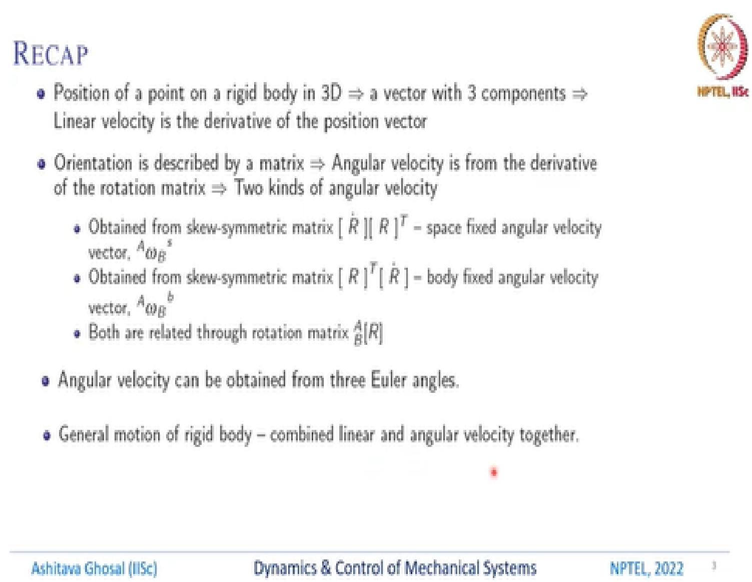
Dynamics and Control of Mechanical Systems
Prof. Ashitava Ghosal
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 05
Motion of Rigid Body and Particles

Welcome to this NPTEL course on dynamics and control of mechanical systems. My name is Ashitava Ghosal, I am a professor in the department of mechanical engineering also in the centre for product design and manufacturing and in the Robert Bosch centre for cyber physical systems Indian institute of science Bangalore. In the last week we had looked at position and orientation of a rigid body and combined motion of a rigid body consisting of translation and rotation.

In this week we will look at the velocity and acceleration of rigid body in 3D space. In this lecture we will look at the motion of rigid body bodies and particles.

(Refer Slide Time: 01:28)



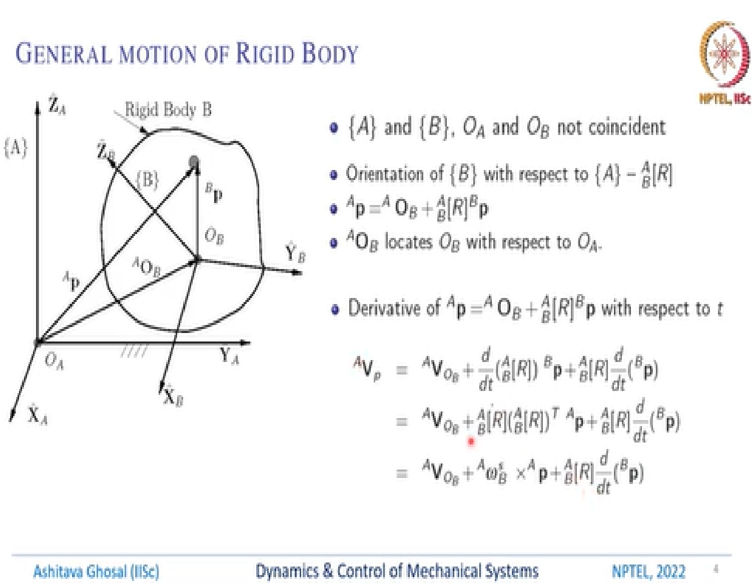
To recapitulate in the last lecture, we looked at the position of a point on a rigid body in 3D space and this is represented by a vector with three components and the linear velocity is the derivative of the position vector. The orientation on the other hand is described by a matrix a 3 by 3 matrix and the angular velocity is derived from the derivative of the rotation matrix and we saw there are two kinds of angular velocities.

One was the body fixed and one was the space fixed. So, in order to obtain the angular velocity, we first obtained the skew-symmetric matrix which is $\dot{[R]}.\ [R]^T$ and this led to the concept of a space fixed angular velocity vector $A\omega_B^s$. We also showed that you could obtain a skew-symmetric matrix from $[R]^T\ \dot{[R]}.$ and this leads to the notion of a body fixed angular velocity vector which is denoted by $A\omega_B^b$.

So, this s and b basically denotes that one of them is a space fixed angular velocity vector and one of them is a body fixed angular velocity vector and I showed you that both are related they are related through the rotation matrix $BA[R]$. The angular velocity of a rigid body can also be obtained from the three Euler angles. So, they are not exactly the straightforward derivatives of let us say $\theta_1, \theta_2, \theta_3$ which are the three Euler angles.

But they are related with cosine and sin of the angles also. In this lecture we will look at the general motion of a rigid body basically we will combine the linear and angular velocity together.

(Refer Slide Time: 03:42)



So, the figure shows a fixed or a reference coordinate system A with $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ and the origin O_A and we have a rigid body in which we have fixed another coordinate system given by $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ and the origin O_B. So, any point on this rigid body this point can be located with reference to the $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ coordinate system by this vector Bp. The location of the origin of this coordinate system attached to the rigid body is given by AO_B.

And we can also find the same point with respect to the A coordinate system. So, basically Ap is nothing but the vector addition of these two but we need to be careful and add it properly. The important thing in this figure is that the origins of O_B and the origin O_A are not coincident, so there is a rotation matrix associated with this rigid body. So, this is the rotation matrix B with respect to A and we have looked at various ways of representing the rotation matrix.

So, for the moment we assume that there is a rotation matrix which is available. So, as I was telling that this vector Apcan be written as AO_B plus this vector but pre multiplied by $BA[R]$ why because we need to make sure that the both the vectors are in the same coordinate system. And as I mentioned earlier AO_B locates the origin of the B coordinate system with respect to the A coordinate system, the origin of a coordinate system O_A.

We now look at the derivative of this vector Ap, so we can take this expression Ap= AO_B + $BA[R]$ Bp and we take the derivative of both sides with respect to time. So, let us go over slowly the derivative of Ap will be denoted by AV_p V means the velocity p means this is this point p. So, first term is the derivative of this AO_B, so this is given by the velocity of the origin of B coordinate system with respect to the A so this is this vector.

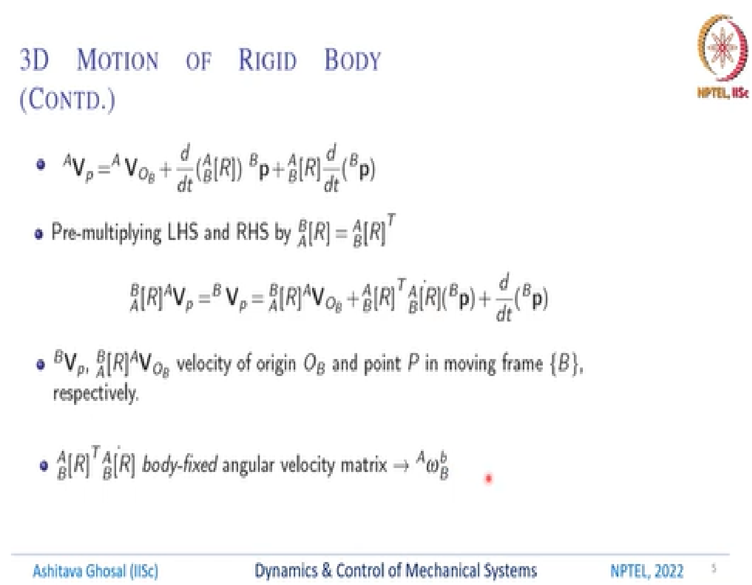
Then we need to use chain rule so the first term is $\frac{d}{dt}$ of this rotation matrix into Bp and then $BA[R]$ into $\frac{d}{dt}$ of Bp . So, let us just simplify so the first term remains same but this derivative can be written in this form so we have $BA[\dot{R}]$ which is denoting the derivative of the rotation matrix. But then instead of writing Bp I can write Ap in pre multiplied by $BA[R]^T$

.

So, you can think of this as B A and then this A will cancel and then we are left with Bp and the last term let us leave it at the moment in this form. So, what you can see is this quantity which is $[\dot{R}]\,.\,[R]^T$ is nothing but the space fixed angular velocity of this rigid body, B with respect to A. So, simplifying what we can show that this velocity vector of this point is nothing but the velocity of this origin plus something like ω× R.

Which you have seen in many undergraduate mechanics courses plus this derivative of this Bp and then pre-multiplied by rotation matrix. So, this is a very useful and important formula which tells you that the velocity of a point on a moving rigid body which is also translating and rotating is given by three terms, one is the velocity of the origin one is something like ω× R and one is the velocity of the point itself in the moving coordinate system B.

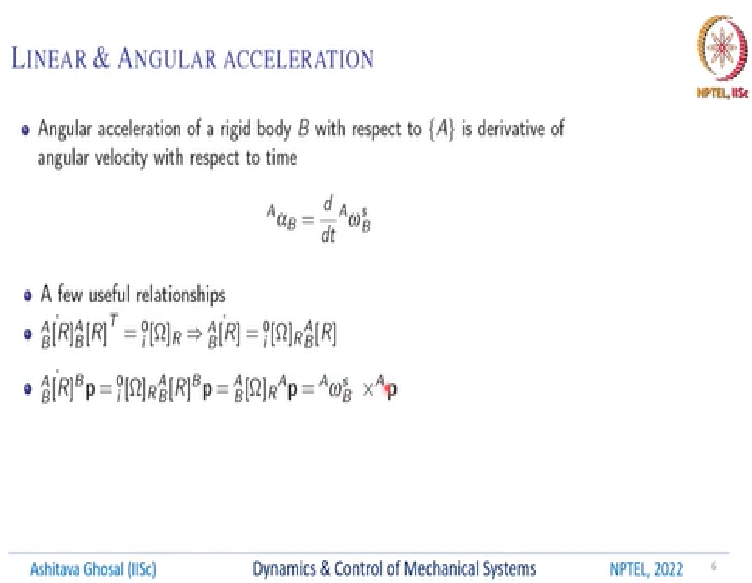
(Refer Slide Time: 08:06)



So, this is just to repeat, the velocity of a point is nothing but the velocity of the origin plus some $\frac{d}{dt}$ $BA[R]$ into Bp plus $BA[R]$ into $\frac{d}{dt}$ (Bp). So, we can also pre-multiply this velocity vector by $AB[R]$, so $AB[R]$ so this again the and it will sort of cancels in your mind. So, we are left with the velocity of this point p in the B coordinate system described in the B coordinate system, so if you go through the process then you have $AB[R]$ into AV_{O_B}.

Then $BA[R]^T$ $BA[\dot{R}]$ into Bp $+ \frac{d}{dt}$ of Bp . So, remember $BA[R]$ and then pre multiply by $AB[R]$ we left with the identity, so this is the simplified expression. So, this is the velocity of this point p described in the B coordinate system again remember this is the velocity of the origin of point p in the moving frame respectively. So, now if you carefully look at this expression what you can see is this is like $[R]^T$ $[\dot{R}]$. $BA[R]^T BA[R]$. This we have seen earlier is the body fixed angular velocity vector of B with respect to A. So, here is that B superscript which tells you that this is the angular velocity vector in the body coordinate system.

(Refer Slide Time: 09:48)



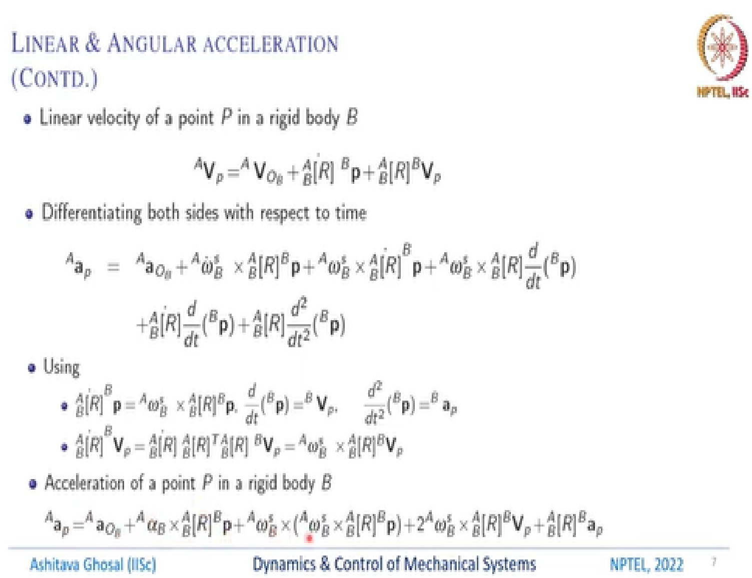
Let us continue, the angular velocity and the linear velocity can be again differentiated to obtain linear and angular acceleration. So, we will start with angular acceleration because that is sort of simple, so the angular acceleration of a rigid body with respect to the A coordinate system is nothing but the derivative of the angular velocity with respect to time. Remember angular velocity is a vector.

And we can take the derivative of each component of the vector and get the angular acceleration. It is simpler than when we went from rotation matrix to angular velocity because now, we are dealing with vectors, a little bit of useful relationships. So, one very useful relationship is that $[\dot{R}]\,.\,[R]^T$ is what we have seen is the right skew multiplication and this is the skew symmetric matrix.

So, $BA[\dot{R}]$ if you can simplify if you post multiply then you can show that this is nothing but this right skew symmetric matrix into $BA[R]$. So, post multiply by $BA[R]$ so you will get $BA[\dot{R}]$ which is what is left this into matrix itself is identity. And this right-hand side is ω into $BA[R]$, likewise another useful relationship is this $\dot{R}$ into Bp $BA[R]$ $\dot{R}$ into Bp is written can be written as ω into $BA[R]$ into Bp.

So, this is also nothing but $BA[\Omega]_R$ into Ap, so this is the space fixed angular velocity vector so this is like $A\omega_B^s$ cross Ap. So, important thing is this $BA[\dot{R}]$ into Bp is same as $A\omega_B^s$ into a cross or product with Ap.

(Refer Slide Time: 12:00)



So, let us continue so we had this linear velocity vector of a point on the rigid body B given in this form which is the velocity vector of the origin plus some $\dot{R}$ Bp plus $BA[R]$ into BV_p, so this is nothing but $\frac{d}{dt}$ of the position vector in the body coordinate system itself, it is $\frac{d}{dt}$ of Bp. So, if you differentiate both sides again with respect to time so the derivative of the velocity is the acceleration of the point.

The derivative of this velocity of the origin is the acceleration of the origin of the B coordinate system. And then this term will come we will get several terms from here so again we need to take the derivative of the first term into this plus this into the derivative of this, so basically, we have to use chain rule here, we also have to use chain rule here. So, from this term and this term so there are four terms here.

So, you can reorganize these fourth terms and then what you can see is you will get one term which is $A\dot{\omega}_B^s$. So, this is like alpha this is like angular acceleration into $BA[R]$ into Bp, so one of the term is this. The second term is that $A\omega_B^s$ cross $B\dot{A}[R]$ into Bp and there is also these two terms, so what you can see is there are 1 2 3 4 5 and one more 6 terms which appear from this.

This we can simplify using these relationships which I told you that $B\dot{A}[R]$ Bp is a ω into $BA[R]$ into Bp we denote the derivative of Bp as BV_p. The second derivative as Ba_p and this $B\dot{A}[R]$ BV_p is $BA[R]$ we can also dot into $BA[R]^T$ $BA[R]$ into BV_p, so you will get $A\omega_B^s$ cross $BA[R]$ BV_p. So, think about little bit it is nothing but simple chain rule and using the derivatives.

And also making sure that we understand what the derivative of a rotation matrix is? So, once more, so the derivative of the left-hand side AV_p will give you the acceleration of the point in the A coordinate system. The derivative of the origin will give you the acceleration of the origin and then from these two terms we have something which is alpha times some distance. So, remember in from undergraduate we have tangential acceleration which is alpha times R.

Then we have something which is $A\omega_B^s$ s into this and one more this and then this is the last term is also very well known, this is nothing but the acceleration of the point in the B coordinate system itself. So, what you can see here is there are two terms which are sort of similar one is this one which is $BA[R]$ into $\frac{d}{dt}$ Bp and then there is one which is $BA[\dot{R}]$ into $\frac{d}{dt}$ Bp.

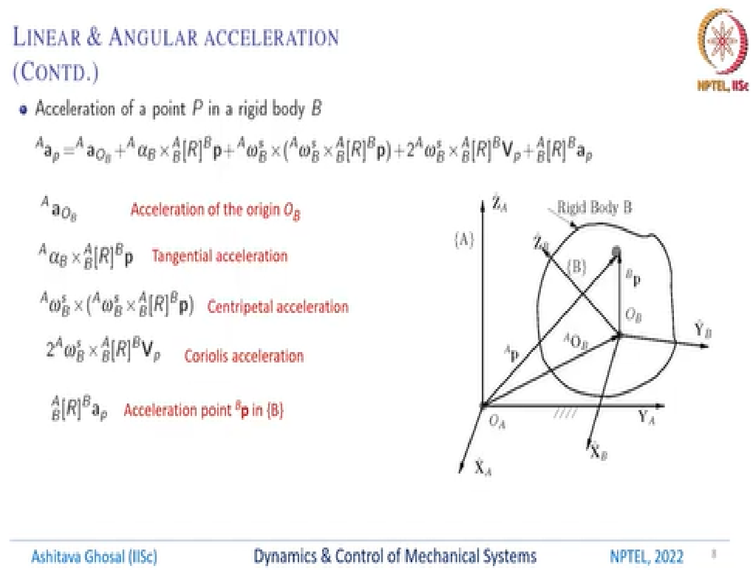
So, these two terms are sort of similar and we will see later that this gives rise to something which is known as the Coriolis acceleration. Whereas this term here it is ω× some derivative of the rotation matrix into some distance, so this is like $\omega\times\omega\times R$

this is the centripetal term. So, as I said we can regroup all these expressions and then we can see that the acceleration of the point p is the acceleration of the origin plus some alpha times R plus some $\omega\times\omega\times R$

this is the centripetal term.

Then there are two of these terms as I said they are sort of similar and you which you will get 2 ω× some relative velocity $BA[R]$ into BV_p. This is the velocity of the point in the B coordinate system and this is the acceleration of the point in the B coordinate system. So, there are 1 2 3 4 5 terms in the acceleration of a point in the a coordinate system when this rigid body is moving translating as well as it is rotating.

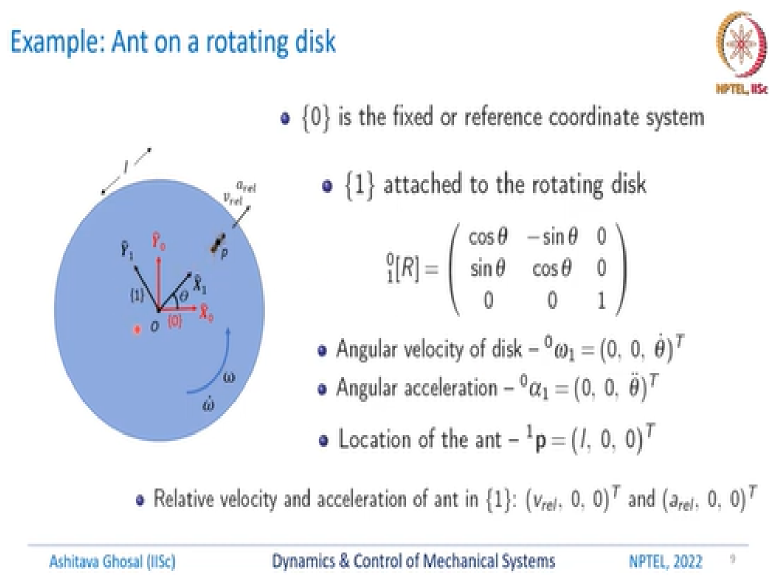
(Refer Slide Time: 17:05)



So, just to recapitulate we have this expression for the acceleration of this point p with respect to the A coordinate system that is given by the acceleration of the origin O_B which is AO_B this term. Then we have this term which is the tangential acceleration this is determined by the rate of change of the angular velocity into some distance. Then we have this term which is $\omega \times \omega \times R$ which is called as the centripetal acceleration.

Then we have one more term which is the Coriolis acceleration which is $\omega \times$ B relative. And then finally acceleration which is this term of the body of the point p in the B coordinate system itself.

(Refer Slide Time: 17:59)



So, let us look at an example and this is a very classic and simple example what we have here is a disk which is rotating with ω *and* $\dot{\omega}$ *or* ω *and* α and there is an ant which is moving on this disk. So, we define a few coordinate systems one is this 0 coordinate system which is $\hat{X}_0$, $\hat{Y}_0$, this is the fixed of the reference coordinate system. Then we have another coordinate system which is 1 which is fixed to the rotating disk.

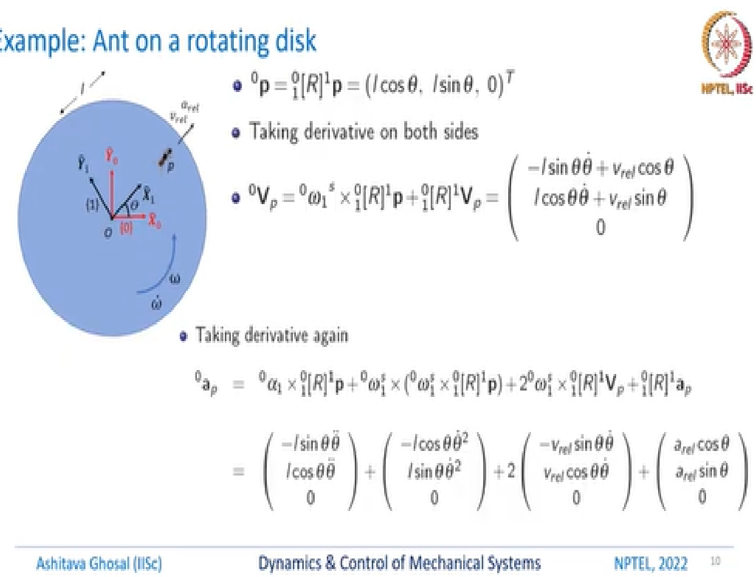
So, you can see it is rotated by some angle θ_1 and $\hat{X}_1$, $\hat{Y}_2$ O but the origins are at the same place. So, we can obtain the rotation matrix of this 1 coordinate system you know this 1 coordinate system with respect to the 0 coordinate system and that is very straightforward because it is a planar rotating disk. The rotation axis is about the Z axis which is coming out so you have $cos\theta$minus $sin\theta$ 0 $sin\theta$ $cos\theta$ 0 0 0 1.

This is the one of the simple rotations about the Z axis the angular velocity of the disk as I said is ω, so it is 0 ω_1 into 0 0 $\dot{\theta}$ so this ω is nothing but $\dot{\theta}$ same as this theta which is shown here. The angular acceleration which is denoted by alpha, so of this rotating coordinate system 1 with respect to the fixed coordinate system 0 is 0 0 $\ddot{\theta}$. The location of this and in the coordinate system 1 is along the X axis.

This is just for simplification it could have been anywhere else. So, what we have is this position vector of this ant in the one coordinate system is l 0 0 it is a column vector with only X component which is l. The relative velocity and acceleration of the ant in this one coordinate system is again assumed for simplicity that it is along the moving X axis. So, the V_{rel} is has only an x component which is V_{rel} 0 0.

And the acceleration of this and in this rotating coordinate system is also along the X axis a_{rel} 0 0. These are just to simplify the algebra and the terms which will appear.

(Refer Slide Time: 20:45)



So, let us continue so the position of this ant in the 0 coordinate system can be obtained by pre-multiplying $1p$ into $10[R]$ and if you we can we saw what is $10[R]$ it was a simple rotation about Z axis. So, it will you will end up with l $cos\theta$ l $sin\theta$ 0 so that is sort of obvious. So, you can see that this is l this distance is l so along the $\hat{X}_0$ axis it is l $cos\theta$ along the $\hat{Y}_0$ axis it is l $sin\theta$.

And there is no Z component because all the motion is happening on the rotating disk which is a planar rotating disk. If you take the derivative of both sides of this one and this one, so the derivative of the position vector is the velocity of this and that can be written as ω× R plus the velocity of the ant itself and we need to make sure that we add these two vectors in the right coordinate system.

So, we have to pre-multiply here with the rotation matrix and here also with the rotation matrix, so remember $1p$ was l 0 0 so 0 1 into $1p$ and then ω across this will give me some term which is minus l $sin\theta$, $\dot{\theta}$ and the y component will be l $cos\theta$, $\dot{\theta}$. And then this term here which is V_{rel} along the x one axis in the origin in the zero coordinate system will be V_{rel} $cos\theta$ and V_{rel} $sin\theta$.

So, this is the velocity of the point looked at from the 0 coordinate system as described in the fixed or reference coordinate system. We can again take the derivative of this which will give us the acceleration so in the last slide I had shown you that there were five terms in the acceleration. So, the same thing is here except one term is not there the origin of the two coordinate systems are at the same place.

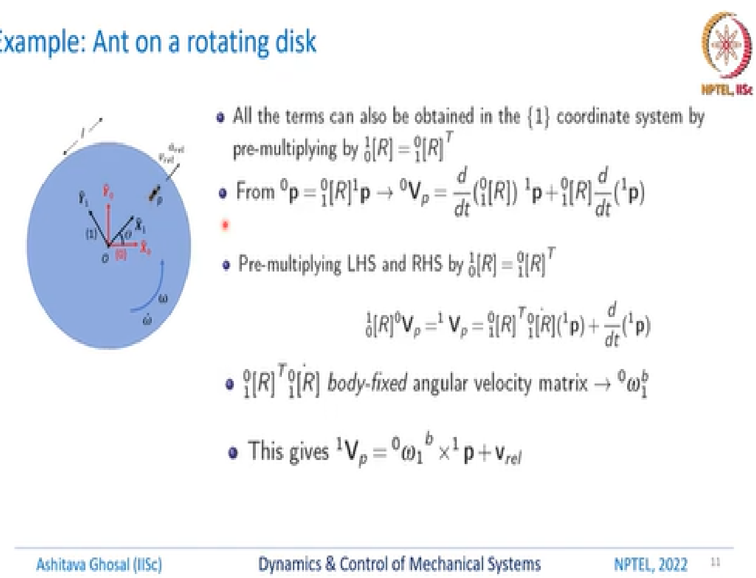
So, the AO_B is not there but the tangential acceleration is there, the centripetal acceleration is there, the Coriolis acceleration is there and the acceleration of the ant itself in the one coordinate system is also there. So, again we see that we can evaluate these terms because alpha is nothing but 0 0 $\ddot{\theta}$ ω is 0 0 $\dot{\theta}$, we know what is $1p$ we know what is the rotation matrix.

We also know what is ω here and we know what is V p, $1V_p$ is nothing but V_{rel} 0 0 and $1a_p$ is nothing but a_{rel} 0 0. And then when you pre-multiply by a rotation matrix $10[R]$ you will get some a_{rel} $cos\theta$ a_{rel} $sin\theta$ 0. Similarly, here this Coriolis term 2 ω× V_{rel} you will get minus V_{rel} $sin\theta$ into $\dot{\theta}$ and then V_{rel} $cos\theta$ into $\dot{\theta}$ and 0. The centripetal term is minus l $cos\theta$ $\dot{\theta}^2$ l $sin\theta$ $\dot{\theta}^2$

And 0 again all the Z component of everything is 0 and the tangential acceleration is minus l $sin\theta$ $\ddot{\theta}$ l $cos\theta$ $\ddot{\theta}$. So, this is a very straightforward example which you may have seen in your undergraduate, I am just trying to show you that we get exactly very similar results but now with this you know different notation and basically, we are trying to use the notion of a rotation matrix.

And how it is body fixed and space fixed angular velocity vector and accelerations and so on. So, this is a different systematic way of deriving or solving the problem of this and which you might have done earlier.

(Refer Slide Time: 25:04)

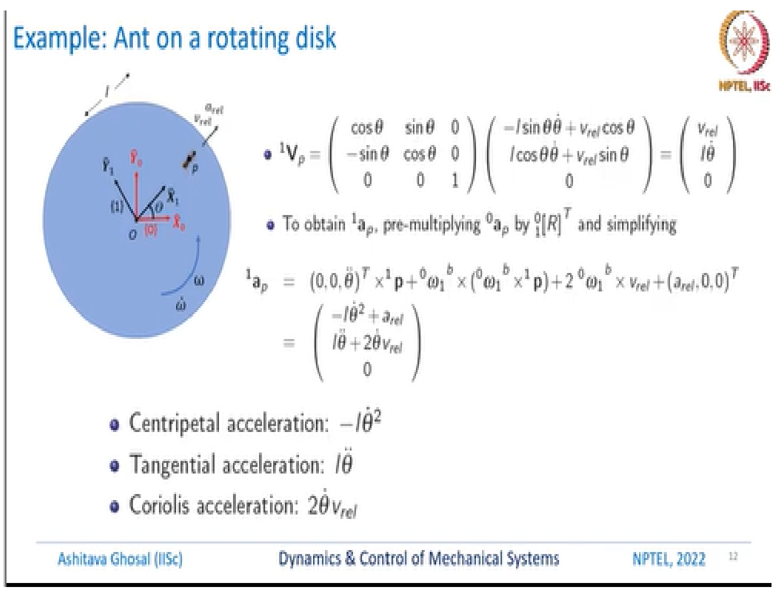


So, this all these acceleration terms we can also obtain in the 1 coordinate system basically in the rotating coordinate system, the coordinate system which is attached to the rotating disk. So, in some sense we need to pre-multiply by a rotation matrix so we need to transform from $10[R]$, so the 2 by pre multiplying by $10[R]^T$ or which is also same as $10[R]$. So, the position vector was given as this $10[R]$ $1p$ the velocity vector was given by derivative of this expression which is $\frac{d}{dt}$ R and then $10[R]$ $\frac{d}{dt}$ $1p$.

So, if you pre multiply both sides of this equation by $10[R]^T$, so you have $10[R]^T$ into 0 V p this is nothing but the velocity of the point in the 1 coordinate system again in your mind you can think of this 0 and 0 will cancel and you are left with 1. So, then if you pre multiply $10[R]$ to the right side then you will get some $[R]^T \, [\dot{R}]$ into $1p$ and this last term will be just $\frac{d}{dt}$ $1p$. So, what is this $[R]^T \, [\dot{R}]$? This is the body fixed angular velocity matrix.

Remember space fixed was $[\dot{R}] \, [R]^T$ whereas body fixed was $[R]^T \, [\dot{R}]$. So, hence the linear velocity of the point p described in the moving coordinate system or the rotating coordinate system is $0\omega_1^b$ $1p + V_{rel}$. So, V_{rel} is what it is just X component so V_{rel} 0 0. And similarly, what is $1p$ it is l 0 0 and $0\omega_1^b$ is 0 0 θ_1 $\dot{\theta}$. So, we can evaluate both the terms on the right-hand side and we can get this velocity of the point in the 1 coordinate system.

(Refer Slide Time: 27:30)



So, continuing with this ant on a rotating disk example, the velocity of this and in the 1 coordinate system is given by this rotation matrix it is nothing but R transpose into the linear velocity component which is - l $\sin\theta$ $\dot{\theta}$ + V_{rel} $\cos\theta$l $\cos\theta$ $\dot{\theta}$ + V_{rel} $\sin\theta$ 0. So, if you simply multiply these two this matrix into this vector and simplify you will get V_{rel} l $\dot{\theta}$ 0.

So, it makes sense because if you are in this rotating coordinate system the velocity of the ant will be clearly along the X axis. And due to the rotation, we have some l $\dot{\theta}$ term. Similar to this velocity vector in the one coordinate system we can also obtain the acceleration of this point in the 1 coordinate system and that again in some sense we need to pre-multiply the acceleration vector in the 0 coordinate system with the $10[R]^T$.

So, if you do the algebra and simplify you will see that the acceleration of this ant in the one coordinate system has four terms 1, 2, 3 and 4. So, we have one which is alpha times R which is $0\ 0\ \ddot{\theta}$ is the angular acceleration $1p$ is the position vector in the 1 coordinate system. Then we also have this Coriolis term which is this one actually and then we have a centripetal term which is $\omega\times\omega\times R$

.

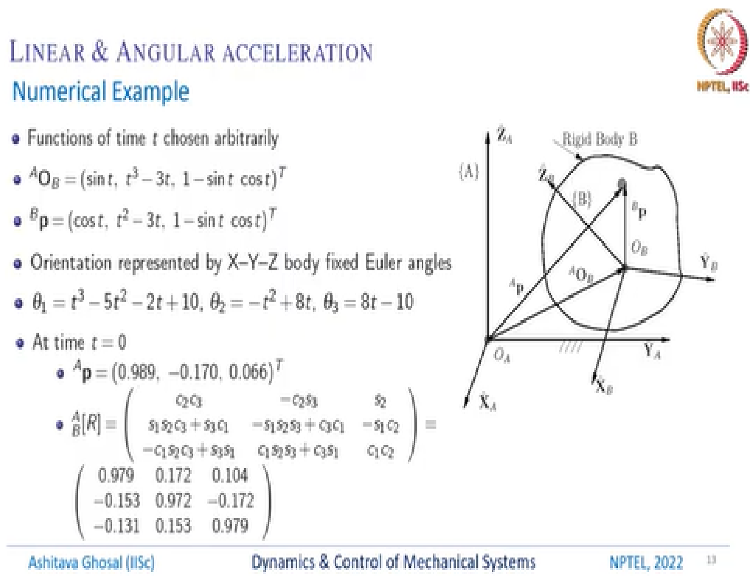
And then we have this acceleration of this ant in the coordinate system one. So, that is nothing but a_{rel} 0 0 we assumed that the ant was traveling along the local or the rotating X-axis. The Coriolis and the centripetal term has this $0\omega_1^b$, so this is nothing but $[R]^T\ [\dot{R}]$. this is derived from the that skew symmetric matrix and we remember it was the body fixed angular velocity vector.

So, if you multiply all this $\omega\times\omega\times R$ and 2 $0\omega_1^b\ V_{rel}$ and then we do all this algebra and simplify we will get a acceleration vector in the 1 coordinate system of this form. So, one is $-l\ \dot{\theta}^2 + a_{rel}$, then you have $l\ \ddot{\theta} + 2\ \dot{\theta}\ V_{rel}$ this is the Y component previous one was the X component and the Z component is 0. So, what you can see is this ant?

There is an acceleration towards the origin and then there is acceleration of the and going away from the origin so that is the X component then we have this $l\ \ddot{\theta}$ which is the tangential acceleration but this is the other term which is the Coriolis term which is acting along the Y direction. So, in summary we have a centripetal acceleration which is $-l\ \dot{\theta}^2$ you have a tangential acceleration which is $l\ \ddot{\theta}$ then we have a Coriolis acceleration which is $2\ \dot{\theta}\ V_{rel}$.

So, I am sure many of you would have seen similar expressions in your undergraduate mechanics but again the whole idea is that we want to do it in a nice systematic formal way and we show that how the space fixed and body fixed angular velocity vectors appear.

(Refer Slide Time: 31:38)



Next, we look at a purely numerical example of obtaining the linear and angular velocities linear and angular accelerations of our rigid body which is rotating as well as translating. So, the picture is here, we have this rigid body which is with a fixed coordinate system B which is attached to the rigid body we have a point on this rigid body and as before we the vector locating the origin of the B coordinate system is given by AO_B.

This point on the rigid body is given by Bp and so on. So, we are going to choose these functions of time arbitrarily, so just there is no particular reason why AO_B as a function of time is chosen as sin t X component Y component is t^3 - 3 t that component is 1 - sin t cos t and so on. It is just to show that given a function of time for all these quantities how the body is translating and rotating.

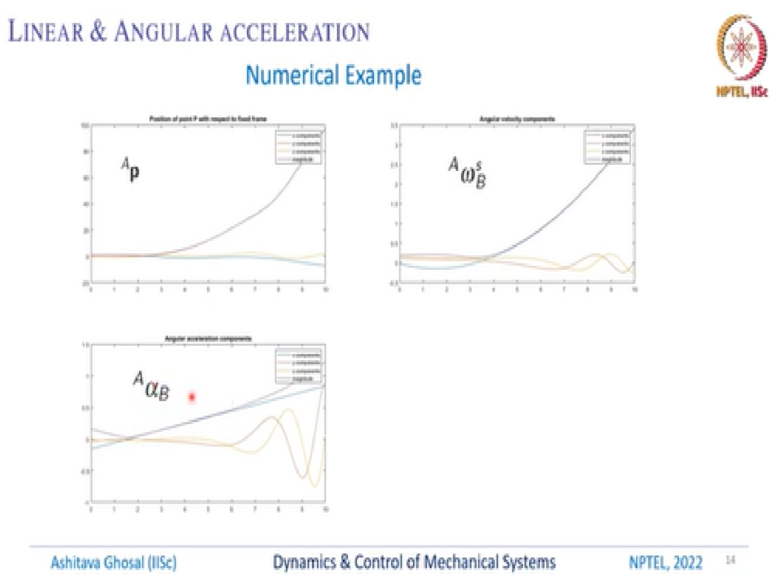
We can find what are the terms of the angular velocity, linear velocity, angular acceleration and linear acceleration. So, the point p in the B coordinate system is also changing as cos t X component t^2 - 3 t is the Y component and that component is 1 - sin t cos t, so as I have said this is chosen arbitrarily. We are also going to represent the orientation of this rotating rigid body using X Y Z body fixed Euler angles.

So, we have a rotation about X given by θ_1 as a function of time, rotation of θ_2 about moving Y which is θ_2 and rotation above the moved Z which is 8t - 10. Again, these functions of time for θ_1 θ_2 and $\theta_3$are chosen arbitrarily, so given this initial set of data at t = 0 we can find the position vector. So, how do I find the positioning vector? It is nothing but this vector plus this vector and pre-multiplied by the rotation matrix.

So, I can find what is the rotation matrix for X Y Z body fixed Euler angles the formula is given, this we have discussed this earlier. So, for example r_{11} is cos θ_2 cos θ_3 r_{33} is cos θ_1 cos θ_2 and so on, so c_2 is cos θ_2 s_3 is sin $\theta_3$and so on. We have discussed this earlier. So, if you substitute t = 0 in these expressions θ_1 θ_2 and θ_3 and then if you compute cos θ_2 cos $\theta_3$or sin $\theta_3$cos θ_1 all the elements of this rotation matrix you will get this rotation matrix.

So, r_{11} is 0.979 r_{23} is - 0.172 so, all these numbers are obtained by nothing but substituting t = 0 in these expressions for θ_1 θ_2 and θ_3 and then evaluating sin and cosine of these theta angles.

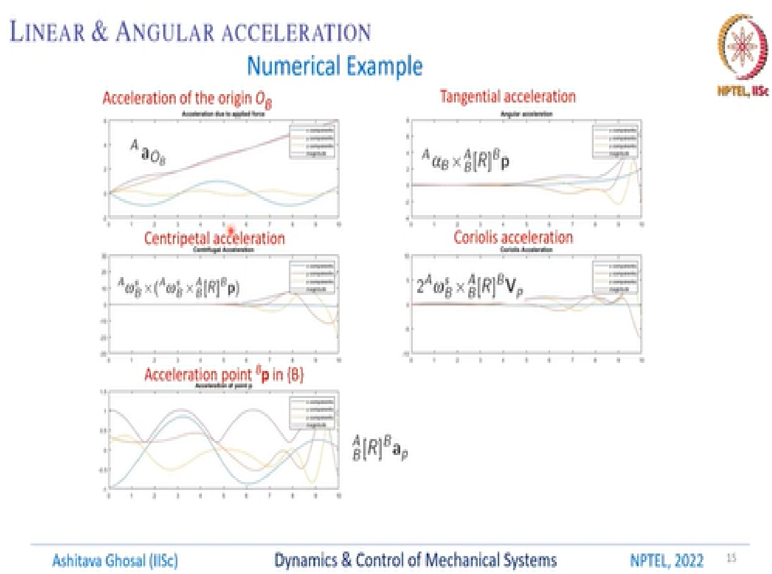
(Refer Slide Time: 35:20)



You can also plot each one of these components, so the position vector of in the A coordinate system can be given by these plots. So, the plots are of the X component which is blue red which is the Y component and yellow is the Z component, so you can see that these plots we can get. Similarly, the space fixed angular velocity vector has three components again X Y and Z, again we can plot this the blue is the X component this slightly reddish yellow is the Y component and yellow is the Z component.

You can also plot the magnitude which is nothing but the $X^2 + Y^2 + Z^2$ the square of the three component and then square root. We can also evaluate $A\alpha_B$ which is the acceleration tangential acceleration alpha is the derivative of the angular velocity vector and again we can plot it.

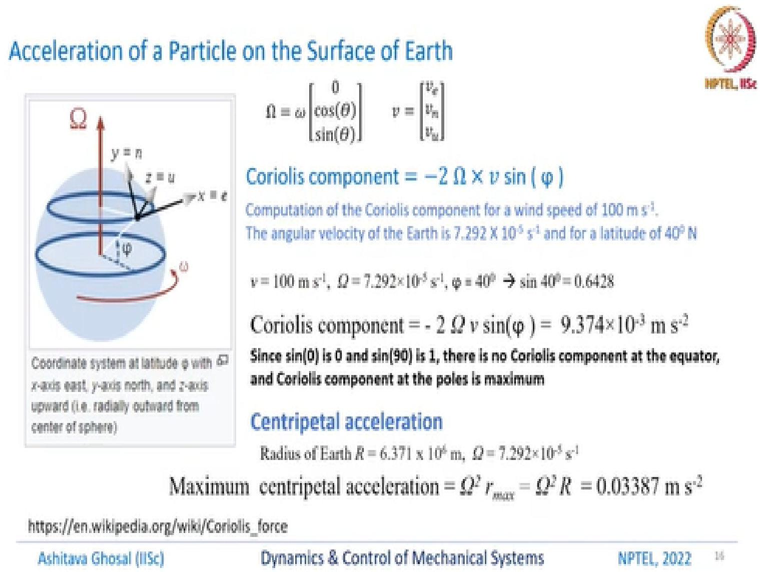
(Refer Slide Time: 36:28)



We can also plot all the terms of the acceleration vector. So, the acceleration of the origin is again given by this. The tangential acceleration can also be obtained and it looks like this again all the three components can be plotted and also the magnitude can be plotted. The centripetal acceleration looks like this the acceleration of the point p in the B coordinate system gives is given by this form.

So, the basic idea is it is nothing very you know deep that we have some way of computing each and every term of the linear and angular velocity and linear and angular acceleration of a point in a rigid body which is both rotating and translating.

(Refer Slide Time: 37:22)



Let us take a different approach, now a slightly more interesting thing than just cooking up numbers and giving some numerical results. So, the picture here shows a sketch of the earth. So, basically what we have is we pick a point on the earth, let us say this point this is at a latitude of 40 degrees north, so this is in the northern hemisphere. So, one standard way to assign A coordinate system on this earth rotating earth is that the X-axis is along the east direction, Y axis is along the north direction and Z axis is the local vertical.

So, again they form a right-handed coordinate system, this is the origin centre of the earth this angle is the latitude angle, the earth is spinning with some ω angular velocity. And what we want to look at is we want to see what are the various components of acceleration of a point which is on the surface of the earth at a latitude of 40 degrees. So, you can do it for any other latitude also, so the angular velocity is given by ω which is 0 $cos\theta$ $sin\theta$.

So, this theta is this angle this is theta. And the linear velocity of a point on this surface of the earth can have three components it can be along the east direction, it can be along the north direction and it can be along the upward direction, so z is u so this can be three components. So, what is the typical value of v let us assume that this is 100 meters per second. So, it is a very large you know some particle of air or some something some object is moving on the surface of the earth at 100 meters per second.

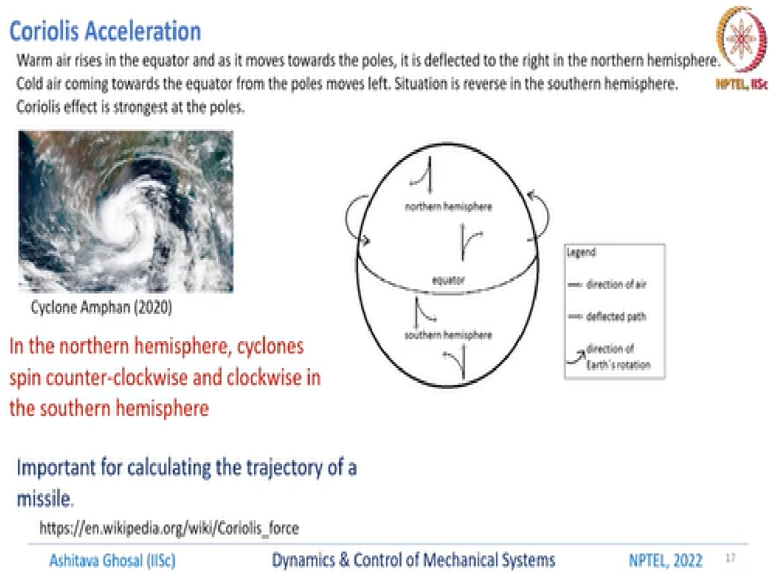
The angular velocity of the earth is can is measured it is very well known it is 7.292 into 10^{-5} per second radians per second. So, this is obtained from various literature is there in fact some of these things are obtained from this Wikipedia. So, if you are given these numbers v is this ω is this the latitude is 40 degrees then we can find what is the Coriolis component. The Coriolis component is nothing but 2 Ω x v sin ϕ.

So, this sin ϕ is coming because of this latitude which is very similar to $\omega\times V_{rel}$. And it turns out this is if you compute this number, it is 9.374 into 10^{-3} meters per second square. The interesting observation is that this is multiplied by sin of this ϕ so where ϕ is this latitude. So, when ϕ is 0 then there is no Coriolis component so there is no Coriolis component at the equator and the maximum Coriolis component is at the poles where phi is 90.

We can also calculate what is the centripetal acceleration of this point on the surface of the earth. So, now we need to know what is the radius of the earth which is given by 6.371 into 10^{6} ω is the same as earlier and the maximum centripetal acceleration is $\Omega\, r_{max}$ which is nothing but $\Omega^2 r_{max}$. Why are we saying r max? Because the radius of the earth is maximum at the equator, at the poles it is slightly smaller.

So, this is at the equator we will get this 0.03387 meters per second square how much so which one is larger this is 10^{-3} whereas this is 0.03387. So, the centripetal acceleration is like 3.3 into 10^{-2} whereas this is 9.4 into 10^{-3}, so the centripetal acceleration is larger.

(Refer Slide Time: 42:14)



Let us look at Coriolis acceleration little bit more, so think of the earth. So, the warm air is rising in the equator and as it moves towards the pole why would it move toward the pole because it is colder there, so the warm air will go towards the colder location. So, it is deflected to the right in the northern hemisphere and cold air coming from the north is deflected to the left, so this is not this is the effect of the Coriolis component.

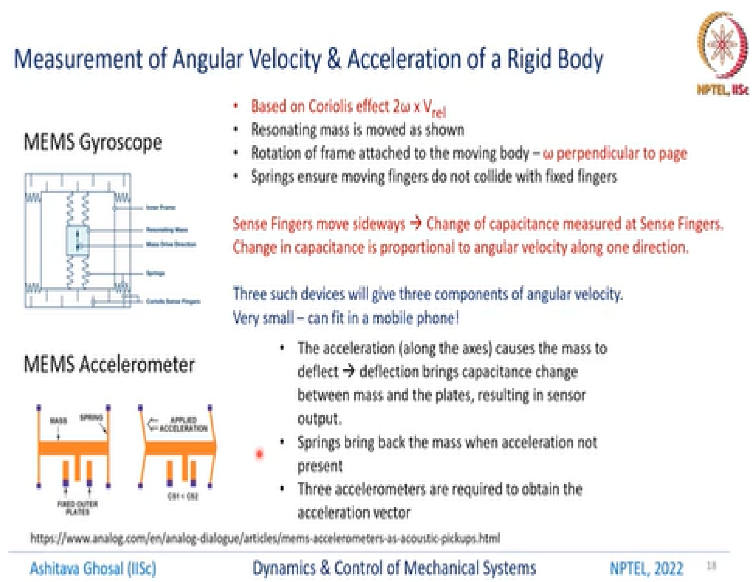
So, we have $\omega \times v$ so ω is pointing outwards v is pointing this way and then it will move to the right. So, why is it to the right? Because there is a minus sign, the situation is reverse in the southern hemisphere. So, if you have a warm air which is going towards the south pole the Coriolis effect will tend to make it in this direction and cold air which is coming from the polar region to the equatorial region will move towards the left.

Another very strong effect or interesting effect of Coriolis acceleration is that if you look at cyclones in the northern hemisphere the cyclone will spin in the counter clockwise direction. And it spins in the clockwise direction in the southern hemisphere, so this is the cyclone which hit India in 2020 this name was Amphan. So, this is a picture of that cyclone very close to the Indian coastline and you can see it is rotating in a counter clockwise direction.

I not showing here but there are pictures of cyclones in the southern hemisphere which rotate in the opposite direction. The Coriolis acceleration is also very important to calculate the trajectory of a missile. So, you can think of a missile being launched from one point on the earth and it is supposed to go somewhere else because the earth is spinning and there is a velocity component of this missile linear velocity.

So, there will be a two ω× v, there will be a Coriolis acceleration and as a because of this Coriolis acceleration the path or the trajectory of the missile will deviate a little bit. So, if there were no Coriolis acceleration it will go and hit somewhere else but due to the Coriolis acceleration it will move to the left or right depending on which way it is moving.

(Refer Slide Time: 45:11)



Let us change topics. We would like now to discuss how to measure angular velocity and acceleration of a rigid body. So, there are various ways to measure angular velocity and acceleration of a rigid body I want to discuss something which is called as a MEMS gyroscope. So, basically a MEMS gyroscope is a very, very small device in fact it is so small that it fits into your mobile phone nowadays.

So, the main components are there is a resonating mass and this mass can be driven up and down in this direction by means of some drive by means of some actuator. And while it is moving this mass it has these springs which are attached so that it does not go and hit the support. And there are a fixed frame and there is a moving frame on both of these frames they have these fingers. So, these are called sense fingers.

So, basically there is a charge between these two fingers and the distance between these two fingers if they change the capacitance will change and you can measure the change in capacitance. So, now let us see what is happening when this MEMS gyroscope is being used. So, I have a oscillating mass which is moving up and down and now if I put it on a surface which is rotating so that means there is an ω.

So, let us assume that the ω vector is coming out of the page. So, we have an ω here and then there is a V_{rel} here. So, due to this $\omega \times V_{rel}$ there will be a sideways force which is Coriolis. So, remember $\omega \times$ v right hand rule it will move to the left or right depending on what is the direction of the velocity vector. And as it moves to the left or right the distance between these two fingers will change.

And you can sense the change in capacitance and these springs make sure that you do not move too much that it does not oscillate too much to the left or right. So, basically measuring the capacitance change I can find out what is the Coriolis effect. So, I can find out what is the Coriolis component of the acceleration, so it is $2\ \omega \times V_{rel}$ I know what is V_{rel} and I know what is $2\ \omega \times V_{rel}$.

So, hence from the change in capacitance so hence I can estimate ω. So, as the sense fingers move sideways the change in capacitance can be measured at the sense fingers and the change in capacitance is proportional to the angular velocity along one direction. Which direction? Perpendicular to this page. So, if you have three such devices mounted in three perpendicular directions.

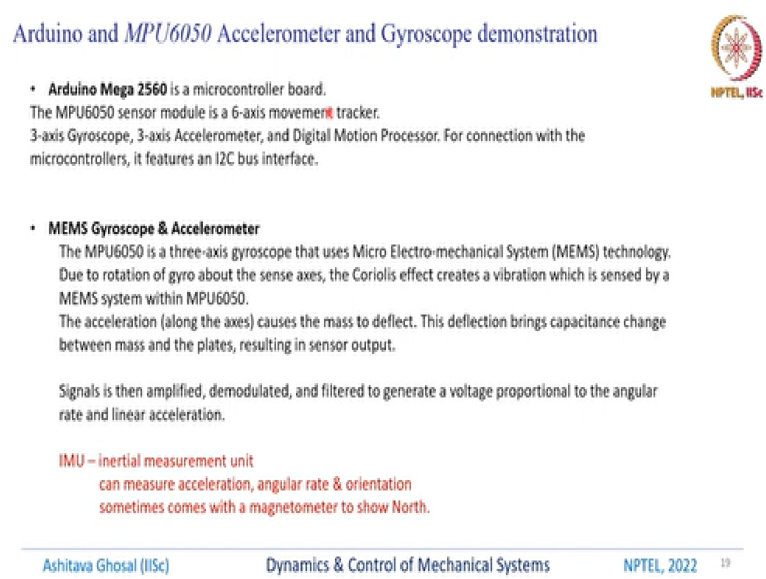
So, you can measure the three components of the angular velocity vector and this is what is used in many gyroscopes nowadays. This is a very small device and it is so small it can fit into a mobile phone. So, in your mobile phone in some good mobile phones you can measure what is the angular velocity of the phone and this is something which is used in many devices. We can also measure the acceleration of a particle or of a rigid body in 3D space.

Again, the idea is that we want to use the change in capacitance when the mass moves and attached to this mass there are these fingers. So, there is one set of fixed plates and then one which is moving plate. So, if you apply an acceleration the distance between these two fingers, for example here has changed. So, hence the capacitance between these two will be different than what it was earlier.

And these things here they are like springs which make sure that you do not go too much. So, by measuring the capacitance between these two fingers, so between this and this and between this and this I can sense what is the acceleration, what is the direction and what is the magnitude. So, this is basically whatever I said is listed here the acceleration along the axis causes the mass to deflect, deflection brings capacitance change between the mass and the plate.

Resulting in a sensor output the springs will bring back the mass when the acceleration is not present. And if you have three such accelerometers mounted in three perpendicular directions, we can get the three components of the acceleration.

(Refer Slide Time: 50:43)



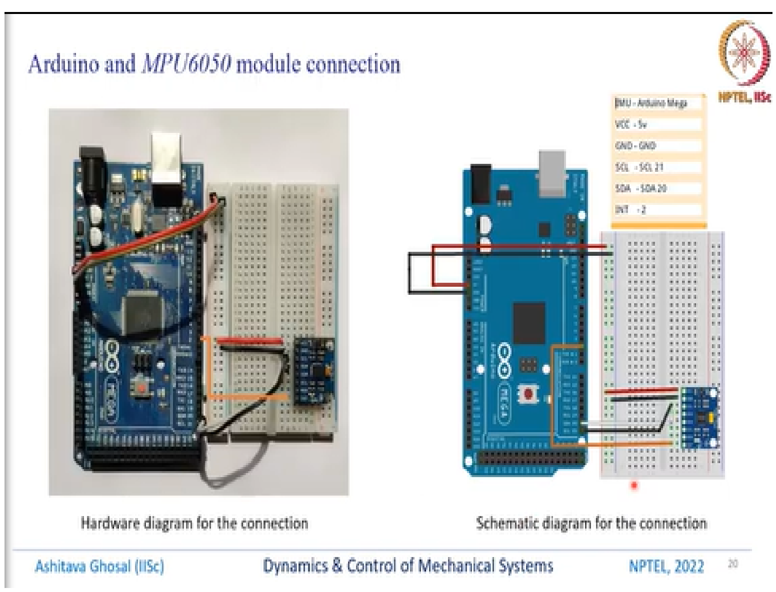
So, we cooked up or rigged up a device in which we can measure acceleration and also the angular velocities. So, this is a setup which was done in the lab it consists of an Arduino mega 2560 microcontroller board. It is a small microcontroller which is very commonly available in many labs nowadays. There is also a sensor which is called MPU6050, it is a 6-axis movement tracker, it consists of three axis gyroscope, a 3-axis accelerometer and some processor.

Because we need to process the signal to find out what is the angular velocity and the acceleration. So, it is connected to this Arduino and microcontroller using some I2C bus. So, let us not worry about the details but it is a small device which is readily available both this Arduino and this MPU6050 sensor. So, this MPUs 6050 is a 3-axis gyroscope that uses MEMS device, it is using something very similar to what I showed in the previous slide.

So, due to the rotation of the gyro the sense axis the Coriolis effect measures creates a vibration which is sensed by a MEM system within this MPU6050. The acceleration along the axis causes the mass to deflect, this deflection brings capacitance change between the mass and plates and again it results in a sensor output. So, all these signals are then amplified, demodulated, filtered and finally it generates a voltage proportional to the angular rate and linear acceleration.

This device is sometimes called as an IMU. It is very commonly seen in many phones and various other places; it is also called inertial measurement unit and it can measure the acceleration angular rate and orientation. Some of these IMUs sometimes also comes with a magnetometer basically to find out what which is the north direction.

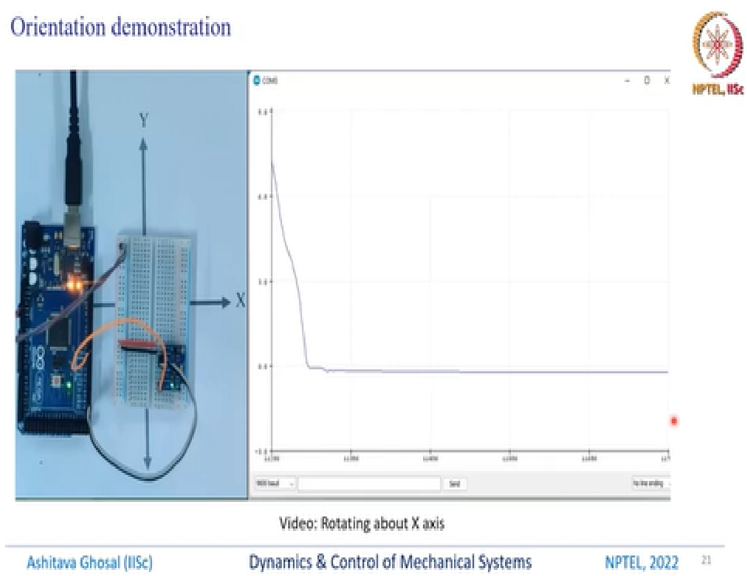
(Refer Slide Time: 53:17)



So, here are the hardware which was done by one of the TA of this course Pramod Pal. So, basically this is the at mega Arduino board there are some processors, this is the sensor this is the MPU6050 module and you connect it up. Let us not worry about the details of connection but it is very straightforward and simple. And then this is showing the hardware diagram and this is also showing how to connect this MPU6050 to the Arduino board.

Which you know which input of the Arduino board is connected to which output and so on. So, some ground is there some VCC is there and so on.

(Refer Slide Time: 54:08)



So, here is a video which shows how this combination of MPU5060 and this Arduino can be used to measure orientation. So, this is a video which is going which I am going to show you which shows how to rotate about X-axis, so I will rotate this breadboard about the X-axis and then I will show you what is the output.

(Video Starts: 54:39)

So, Pramod is rotating this breadboard with the sensor. So, there is some angular orientation some angular velocity.

(Video Ends: 54:53)

And as you saw here it shows what happens to the plot of this angular velocity as a function of time. So, once it is stationary it comes down to 0. This is the same thing I am going to rotate now about the Y-axis, so this is the Y-axis this is the X-axis. Now we will rotate about the Y-axis, so at t = 0 nothing is moving so it is at 0 the angular velocity is 0.

(Video Starts: 55:28)

So, as you rotate about the Y-axis you can see this plot or the output of this sensor and it shows how the angular velocity along the Y-axis is changing so as you can see there was some jerk while moving up and down jerk that is faithfully captured in this output.

(Video Ends: 55:42)

Here is a demonstration of rotation about the Z-axis.

(Video Starts: 56:00)

So, as you can see, he rotated by some amount and then it came back about 90 degrees and then it went to 90 degrees and it came to 0.

(Video Ends: 56:12)

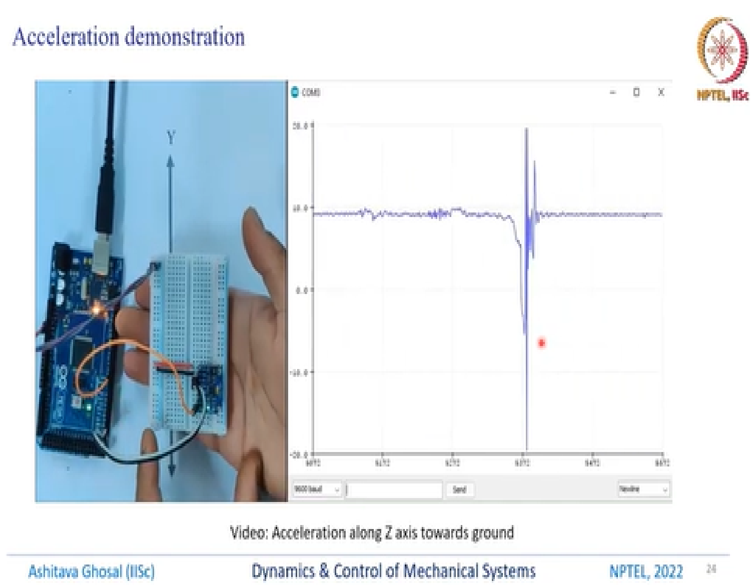
So, we can show all these angles and rotations by this simple microcontroller you know Arduino board and this device. This is the demonstration of acceleration. So, again this is the same Arduino board it is the same MPU5060 and the acceleration data is a little bit noisier as you can see this is some small value is there, so this value is nothing but 9.8

(Video Starts: 57:01)

And what he is going to show you is that he will lift it to some level and drop it.

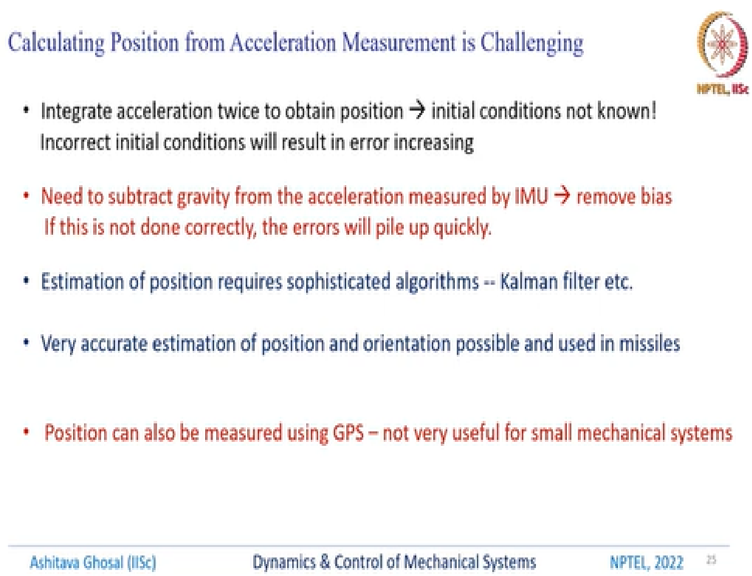
(Video Ends: 57:12)

(Refer Slide Time: 57:14)



So, while it is being dropped what it should come back you should measure this 9.8 g which is basically what is being measured. So, hence at normal thing it is like 9.8 and when it is falling freely you can see that it is very close to 0.

(Refer Slide Time: 57:28)



Once we know what is the acceleration by this device can we calculate position of this device because eventually we would like to have position velocity and acceleration linear components. So, it turns out that it is very hard it is not so easy to calculate or compute the position of this object if I know the acceleration and the basic problem is that we need to integrate acceleration twice to obtain position and we do not know the initial conditions.

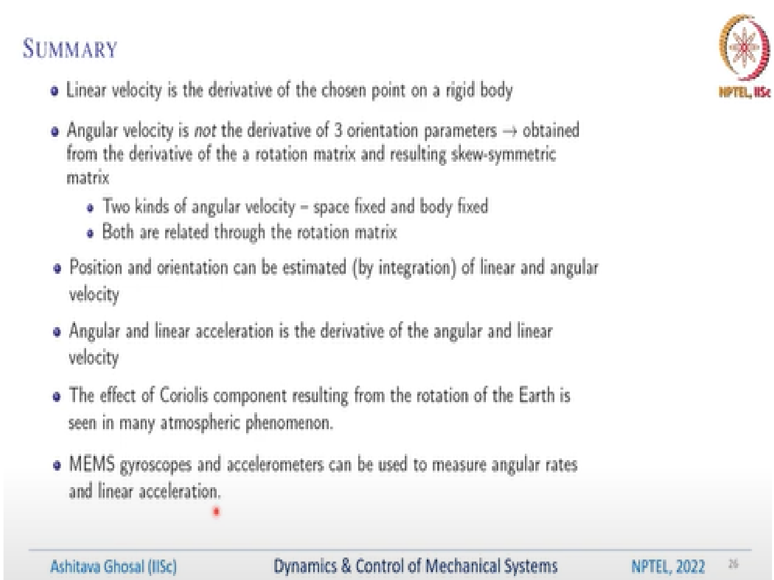
So, simple calculus if I know acceleration if you integrate once you will get velocity. But then you have an integration constant which is c_1 then when you integrate it again you will have another integration constant which is c_2 and the c_1 will become c_1 into t but we do not know c_1 and c_2 very well. So, the incorrect initial conditions will result in errors and these errors will increase with time because if I do not know c_1 remember c_1 is multiplied with t.

So, if c_1 there is an error, then as time progresses c_{12} will keep on increasing the error will increase. We also need to subtract gravity from the acceleration measurement by the IMU, so this is a bias and if this is also not done correctly the errors will pile up. So, measuring position from accelerometer or from acceleration is non-trivial it is requires sophisticated algorithms and there is a whole field with of using algorithms.

And using you know processes and techniques such as Kalman filters which can be used to obtain the position from acceleration readings. It is not impossible, very accurate estimation of position and orientation is possible and it is used in many missiles and various other devices and systems. The position can also be measured using GPS. So, nowadays GPS is available all over the world.

So, you can from your mobile phone you know exactly where you are but it is not very useful for small mechanical systems. So, if you have a small robot or if you have some small mechanical system and you want to find out what is the position of some part as this mechanical system is moving GPS is not very useful. The IMU which I showed you can be used provided you do it properly and you can estimate the position and orientation correctly.

(Refer Slide Time: 1:00:34)



So, in summary the linear velocity is the derivative of a chosen point on the rigid body. So, the angular velocity is not the derivative of three orientation parameters, so if you had X Y Z Euler angles and say let us say θ_1 about X θ_2 about Y $\theta_3$about Z so the three parameters are θ_1 θ_2 theta

3. The angular velocity is not simply $\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3$, we need to obtain the derivative of a rotation matrix.

We will get a resulting skew symmetric matrix and from that we can obtain the angular velocity. There are two kinds of angular velocities one is the space fixed and one is body fixed, both are related to the root through the rotation matrix. The position and orientation can be estimated by integration of linear and angular velocity. The angular and linear acceleration is the derivative of the angular and linear velocity.

The effect of the Coriolis component resulting from the rotation of the earth is seen in many atmospheric phenomena, it is seen in cyclones and it is also seen in motion of a missile. And we also have nowadays very simple and cheap IMUs inertial measurement unit. So, these are basically MEMS gyroscope and accelerometers which can be used to measure angular rates and linear acceleration.

And then using some reasonably sophisticated algorithms you can find the position of a rigid body in 3D space and the orientation of a rigid body in 3D space. So, in this lecture we looked at position and orientation of a rigid body in 3D space the, linear and angular velocity is an acceleration. In the next lecture we look at multi-body dynamical systems.