are non-unique.

So, I can describe one mechanism or a multi-body system with one set of generalized coordinates and somebody else can use another set of generalized coordinates and I will show you examples as we go along. So, what is the use of generalized coordinates? We can choose to minimize the number of variables and to for ease of analysis. So, generalized coordinates are chosen such that it is easier to analyse we can easily obtain the equations of motion.

And we can also solve those equations of motion. So, we will come to this notion of equations of motion when we come to dynamics, so we will choose generalized coordinates in a way such that the equations are motion are very simple and I can solve them. The number of generalized coordinates is not necessarily the minimum or equal to the degree of freedom of a system. So, I can have in a forward mechanism one degree of freedom.

But it is often useful to use two or three generalized coordinates to describe this four-bar mechanism. If the generalized coordinates are independent then the number of generalized coordinates equal to degree of freedom. So, in the case of a serial chain it is sensible and useful to choose the generalized coordinates as only the joint variables. So, there are six joint variables in the case of that six degree of freedom puma robot.

And the number of generalized coordinates are also six and both of them are equal to the degree of freedom. The time derivatives of the generalized coordinates are called generalized velocities, so if I have a theta as a generalized coordinates $\dot{\theta}$ will be called as generalized velocities.

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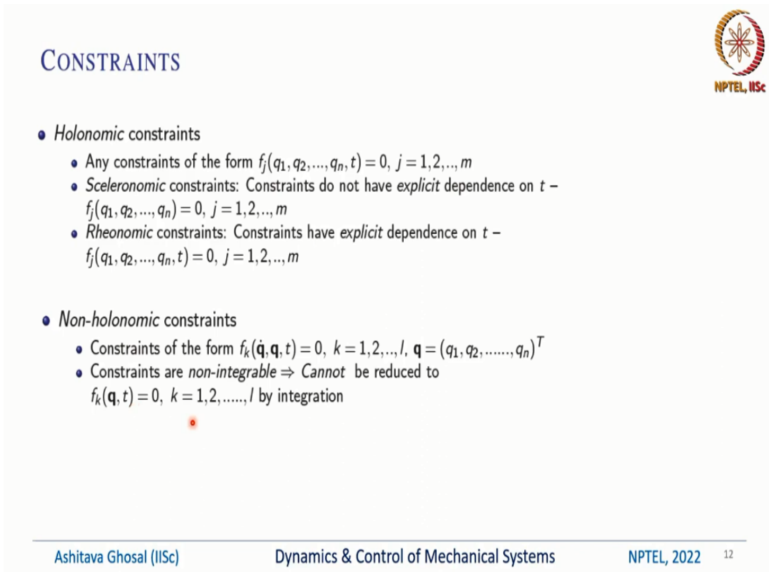

So, once we have selected or chosen a set of generalized coordinates say q_1 through q_n, then there could be some functional relationships which relate these n generalized coordinates. So, there are two kinds and these are called constraints and there are two kinds of constraints one is what is called as an holonomic constraints and these constraints have the form $f\,(q_{1,}\, q_2,\ \ q_n, t) = 0$. So, there could be m of these constraints so you can see $J\ =\ 1,\ 2\ ,..,\ m$.

So, important thing is these are could be non-linear functional relationships. So, $f\,(q_{1,}\, q_2,\ \ q_n, t)$ could be any non-linear function in these holonomic constraints there are again two sub categories one is what is called as a scleronomic constraints. In this the constraints do not have explicit dependence on time, so what you can see here is $f\,(q_{1,}\, q_2, ...,\ \ q_n)$ the time which was there in general form is not present and again that could be m such constraints.

We can also have what are called as rheonomic constraints, these constraints have explicit dependence on time t. So, the functional form of these constraints are $f\,(q_{1,}\, q_2,\ \ q_n, t)$ and again there could be m such constraints. So, important thing in these constraints is it is only a function of the generalized coordinates q's and possibly time. We can also have one more class of constraints in mechanical systems these are called non-holonomic constraints.

The non-holonomic constraints are of the form $f_k(\dot{q}, q, t) = 0$ So, important thing is this $\dot{q}$ so not only the functional form of the constraint contains q may be q_1 through q_n but also the derivatives time derivatives of q and again there could be l of these, so you can see k is 1,2,…,l and there are n generalized coordinates q_1 through q_n and in general this functional relationships could contain both $\dot{q}, q$ as well as time.

The important things about non-holonomic constraints are that these constraints are non-integrable. What do we mean by non-integrable? So, this has $\dot{q}, q, t$, I cannot integrate these equations and reduce it to form $f(q, t) = 0$. So, I cannot remove these $\dot{q}$, if I could integrate these expressions or this functional form then it will become holonomic, so non-holonomic constraints by definition are non-integrable. So, there is no way to reduce it from $f_k(\dot{q}, q, t) = 0$ to $f(q, t) = 0$.

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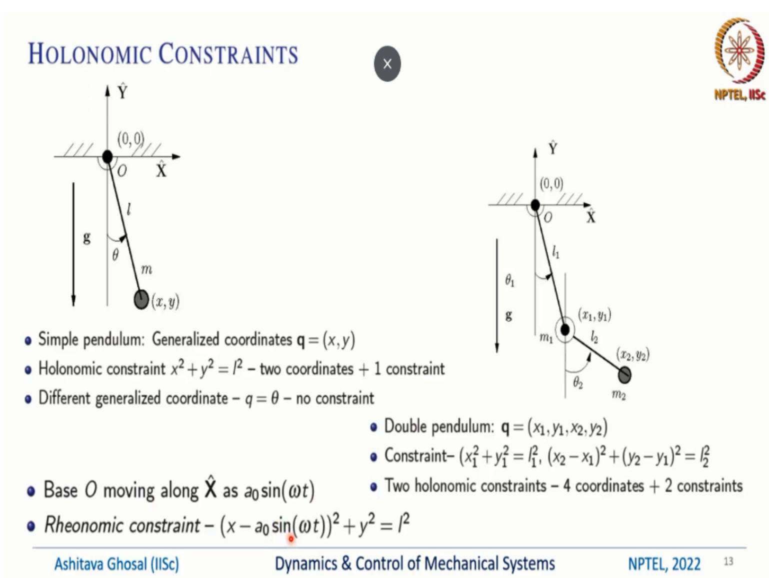

Let us take a look at some common examples of holonomic constraints. So, this picture shows here is a pendulum. So, basically, we have a bob of mass m it is suspended from some point 0, 0 and of length l, so maybe it is a string and we locate these coordinates of this mass by x, y. We can also locate this link or this wire by an angle theta from the vertical and there is also gravity acting. So, in this simple pendulum we could choose the generalized coordinates q as x, y.

So, one possible choice is x, y but however this x, y is related by this constraint which is $x^2 + y^2 = l^2$. So, the distance between this enter of this bob and this origin is l, so $x^2 + y^2 = l^2$ So, there are two generalized coordinates x and y but there is now one constraint. I could have also chosen to describe this simple pendulum with a single variable θ with a single generalized coordinate θ.

So, then we would have no constraints. Let us look at another example a little bit more complex, so this is called as a double pendulum. So, basically, we have one pendulum here with a mass at one point and then it is connected to another mass which is m_2. So, and then we could measure the angle of this link from the vertical with θ_1 we could also measure the angle from this second link as θ_2.

One way to describe the most the configuration of this double pendulum is pick a point here which is $x_1\ y_1$ and then we have another point which is $x_2\ y_2$. The rest of the figure is similar, so there is gravity acting later on we will see why we need this gravity when we derive the equations of motion. So, in the case of a double pendulum one possible choice of generalized coordinates at $x_1,\ y_1,\ x_2, y_2$, so the coordinates of these two masses.

The constraint in this case is that we have $x_1^2 + y_1^2 = l_1^2$, $\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 = l_2^2$. So, this distance is constant and this distance is also constant so what we can have is two holonomic constraints because remember these are functional relationships containing queues only there is no $\dot{q}$. So, there are four coordinates $x_1,\ y_1,\ x_2, y_2$ but there are two constraints.

If you have this base which is moving for this simple pendulum so instead of a fixed pivot suppose it was on some kind of a cart or something in which it is moving and it is moving such that it is $a_o sin(\omega t)$. So, this base is oscillating so in that case your constraint is $\left(x - a_o \sin sin\,(\omega t)\right)^2 + y^2 = l^2$. So, this is now a function of time so as I mentioned the constraint now contains the generalized coordinates which is x and y or q, t = 0. So, it is a functional form which contains t. Hence it is a rheonomic constraint.

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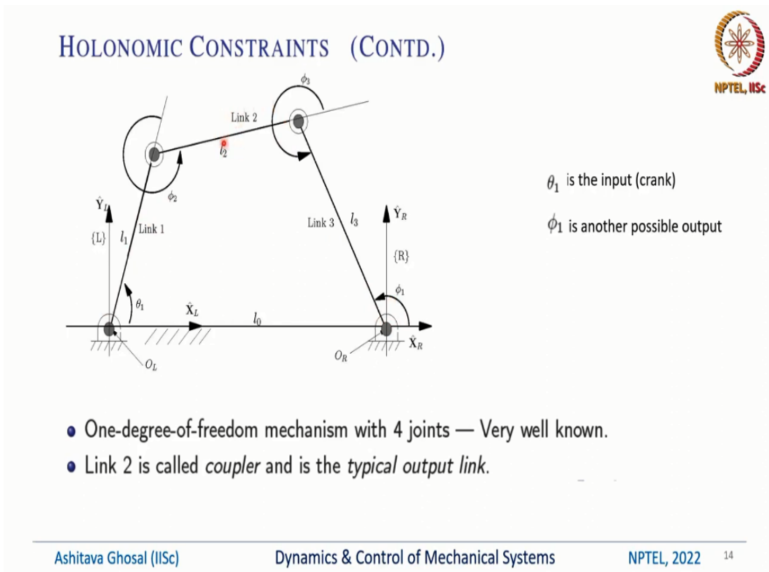

Let us look at another very classical example of a four-bar mechanism. So, this is a parallel mechanism. So, we start from one fixed base and we can go around and we come back so there is one loop. So, for a four-part mechanism there are two well-known ways there are several well-known ways but to show this mechanism. One is that we can look at this θ_1 as one of the rotations of this so, called link one.

This is sometimes called as the input or the crank and this ϕ_1 could be one of the possible outputs. So, when you rotate this link then we want to see how the length 3 is rotating, so this is sometimes called as a crank and this is sometimes called as an output link. So, this is one way of describing this four-bar mechanism, it is a one degree of Freedom mechanism with four joints. Sometimes instead of this output ϕ_1 you could pick a point on this link 2 which is called as the coupler.

And then we could have that as an output link. So, in that case we have x and y as x, y as one of the output variables. So, let us continue.

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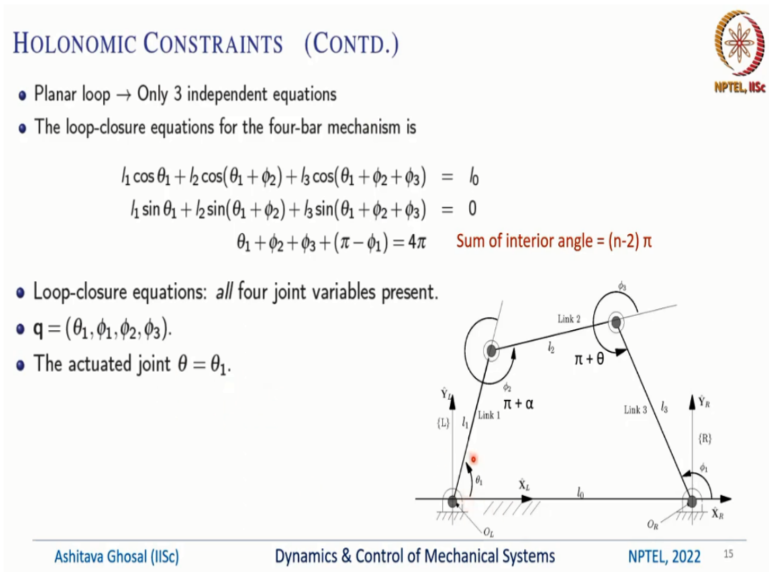

So, I want to find out what are the constraints and how we can describe this four-bar mechanism. So, this is a planar loop, it has one degree of freedom, so there should be only three independent equations. So, let us look at these equations which will tell us what are these constraints so one possible way of looking at these variables is that we go from here to here and from here to here and from here to here and this will be equal to l_0.

So, the X coordinate is $l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2) + l_3 \cos(\theta_1 + \phi_2 + \phi_3) = l_0$. So, remember ϕ_2 is this external angle, so it is actually some $\pi + \alpha$ we could have also looked at this angle then it would be α but then this is a nice way of representing we do not really care whether it is the included angle or the full angle. The Y coordinate is $l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \phi_2) + l_3 \sin(\theta_1 + \phi_2 + \phi_3) = l_0$.

Because we start from here, we go here that is so this one is $l_1 \sin\theta_1$ perpendicular distance then we go from here to here then this is $l_2 \sin\theta_1$ and then we come back to here. So, the Y coordinate is 0 again. The third constraint equation is $\theta_1 + \phi_2 + \phi_3 + (\pi - \phi_1) = 4\pi$. How do we get this? This is a well-known result that the sum of the interior angles of a quadrilateral are for that matter any convex polygon of sides n is n - 2π.

So, in this case there are four sides so it is 4 - 2 which is 2π. But you can see what are the interior angles? This is θ_1. I should have actually counted this α and I should have actually counted this theta and I should have actually counted $\pi - \phi_1$. So, $\pi - \phi_1$. is already here but ϕ_2 is actually $\pi + \alpha$, so there is another π which is coming here and similarly ϕ_3 is $\pi + \theta$.

So, you can see one π is coming from here, one π is coming from here, then the sum of all these angles is 2π. So, this whole thing will become 4 π. So, what am I trying to say here in this four-bar mechanism there are these generalized coordinates θ_1 ϕ_2 ϕ_3 and ϕ_1 and there are it is a one degree of freedom system, so there must be three constraints and these are the three constraints.

So, the X component will add up to l_0 the Y component is 0 and the sum of the interior angles is 2π. So, in this loop closure constraint equations in this form all the four joint variables are present, so the generalized coordinates are θ_1 ϕ_1 ϕ_2 ϕ_3. It is a one degree of freedom, so we could have said that there is only one independent generalized coordinate and most of the time it will be θ_1 which is the actuated joint.

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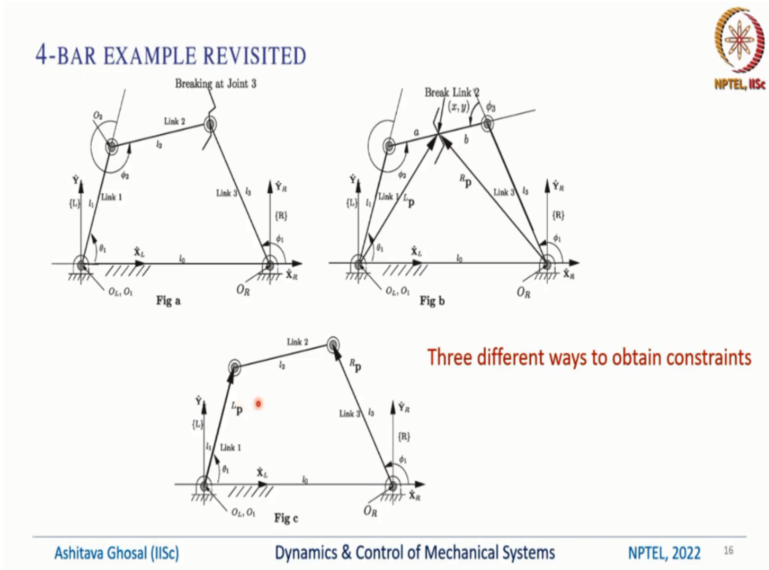

If you look at the four-bar mechanism again, there are various ways of representing these four bars or there could be a various combination of generalized coordinates. So, in the previous case we went around the loop and then we said this is equal to l_0 the Y coordinate to a 0 and so on. I could also find another way of representing this four bar or finding out these constraints by breaking these 4 bars at joint 3.

I can also break it in the coupler at some point which is a and b from both ends. I could also break the four bar at these two places, so all of these different ways of representing the constraints will give rise to different kinds of generalized coordinates. So, in this first one if I break it here, we will see that only θ_1 ϕ_2 and ϕ_1 will show up if I break it in the coupler link then θ_1 ϕ_2 ϕ_1 and $\phi_3$will show up.

And if I break it at both these two places then this will be θ_1 and ϕ_1 will show up. So, what is the basic goal that we want to find the best way to represent the constraints which are inherent in a four bar. So, we have one degree of freedom, so only one single variable is enough to describe the configuration of this, four bar but there are different ways or simple ways of choosing variables in this four bar.

So, if I choose the variables θ_1 ϕ_2 and ϕ_1 so 3 of them there must be two constraints, if I choose the variables θ_1 ϕ_2 ϕ_3 and ϕ_1, 4 variables there must be three constraints and if I choose θ_1 and ϕ_1 as the variables to represent this 4 bar or to show the configuration of the four bar then there must be only one constraint.

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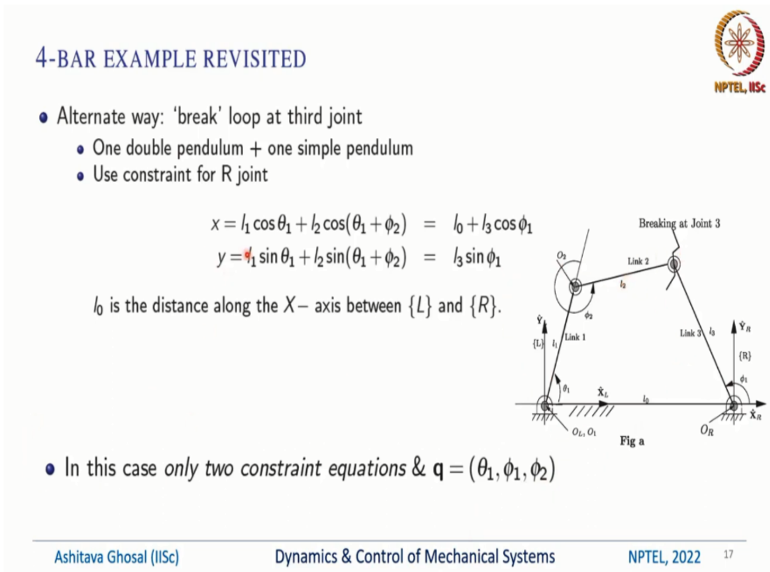

So, let us continue, so I want to see what are the constraints if I were to break this 4 bar at this joint 3 and then I want to show that there are two constraints involving θ_1 ϕ_2 and ϕ_1. What is the way to get those two constraints is explained next. So, if you were to break it at this third joint then basically what you have is a double pendulum. So, this is like a pendulum with two links and this is like a single pendulum.

And what can I do we can impose the constraint that this is an R joint. And what is the constraint for an R joint? Remember we had discussed the constraints that in this case of course it is planar, so we do not have the rotation matrix part. So, what is this? That this vector from origin to this point will be equal to this plus this vector. So, and that is what is mentioned here, so if you see this points x, y.

So, this x, y will be given as $l_1 \cos\theta_1 + l_2 \cos\left(\theta_1 + \phi_2\right) = l_0 + l_3 \cos\left(\phi_1\right)$ and likewise the Y coordinate will be $l_1 \sin\theta_1 + l_2 \sin\left(\theta_1 + \phi_2\right) = l_3 \sin\left(\phi_1\right)$. So this point X, Y where we are breaking a root a point on the rotary joint 3, we impose the constraints that the point can be reached in two ways and it is the same point.

And this l_0 is the distance between the left coordinate system and the right coordinate system. Remember we had discussed how to obtain the constraint for a R joint in a loop and that is exactly what we are using there is nothing much new. So, in this case we have three variables θ_1 ϕ_2 and ϕ_1, so and the two constraints are this that $l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2) = l_0 + l_3 \cos(\phi_1)$, $l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \phi_2) = l_3 \sin(\phi_1)$.

So, if the generalized coordinates are chosen as θ_1 ϕ_1 and ϕ_2 to represent this four bar mechanism, we get these two constraints.

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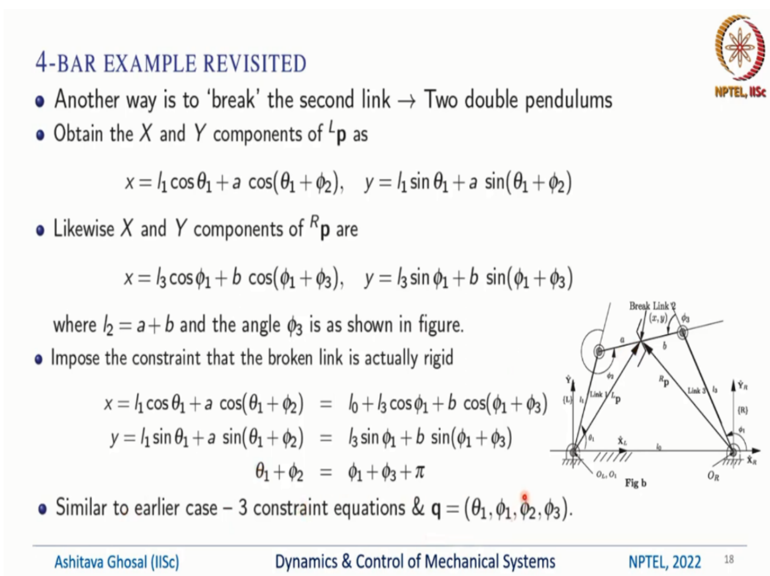

Let us continue. So, another way to break is the four bar is to break it at the second link. So, we break this link here and now what we will do is we will impose the constraint that this vector will be equal to this vector and the orientation of this link or this rigid body going this way and going this way will be same. So, basically if you break it at a distance a and b from a from here and b from here and let us call this point as x, y so what do we have we have two double pendulums.

So, basically very similar to the constraint imposed by the double pendulum so the x component of this point from this coordinate system the left coordinate system is $l_1 \cos\theta_1 + a\cos(\theta_1 + \phi_2)$ the y component or the y coordinate is $l_1 \sin\theta_1 + a\sin(\theta_1 + \phi_2)$. Likewise, the variable x and y can be reached from this right coordinate system that is $l_3 \cos\phi_1 + b\cos\left(\phi_1 + \phi_3\right)$ and y is $l_3 \sin\phi_1 + b\sin\left(\phi_1 + \phi_3\right)$ and this $l_2 = a + b$.

Because remember this link was broken into at this point a and b from two ends. So, now we want to impose the constraint of this link which is what that the vector which is going from here to here must be equal to vector which is going from here to here from this along l_0. Then from here to here and then from here to here which is exactly what is shown here in these two equations.

First two equations which is that l_1 the x component of this vector must be also equal to x component of these some of these three vectors which is this and what you can see here is the left-hand side is $l_1 \cos\cos\theta_1 + a\cos\cos\left(\theta_1 + \phi_2\right)$. And the right-hand side is $l_0 + l_3 \cos\phi_1 + b\cos(\phi_1 + \phi_3)$. The y component is involving signs so is $l_1 \sin\theta_1 + a\sin(\theta_1 + \phi_2) = l_3 \sin\phi_1 + b\sin(\phi_1 + \phi_3)$.

It is basically very similar we are just equating the X and Y components of this point reached in from the left as well as from the right like this. Now we must also ensure that this link the rotation of this link obtained from this way and the rotation of this link are related. How are they related? So, what is the orientation of this link that is $\theta_1 + \phi_2$ you can see that, so if I were to draw one line from the horizontal here.

So, this angle from the horizontal of this link is $\theta_1 + \phi_2$. So, you should ignore the fact that this is there is a big angle here but you know it will adjust automatically, so this is actually some large angle and this is θ_1 but any two link or a double pendulum the orientation of the second link is nothing but the some of the relative rotations. So, this side it is $\theta_1 + \phi_2$ and if you go from this side, it is $\phi_1 + \phi_3$.

But the orientation of this vector is pointing this way and the orientation of the axis is pointing this way. So, they are separated by π, so hence $\theta_1 + \phi_2$ is $\phi_1 + \phi_3 + \pi$. So, please let us go over it once more, what do I have in this way of breaking a four bar on the coupler link at a point x, y which is a from one end and b from the other end, the generalized coordinates are theta1 ϕ_2 ϕ_1 and ϕ_3, so there are four generalized coordinates q's the dimension of q is 4.

So, hence we must have three constraints because this four bar mechanism is clearly has only one degree of freedom. So, let us say if θ_1 is the root motor so that is the only degree of freedom. So, we must have three constraints relating these four generalized coordinates and these are the three constraints, one is we equate the x component going from two sides, then one we equate the y component going from two sides.

And then we also need to make sure that the orientation of this link and orientation of this link are same but after all it is only one link. So, similar to the earlier case there are three constraints out of these four generalized coordinates.

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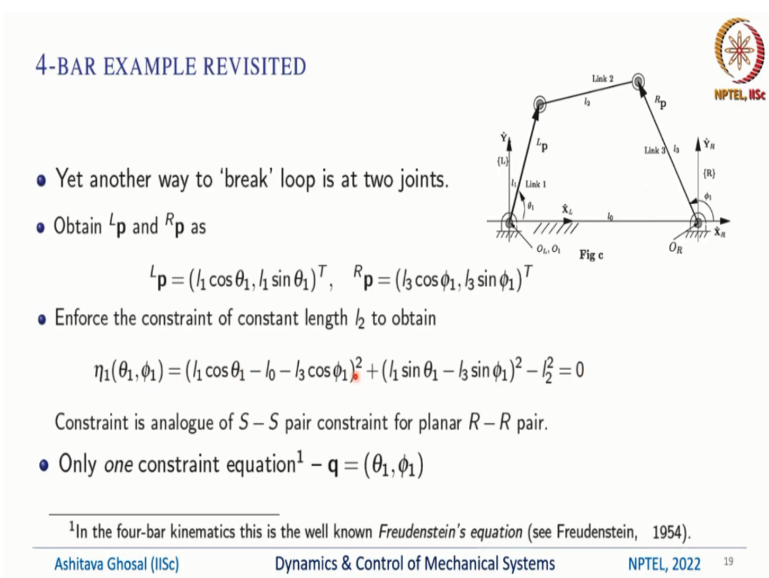

Let us look at the four bar in one last way. So, another interesting way is to break up this four bar at both these places, so I can break it up at this rotary joint as well as this rotary joint. So, what do I have? I have a vector from this 0 which is the left coordinate system to this point this is one vector and similarly I have another vector from the right coordinate system to this point. So, what is this vector? This is $l_1 \, cos \, \theta_1$ and this is $l_1 \, sin \, \theta_1$ the y component.

What about this vector? It is $l_3 \cos cos$, $l_3 \sin sin \, \phi_1$. So, from here it is nothing but x component and then this is the y component, same thing here. Now what we can do is we can import the constraint that the distance between these two points is constant which is nothing but the length of this link 2 which is l_2. So, what is the distance from here to here which is $\left(x_1 - x_2\right)^2 + \left(y_1 - y\right)^2 = l_2^2$.

Just simple Euclidian distance and that is sort of distance square. So, that is exactly what is written here, that we have $l_1 \, cos \, \theta_1$ and then in order to get to this point we have to write l_0 + $l_3 \, cos \, \phi_1$, so $\left(l_1 \, cos \, \theta_1 - l_0 - l_3 \, cos \, \phi_1\right)^2 + \left(l_1 \, sin \, \theta_1 - l_3 \, sin \, \phi_1\right)^2 - l_2^2 = 0.$

So, this is very similar to the S S pair constraint which I showed earlier, S joint in a plane is more or less similar to a R joint. So, the distance between two S joints was the constraint which is constant here the distance between two R joints is a constraint that the distance is constant which is l_2^2 in this case. So, now we have generalized coordinates θ_1 and ϕ_1, so what are the constraints, this is just single constraint.

So, we could have describe this four bar mechanism using two generalized coordinates θ_1 and ϕ_1 and this is the constraint so this is a very famous equation this is known as the Freudenstein's equation and it was invented or discovered by Freudenstein in 1954. So, this equation can be used to analyse the motion of a four bar that is obvious because if I rotate θ_1 I know ϕ_1 by this equation.

Moreover, I could also design a four-bar mechanism to obtain certain special motions of ϕ_1. So, one last question how I am constantly saying that with these two generalized coordinates I can show you or describe the configuration of these four bars. So, is that true? Yes, because if I know what is θ_1, I can reach this point because this l_1 is known the length of link 1 is known l_0 is known.

So, I can come to this point and then I can compute ϕ_1 from this equation I can solve this equation and compute ϕ_1. So, and then from ϕ_1 I can draw this vector which is l_3, so these two points are known, so hence this complete configuration of the four-bar mechanism is known. So, if I choose these generalized coordinates if I tell you this is the value of θ_1 and this is the corresponding value of ϕ_1.

I could draw this four bar mechanism on a sheet of paper, which is again very important concept. That there is only one degree of freedom there is only one independent variable θ_1 which completely describes the four bar mechanism. But it is easier to either choose two generalized coordinates, sometimes it previously I showed you I could choose three generalized coordinates, I could even choose four generalized coordinates.

If I choose larger number of generalized coordinates to describe this forward mechanism then I need more constraints and these are all holonomic constraints because they are only functions of q they do not contain $\dot{q}$.

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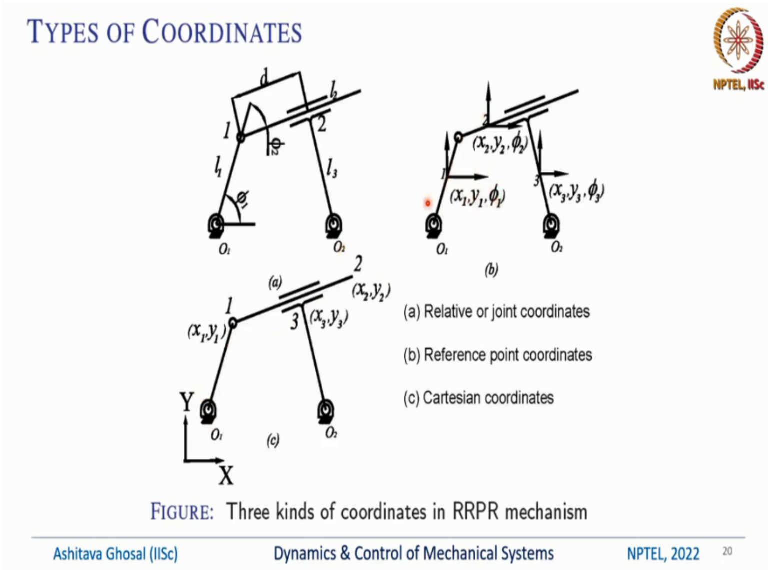

FIGURE: Three kinds of coordinates in RRPR mechanism

So, with this notion of generalized coordinates and constraints, there are several ways to or several kinds of coordinates which are in use in parallel mechanisms and closed loop mechanisms and I want to show you that there are these three main or three different kinds of coordinates which are in use through this example. So, this is a RRPR mechanism. So, what does RRPR means?

That there is a first rotary joint then another rotary joint, the third joint is the sliding joint it is a prismatic joint and then there is a fourth rotary joint. And I would like to describe these RRPR mechanism using different kinds of coordinates. So, the first is what is called as a relative or joint coordinates. So, what is the relative or joint coordinates? So, we see that this is the rotation at this joint which is denoted by ϕ_1.

This is the rotation at the second rotary joint which is denoted by ϕ_2 and we can also have the displacement of this prismatic joint or sliding joint with d. So, these three describe the relative or joint coordinates for this RRPR mechanism. The reference point coordinate is a different kind of representation of this parallel mechanism or closed loop mechanism. In which basically what you do is for each link you specify the position and orientation of the length of some point on the link.

So, for example the centre of this link I could describe by some $x_1\ y_1$ and the orientation of this link which is ϕ_1. Likewise for the second moving link I could say this is $x_2\ y_2$ and ϕ_2, this is the location of some point a chosen point on this link and this is the orientation of this link. And the third link which is moving here in this case again I can pick another point 3 which is $x_3\ y_3\ \ \phi_3$, so this is called as reference point coordinate.

So, I pick a reference point in each of the moving links and for each of these links I say what is the position and orientation, so in the case of plane I have only x y coordinates and a single orientation. In 3D you would go to x y z and some three other angles. The last way of representing the same RRPR mechanism is that you move this reference points to the joint. So, this $x_1\ y_1$ which was here and this orientation ϕ_1.

If I move this point to the joint then all I need is just the position of this point which is $x_1\ y_1$. I can move this coordinate system to this centre of this sliding joint, so we get $x_2\ y_2$ and then we have $x_3\ y_3$. So, these are three kinds of coordinates which can be used possibly for this RRPR closed loop or parallel mechanism.

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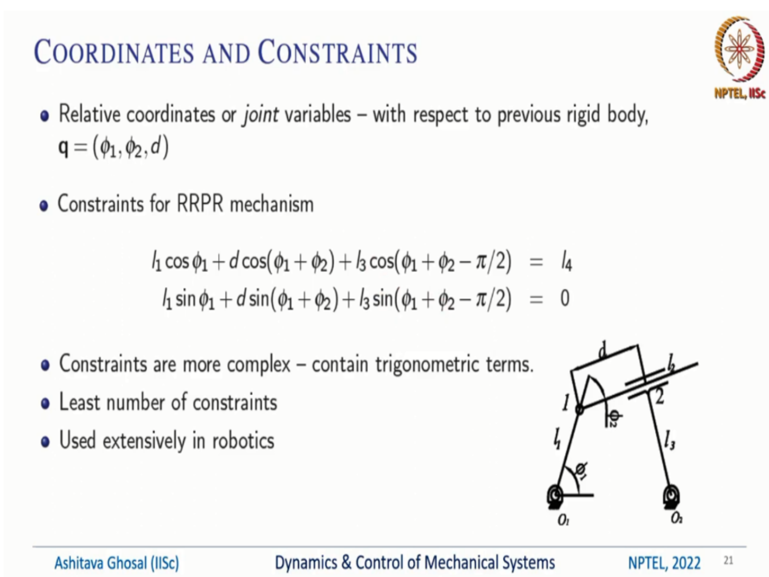

So, let us look at what are the constraints for each of these different ways of representing the same RRPR mechanism. So, in the first one which is relative coordinates or joint variables, so these are the variables with respect to a previous rigid body. So, ϕ_1 is with respect to the fixed coordinate system or fixed reference. So, ϕ_2 was the rotation of this link with respect to this link and d was the translation of this link with respect to this link.

So, what are the generalized coordinates? In this case it is ϕ_1, ϕ_2, d, remember this also has one degree of freedom, we can compute the degrees of freedom using the Grubler Kutzbach's criteria and we will see it as one degree of freedom. So, meaning that there is only one independent variable to describe this mechanism. So, although we have chosen three generalized coordinates there is only one independent variable which implies that there are two constraints.

And what are the constraints for this RRPR mechanism? We can easily find out which is that the X component $l_1 \cos\phi_1 + d\cos(\phi_1 + \phi_2) + l_3 \cos(\phi_1 + \phi_2 - \pi/2) = l_4$. So, think a little bit this is the x component which is going from here to here to here, this must be I am coming down this must be equal to l_4. The y component involves sin so $l_1 \sin\phi_1 + d\sin(\phi_1 + \phi_2) + l_3 \sin(\phi_1 + \phi_2 - \pi/2) = 0$.

So, this $\pi/2$ is coming because of this sliding joint, the angle between this last link and the sliding joint is $\pi/2$. So, if you work it out you will see that we will have some $\pi/2$. So, in this case the constant constraints are more complex they contain trigonometric terms basically you have cos ϕ_1 ϕ_2 and also multiplied by d. So, but the number of constraints is very low, so there are only two constraints for these three generalized coordinates.

So, it is a little bit more complex because not only you have this variable d but this d and this rotation is coupled. So, one way of thinking about this for this mechanism RRPR mechanism is that we have a single degree of freedom which is this ϕ_1. So, this d and ϕ_2 should be related but they are in some ways reasonably complicated way of relating d and ϕ_2 using trigonometric terms.

This kind of relative coordinates or joint variables is used extensively in robotics. It also gives the least number of constraints. So, I have three generalized coordinates and I have two constraints and hence it has one degree of freedom. So, hence it is consistent with what we know.

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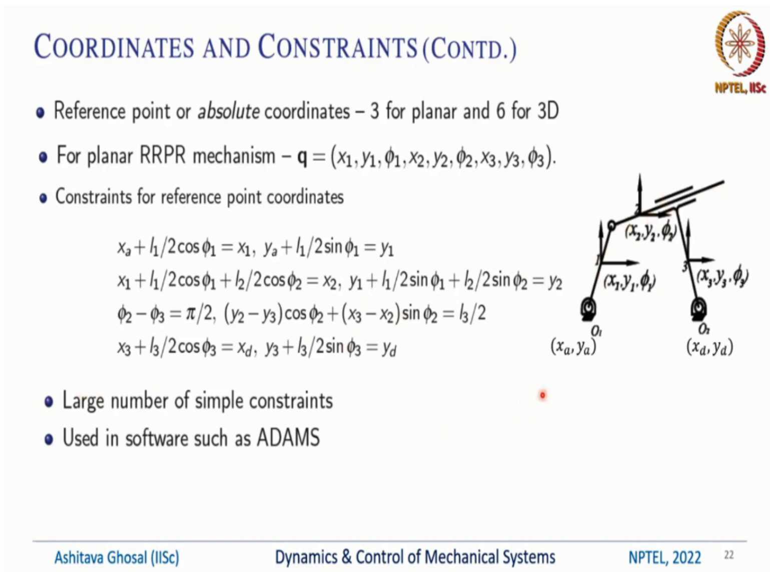

Let us look at also now the other which was the reference point or absolute coordinates. So, for each of these links we now have x_1 y_1 ϕ_1 and here this link is x_2 y_2 ϕ_2 for third link it is x_3 y_3 ϕ_3. So, each link is represented by a chosen point the x, y coordinate of the chosen point and the orientation of that link. So, as you can see here there are three here, three here, three here.

So, there are nine generalized coordinates which represents this for RRPR mechanism again. And so, if you have nine of these but again remember there is only one degree of freedom. So, basically, we need to find eight constraints and all these constraints will be in terms of these generalized coordinates. So, let us look at a few all these eight constraints. So, one is if I start from some origin let us call it x_a y_a this is the left one and then this is the other one which is x_d y_d.

So, what you can see is $x_a + l_1/2 \cos\phi_1 = x_1$. So, if I go from here to here so this is x a plus the x component is l/ 2 this reference point is chosen at the midpoint of this link which is why it is l_1 by two into $cos\,\phi_1$ will be x_1. So, the y coordinate will be $y_a + l_1/2\,sin\,\phi_1 = y_1$., so these are two constraints. Likewise, $x_1 + l_1/2\ cos\,\phi_1 + l_2/2\ cos\,\phi_2 = x_2$.

So, I go from here to here and then you can show that this is x_2. Then we have $y_1 + l_1/2\ sin\,\phi_1 + l_2/2\ sin\,\phi_2 = y_2$. So, this is two more constraints and then we can show that the orientation of this link which is ϕ_1 and ϕ_2 and ϕ_3. These are the angles which determine the orientation of these three links this orientation ϕ_2 - ϕ_3 should be equal to $\pi/2$. What was ϕ_2 which was this angle and what was ϕ_3 it was some angle here.

So, then it will become this is perpendicular we assume that the sliding joint is perpendicular to this link. So, again this $\pi/2$ will show up. Then we have this $\left(y_2 - y_3\right)cos\,\phi_2\ and\ \left(x_3 - x_2\right)sin\,\phi_2 = l_3/2$. Does that make sense? Yes so, the y coordinate is $y\left(y_2 - y_3\right)cos\,\phi_2\ and\ \left(x_3 - x_2\right)sin\,\phi_2 = l_3/2$ and so these are two more and the last two are $x_3 + l_3/2\ cos\,\phi_3 = x_d.\ So, x_3 + l_3/2\ cos\,\phi_3 = x_d\ and\ y_3 + l_3/2\,sin\,\phi_3 = y_d$.

So, basically, we are going around this loop. So, we had all these generalized coordinates. So, there are nine of them and I have written down here eight of these constraints. So, which basically make sure that this mechanism has one degree of freedom. So, as you can see if you were to choose this way of representing this RRPR mechanism with nine generalized coordinates then we have eight of these constraints.

So, these are much more complex, so many more constraints and many more generalized coordinates. The constraints themselves are very simple, there is no product of d and sin and cos phi, so these are little bit simpler. This is representation which is used in ADAMS. So, in ADAMS every link has these x, y and ϕ in 3D it will be x, y, z and the three orientations and then we impose these constraints, distance constraints or orientation constraints and so on.

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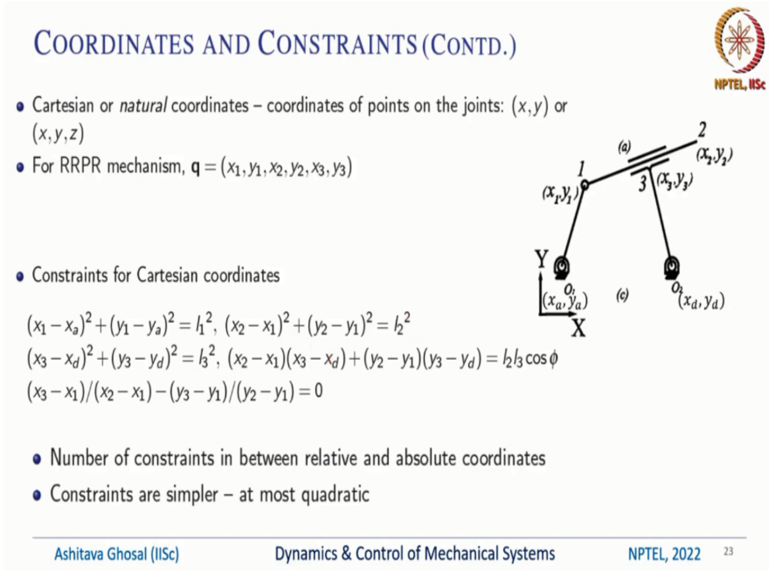

Last, we look at these Cartesian coordinates or sometimes also called natural coordinates. So, basically what we do is we move the reference point to the joint. So, hence what we have here is this point will have x_1, y_1 this point will have x_2, y_2, this point will have x_3, $y_3$and then let us start from some $x_a y_a$ and this is $x_d y_d$. So, what are the generalized coordinates in this case? It is x_1, y_1, x_2, y_2, x_3 , y_3, so there are six of these generalized coordinates.

Again, this is one degree of freedom. So, there must be five constraints and what are these constraints, in this case we can also see that $x_1 - x_a$, so this length should be l_1^2 so $(x_1 - x_a)^2 + (y_1 - y_a)^2 = l_1^2$. Similarly, $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$, then $(x_3 - x_d)^2 + (y_3 - y_d)^2 = l_3^2$ and then these two vectors $(x_2 - x_1)(x_3 - x_d) + (y_2 - y_1)(y_3 - y_d) = l_2 l_3 cos\phi$.

And then these slopes of this point to this point and this point of this point must be equal so $\frac{(x_3 - x_1)}{(x_2 - x_1)} - \frac{(y_3 - y_1)}{(y_2 - y_1)} = 0$. So, we have 1 2 3 4 and this is the fifth constraint. So, again if you were to use cartesian coordinates which are nothing but x and y coordinates of these joints, you will have six generalized coordinates and you left five constraints.

So, in this case the number of constraints is between the relative and absolute coordinates. So, when we use joint level constraints we had the least number of constraints, when we would use the previous case which is used in ADAMS, we had the largest number of constraints. Remember we had eight constraints and nine variables. In this case there are six generalized coordinates and five constraints.

In the first case when the relative or joint variables we had three generalized coordinates and two constraints. So, however in this case the constraints are very simple, so the constraints are at most quadratic. So, we can use later on this fact that it is very simple either linear or quadratic constraints effectively, it does not have sin and cosine, this phi is a constant so there are no trigonometric terms in these constraints.

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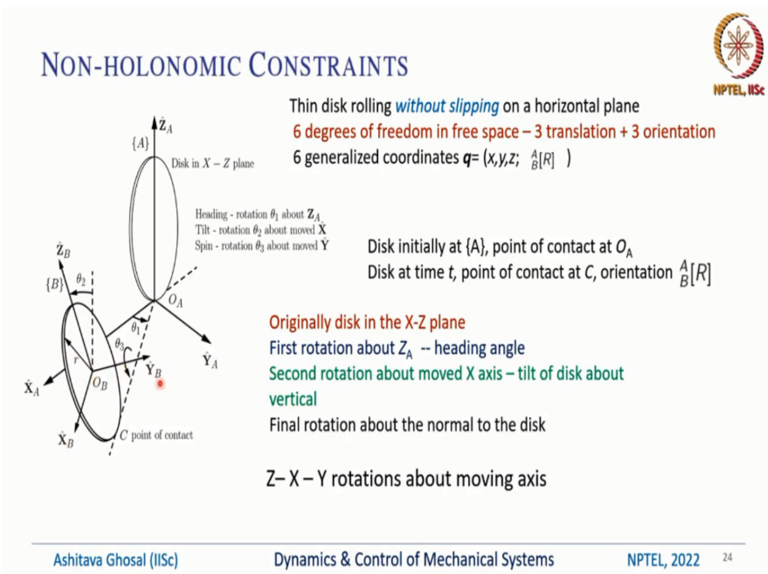

Now let us look at an example of a non-holonomic system. So, this is a very well-known example of a disk which is rolling without slipping on this ground. So, it is a thin disk which rolls on this X, Y plane without slipping and this X, Y plane is the horizontal plane. So, at this in 3D space without any contact with the ground will have six degrees of freedom. So, we will have three translation and three orientation, it is just like any other rigid body.

So, what we have here is some x, y, z and some rotation matrix $BA[R]$ which are the six generalized coordinates for this thin disk. In $BA[R]$ there are three independent parameters, three parameters which represents the orientation of this thin disk. So, the disk is initially in this X, Z plane and it is in contact at this origin O_A and then at some time t this disk has come to this place. So, the contact point is now at C and how do we show this disk at some other time.

So, basically it has rolled along some direction in some path and it has come here. It is also tilting by angle θ_2 and it is moved in a direction which is θ_1. So, this θ_1 is a rotation about Z axis, this is also sometimes called as heading, this tilt is the motion about the X axis so it is tilting from the vertical by θ_2 and the disk is rolling. So, in the sense that it is spinning about a Y axis, so θ_3 is about the Y axis.

So, what we can see is if I want to describe the position and orientation of this disk, I need to worry about the θ_1 θ_2 and θ_3 and also the location of this origin of this disk which is now at O_B. So, how do I find the orientation? In this example as I have shown you it means there is a rotation about the Z axis by θ_1 then there is a rotation about the X axis the new moved X axis by θ_2 and then there is a rotation about the moved Y axis by θ_3.

So, this is the well-known Z X Y Euler angle rotations and it is about moved access all the time, so the final rotation about the normal to the disk is θ_3. So, the first rotation is θ_1 about Z as I said it is called the heading angle. The second rotation is above the moved X axis, this is the tilting of the disk and the final rotation is the normal to this risk about Y B.

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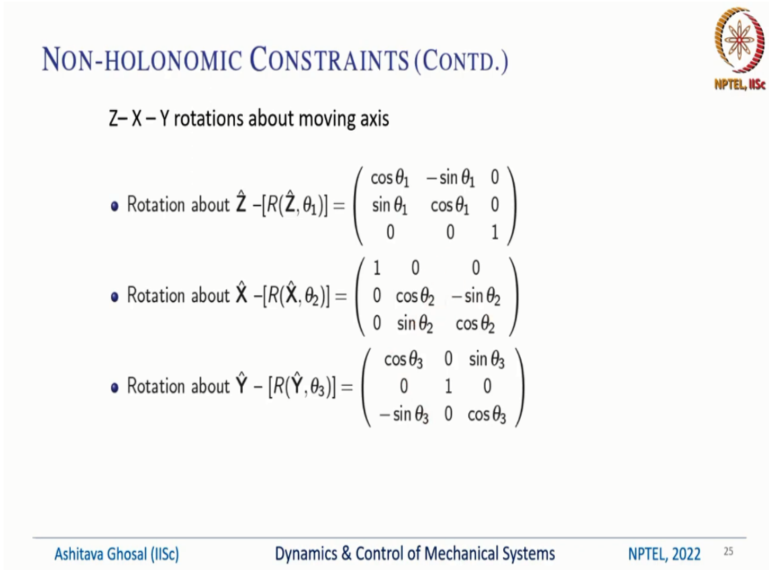

So, I can find out what this rotation matrix is by finding the rotation about Z by θ_1 which is the simple rotation which we have seen earlier. So, this is cos θ_1 - sin θ_1 0, sin θ_1 cos θ_1 0 and 0 0 1. Similarly, the rotation about X is again a simple rotation so first column is 1 0 0 and this is cos θ_2 sin θ_2 0 and so on. So, we have seen this that simple rotations about X Y and Z axis are given by these formulas.

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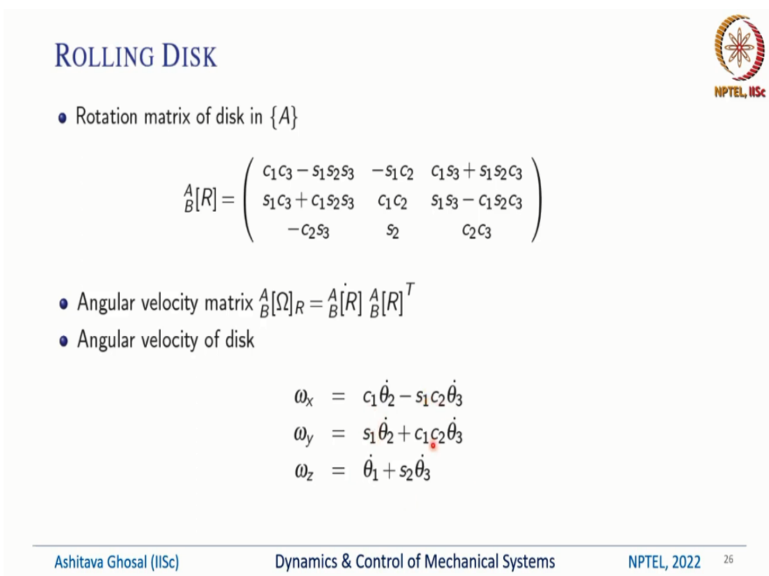

So, in this case we have the rotation matrix is given by product of Z X and Y. So, if you multiply those three rotation matrices we will get a final rotation matrix of the disk. So, B with respect to

A is given in terms of θ_1, θ_2 and θ_3. So, again here cos c_1 means cos θ_1, s_2 means sin θ_2 and so on, s_3 means sin θ_3. So, you will get this rotation matrix which contains cosine and sin of this θ_1 θ_2 and θ_3 angles.

The angular velocity matrix the $[\dot{R}]$ $[R]^T$ which is the space fixed angular velocity matrix can be also obtained. So, we can find out what is $[\dot{R}]$ we take the derivative of each one of these terms, so for example sin θ_2 will be cos θ_2 into $\dot{\theta}_2$ and so on. And for others we have to use chain rule and then you do $[\dot{R}]$ $[R]^T$ the X component of the angular velocity can be extracted from this Q symmetric matrix.

The Y component also and the Z component also and they will look like this so the X component of the angular velocity is given by $c_1\dot{\theta}_2 - s_1 c_2 \dot{\theta}_3$. So, c_1 here means cos θ_1, c_2 means cos θ_2 s_1 means sin θ_1. So, we can obtain this by doing this operation.

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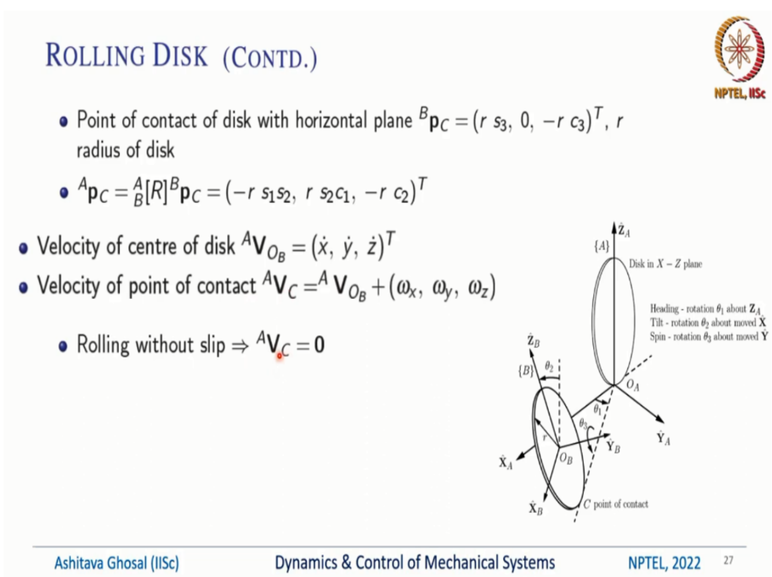

The point of contact of the disk with the horizontal plane can be given in terms of(r s_3, 0, - r c_3) , where r is the radius of the disk. So, you can see that the centre of the disk is the origin and it is rotating about the θ_3 axis. So, hence the Y it is rotating about the y axis so this is still 0 and then we can see the X component is r sin θ_3 and - r cos θ_3 and the position vector of this centre of the disk is given by $BA[R]\ Bp_c$.

So, I am converting from that coordinate system which is attached to the rotating disk to the coordinate system fixed reference coordinate system. And we will get this vector so it is $(-\ r s_1 s_2\ +\ r s_2 c_1\ -\ r c_{2)}$, so this picture again shows the same thing. So, the velocity of the centre of the disk which is given by the position vector to this O_B and the derivative of the position vector, so we can write it as del let us denote it by $\dot{x}\ \dot{y}\ and\ \dot{z}$.

So, what is the velocity of the point of contact? It is this velocity of this origin or the centre of the disk plus $\omega \times\ r$ and we do this cross multiplication in the A coordinate system. So, we have $\omega_x\ \omega_y\ \omega_z$ which we derived earlier cross r Ap_c we have derived this Ap_c right now the point of contact and the velocity of the centre of this disk. So, now comes the constraint.

So, if this disk is rolling without slipping then the velocity of the point of contact will be 0. So, if it is rolling without sleeping then that is this point of contact has 0 velocity and that is what this expression is showing that $AV_c = 0$.

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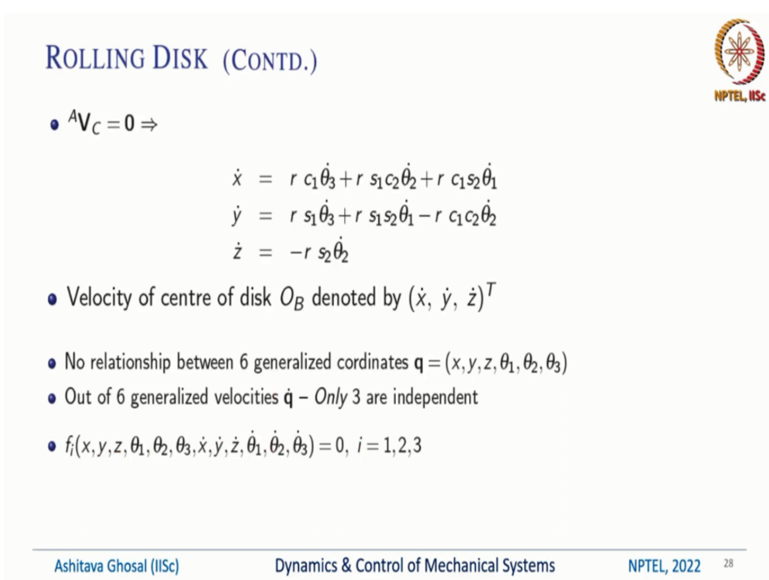

And $AV_c = 0$ can be written in this form which is that $\dot{x} = r c_1 \dot{\theta}_3 + r s_1 c_2 \dot{\theta}_2 + r s_2 c_1 \dot{\theta}_1$ $\dot{x} = r s_1 \dot{\theta}_3 + r s_1 s_2 \dot{\theta}_1 - r c_2 c_1 \dot{\theta}_2$ and $z =- r s_2 \dot{\theta}_2$. So, what is x y z? They are the location of the centre of the disk and he said that the velocity of the point of contact is 0. So, the velocity of the point of contact was nothing but the velocity of the centre plus omega cross r and which is what I am equating to 0.

So, hence we can find what is x dot y dot z dot in terms of θ_1 dot θ_2 dot and θ_3 dot. So, what have we achieved? So, we have six degrees of freedom of this disk x y z of the centre and three rotation angles θ_1 θ_2 θ_3. So, it has six degrees of freedom but then because of no sleep because of imposing the fact that it is rolling without sleeping the $\dot{x}, \dot{y}, \dot{z}$ and $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$ are related by these three equations.

And note that the equations contain derivatives of the generalized coordinates x y z and θ_1 θ_2 θ_3 and also θ_1 θ_2 and θ_3, θ_3 is not showing up here but the q's are θ_1 θ_2 θ_3 x y z and q dots are the derivatives of those and there is the relationship between $\dot{x}, \dot{y}, \dot{z}$ and the thetas and their derivatives. So, as I said the velocity of the centre of the disk is denoted by $\dot{x}, \dot{y}, \dot{z}$ column vector. So, what is the summary?

There is no relationship between the six generalized coordinates as such but out of those six derivatives of the generalized coordinates $\dot{x}, \dot{y}, \dot{z}, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$ there exists three equations. So, there are sixth generalized coordinates x y z θ_1 θ_2 θ_3 and their derivatives $\dot{x}, \dot{y}, \dot{z}$ so these are like $f(\dot{q}, q) = 0$ and there are three of these, these are the three equations.

So, as you can see this is very different from the four bar mechanism constraints. Because in this case we have the constraints involving the derivatives of the generalized coordinates. So, these are we will show that these are called non-holonomic constraints but before that we have to prove that this cannot be integrated. So, if I could integrate these equations and bring it to the form of $f(q) = 0$ then they are not non-holonomic, they will become holonomic constraints.

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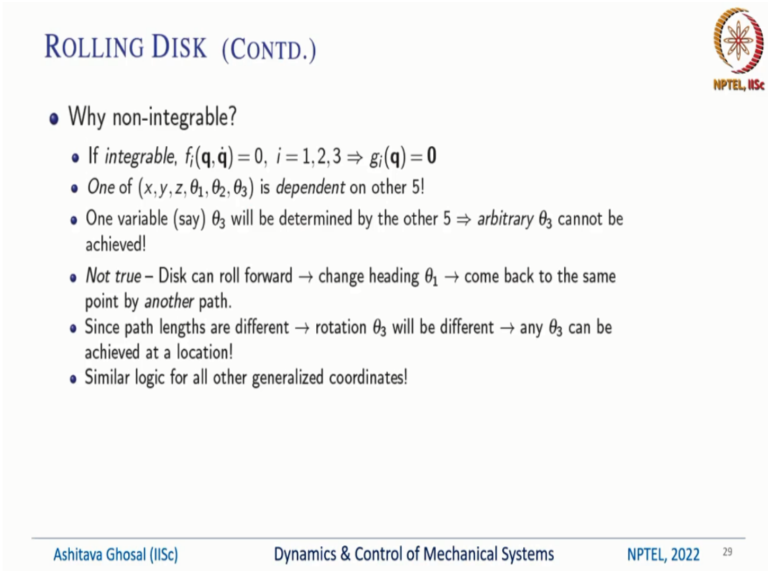

So, why are these non-integrable? So, let us look at it in a different way let us assume they are integrable. So, what do we have? We have $f(q, \dot{q}) = 0$ there are three of these and let us assume they are integrable which basically means that I can get rid of $\dot{q}$ or I can integrate $\dot{q}$ and from $f(q, \dot{q})$ I can get $g_i(q) = 0$. So, what is q here? q is $x\ y\ z\ \theta_1\ \theta_2\ \theta_3$, so if I could integrate and get one expression of $g_i(q) = 0$.

So, basically one of these generalized coordinates is now dependent on the other five, it could be more than one which is dependent on the generalized coordinates. So, let us assume for the moment that θ_3 is now determined by the other 5, so θ_3 is dependent on x y z θ_1 θ_2 so which means what at any point the centre of the disk is at x, y, z and it is headed heading direction is θ_1.

The tilt direction is θ_2, θ_3 is automatically determined because of this integrated constraint equation. So, I am going to show you that that is not possible that I can still get arbitrary θ_3 this if it were integrable then θ_3 will be dependent on the other five and θ_3 will not be arbitrary. So, let us look at this example that I want to show you that θ_3 can be arbitrary, so what is the basic idea that this can roll forward.

It can change heading and it can go to some other place and it can roll backwards by another path. The important thing is it is going forward and it is taking some path and then it is coming back to another path. So, the path length for going forward and coming backwards is different, so if the path lengths are different the rotations θ_3 will be different because it is not sleeping so whatever it goes forward by rolling and if it comes backward also by rolling but with different path lengths the θ_3 will be different.

So, hence I can get any θ_3 at any location so I start from some point I roll forward in go in a path and then I come back by a different path and I will get a different θ_3. And whatever I can choose infinitely many different paths to come back to the same point and I will get whatever θ_3 I want. So, what I am showing you here is that this equation if it were to be integrable and if θ_3 could be a function of this.

Then θ_3 would not be arbitrary and I am giving you a logic and I am trying to justify that I can make θ_3 arbitrary. The same thing I can show you that I can at any position I can get arbitrary θ_1 arbitrary θ_2 and so on. Again, by going on a different path and coming back or at same place I can roll you know I can tilt at different angles. So, clearly θ_2 is arbitrary at any x, y, z.

How about θ_1 being arbitrary? Yes, I can point that disk at some any different angle. So, θ_3 is the most interesting one which is the hardest one to achieve because if it was sleeping then θ_3 could be anything, θ_3 is the last rotation. We are not allowing it to slip but we are allowing it to go in a path and coming back by another path. So, I can show or I can argue that this equation cannot be integrated.

Because if it could be integrated then θ_3 would not be arbitrary but I am giving you enough reason or a logic or an argument that θ_3 can be arbitrary.

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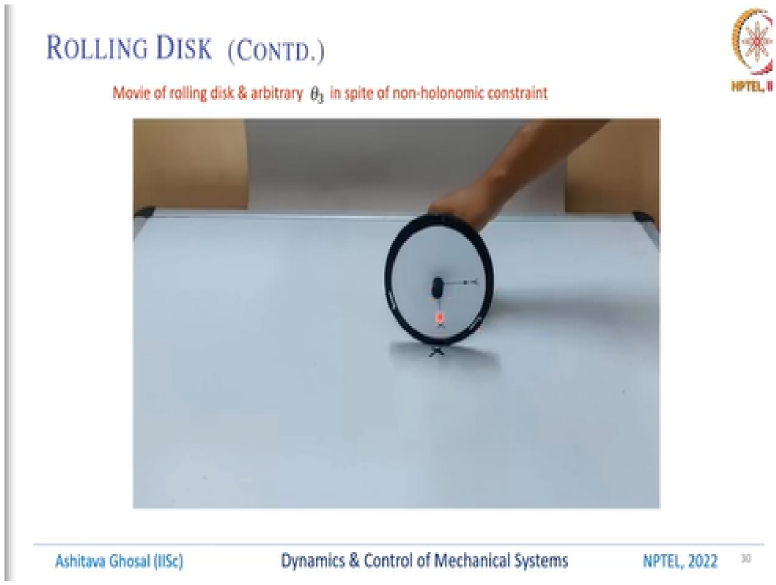

So, here is a video which actually shows you that this rolling of this thin disk I can achieve arbitrary θ_3 in spite of this constraint that it is rolling without slipping. So, in this video what you will see is that this is a thin disk that is an X and a Y coordinate which is marked this is the point of contact this is from where we are starting. So, what this video done by one of the teenagers is that it will roll forward in some direction and it will come back by another forward another path backwards.

So, what you can see is that once it comes back to the same place the X will no longer be pointing downwards and Y will no log out be horizontal. So, which basically means that this θ_3 which is the rotation of this disk about this line which is perpendicular to the disk can be arbitrary.

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So, it started with Y horizontal and X pointing down but now I have taken this on a path and then now I am showing you that X is in some angle Y is in some angle which basically means I have achieved another θ_3 at the same place. And I could have chosen some other path then I could have got some other θ_3. So, what it means is this equation which involves both $\dot{\theta}_3$ and $\dot{\theta}_1$ and $\dot{\theta}_2$ and all the derivatives of the generalized coordinates cannot be integrated.

I hope it is clear that, if it could be integrated then θ_3 would not be arbitrary but I am giving you enough reasoning and logic to show that θ_3 can be arbitrary.

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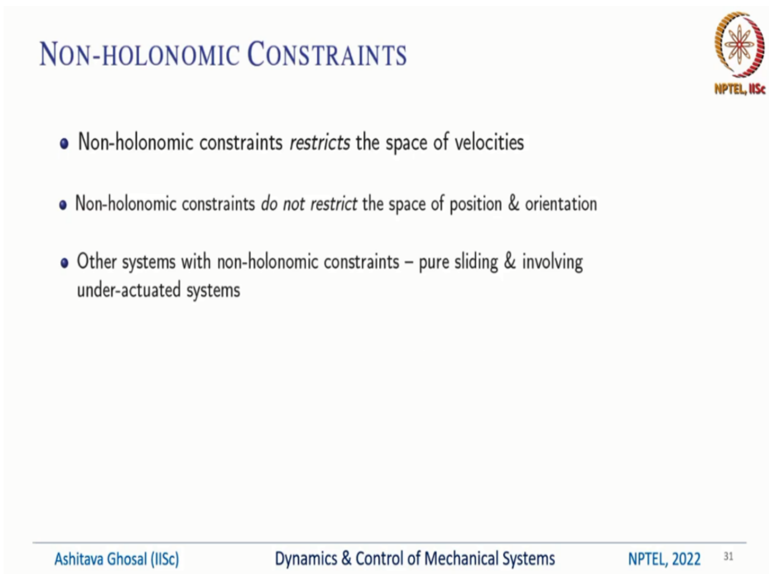

So, non-holonomic constraints restricts the space of velocities, it does not restrict the space of the generalized coordinates it restricts the space of q dots, they do not restrict the space of position and orientation they do not restrict cues but they restrict the space of velocities meaning that only certain velocities are possible. So, this is a very different kind of constraints in mechanical system.

So, I had generalized coordinates and I could have a constraints in as a function of this generalized coordinates. So, $f(q) = 0$ it could be $f(q, t) = 0$ so that those are holonomic constraints but then you have these non-holonomic constraints which are $f(\dot{q}, q, t) = 0$ and those constraints are not integrable meaning that I cannot reduce or I cannot remove those q dots by integration. There are other systems with non-holonomic constraints you can have what is called as pure sliding.

What I showed you is pure rolling and also systems which have under-actuated. So, the number of motors or number of actuators are less than what it can be allowed.

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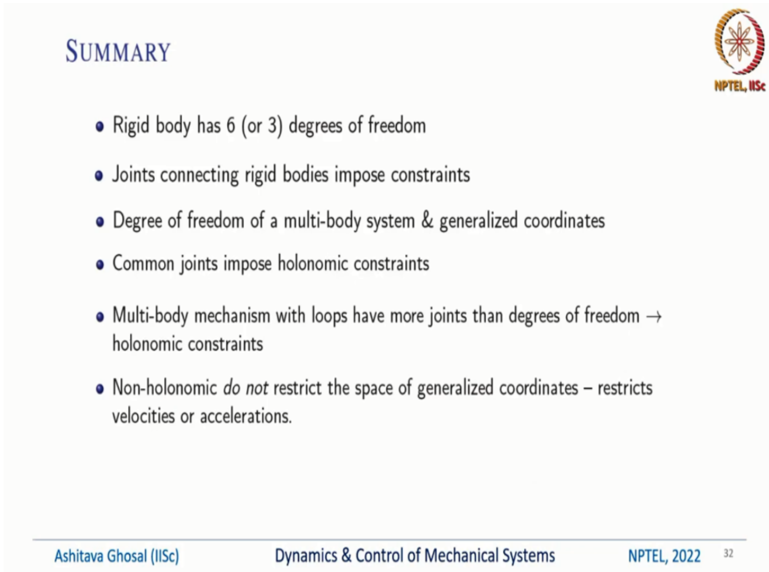

So, in summary a rigid body has six degrees of freedom in 3D space or it has three degrees of freedom in the plane. You have joints connecting rigid bodies, so if you have two rigid bodies you should have 12 degrees of freedom but if you connect these two rigid bodies by a rotary joint you have only 7 degrees of freedom. So, this rotary joint is imposing constraints and how many constraints five constraints.

So, the degree of freedom of a multi-body system can be determined by using some very well-known formula. These are called the Grubler Kutzbach's formulas we can also represent a rigid body or a set of rigid bodies using generalized coordinates. So, as I said the generalized coordinates are some mixture of Cartesian coordinates and joint variables and any other variables which allows us to obtain the configuration of this mechanical system or multi-body system.

So, common joints impose holonomic constraints, so as I said the rotary joint will impose five constraints. So, the sum of two rigid bodies with a rotary joint will have only 7 degrees of freedom, so these constraints imposed by a rotary joint are of the form $f(q) = 0$ where q's are the generalized coordinates. So, multi and an example of holonomic constraints other than those imposed by a single joint are this multi-body mechanisms with loops.

So, I showed you the example of a four bar mechanism, there are four links there is a loop it has only one degree of freedom. So, depending on how I derive the constraints for this forward mechanism I could have one constraint, I could have two constraints or I could have three constraints and this kind of things happen in all multi-body mechanisms with loops, all parallel chains.

We can also have non-holonomic constraints, the non-holonomic constraints do not restrict the space of generalized coordinates they restrict the velocities or accelerations. So, a non-holonomic constraint is given by $f(\dot{q},\ q,\ t)\ =$ and they are not integrable