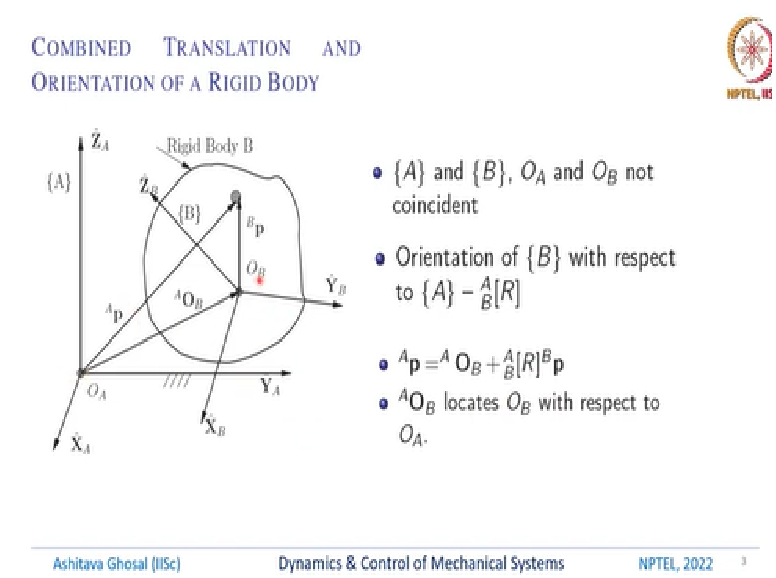
Dynamics and Control of Mechanical Systems
Prof. Ashitava Ghosal
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 03
Homogenous Transformation

Welcome to this NPTEL course on dynamics and control of mechanical systems. My name is Ashitava Ghosal, I am a professor in the department of mechanical engineering and in the centre for product design and manufacturing and Robert Bosch centre for cyber physical systems, Indian institute of science Bangalore. In this course we will start with the representation of rigid bodies in 3D space. So, the last topic this week is what is called as homogeneous transformation.

(Refer Slide Time: 01:06)



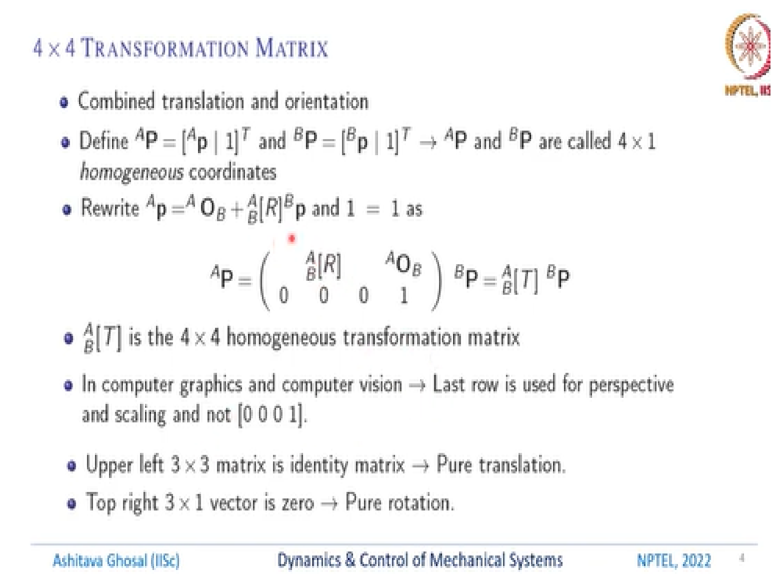
So, homogeneous transformation as it names implies it is a combined translation and orientation of a rigid body. So, we have this reference coordinate system $X_A Y_A Z_A$ which is origin O_A and we have a rigid body but this rigid body origin is not at O_A. It has gone to some another point which is located by this vector AO_B. So, as you can see that there is $X_B Y_B Z_B$ is different from $X_A Y_A Z_A$ as well as the origin has gone from O_A to O_B.

And in this figure, I am showing you a point on this rigid body which is given in the B coordinate system by Bp. So, we will see later how we can express this vector in the A coordinate system. So, A and B are not coincident, orientation of B with respect to A is again the familiar rotation matrix $BA[R]$, we can write this vector to this point on the rigid body from the fixed coordinate system O_A.

This vector Ap can be written as AO_B plus this but then if you want to add these two vectors you have to make sure they are in the same coordinate system, so AO_B and Bp should be in the same coordinate system. We can do that by pre-multiplying Bp by this rotation matrix, so again if you think a little bit or in your mind you can see that this B and B cancels and you are left with Ap . So, now we can add these two vectors and we can get this vector.

So, this nothing but vector addition but vector addition where the vectors are in the same coordinate system and this AO_B locates the origin of the rigid body with respect to the fixed coordinate system or the reference coordinate system.

(Refer Slide Time: 03:12)



We want to represent the combined rotation and translation of a rigid body in 3D space. So, in order to do that we define a new quantity which is AP which is nothing but Ap which is the position vector of a point in a coordinate system. We add 1 to it so hence this AP is now a 4 by 1 vector, so there are four elements. Likewise, we do that for BP so we have a vector p in the B coordinate system we add the 1 to it and then hence this AP and BP are vectors with four elements.

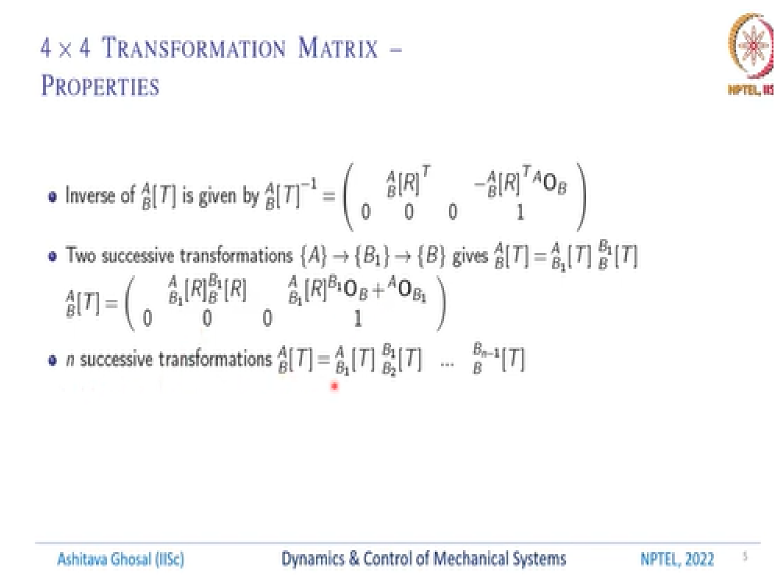
The last element is one. These are called homogeneous coordinates. So, we can rewrite this vector equation which was Ap is equal to the vector locating the origin of the B coordinate system then some rotation matrix times the vector in the B coordinate system. We add one more equation which is 1=1. So, now we can write so this we can write these three equations and this one equation which is evidently true in this form.

So, we have AP which is now a matrix which is a 4 by 4 matrix the top 3 by 3 is a rotation matrix. Then this is a column vector which is AO_B and the last row is 0 0 0 1 into BP, so we are going to write this 4 by 4 matrix as $BA[T]$. So, you can clearly see that AP will be equal to $BA[T]$ into BP, if you expand it, you will get this equation and 1 = 1.

So, $BA[T]$ is the 4 by 4 homogeneous transformation matrix. In computer graϕcs and computer vision the last row is not 0 0 0 1. If you want to show perspective and scaling then we need to change this last row into some other numbers. This is a very useful way of representing combined translation and orientation of a rigid body in 3D space. As I said the upper 3 by 3 matrices here which is $BA[R]$ represents rotation.

So, if this matrix is a pure is an identity matrix, then this 4 by 4 homogeneous transformation matrix represents pure translation. Similarly, if this last column vector which is AO_B with 1 is 0 0 0 and then 1 then this represents pure rotation, so this 4 by 4 transformation matrix can represent all aspects of rotation as well as translation of a rigid body in 3d space.

(Refer Slide Time: 06:45)

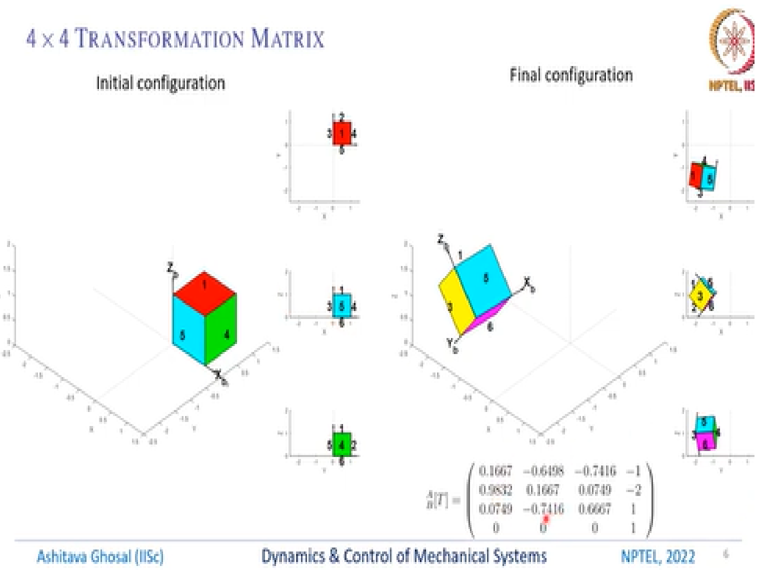


There are some nice interesting properties of this homogeneous transformation matrix. For example, the inverse I do not have to do an inverse of 4 by 4 matrices, so the rotation part is transpose and the translation part is this transpose into AO_B with a minus sign. Likewise, if you have two successive transformations A to B_1 and B_1 to B then the product A B_1 into B_1 B gives the resultant transformation matrix.

And the resultant transformation matrix can again have nice simple closed forms expressions. The rotation part is A B_1 into B_1 B it is exactly similar to rotation two successive rotations and the translation is again addition of two vectors as long as you make sure that they are in the same coordinate system. So, pre-multiplying by this rotation matrix brings this $B_1 \;\; O_B$ to the A coordinate system.

And likewise, if you have n successive transformations then you can just multiply all these transformation matrices.

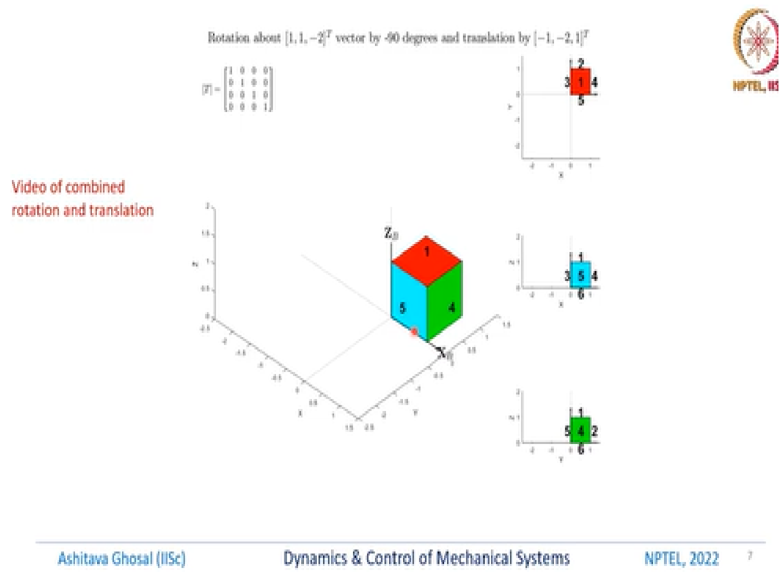
(Refer Slide Time: 07:58)



So, here is an example of 4 by 4 homogeneous transformation matrix again obtained using Matlab and this example of the dice. So, we have this initial configuration which is here you can see it is at 0 0 0 with some at this point. So, not only the origin has gone from here to somewhere some other place but there is also rotation and now it is again the same dice so it is 1 5 4 and then this 1 5 and 3 and then something is visible here.

So, we are not seeing this 4 anymore face with the 4. So, for this example I can find the homogeneous transformation matrix and it looks like this. So, as you can see, we have gone from 0 0 I think it is some other number Z coordinate so it has gone to - 1 - 2 and 1 and the rotation matrix top 3 by 3 rotation matrix is given in this form.

(Refer Slide Time: 09:16)



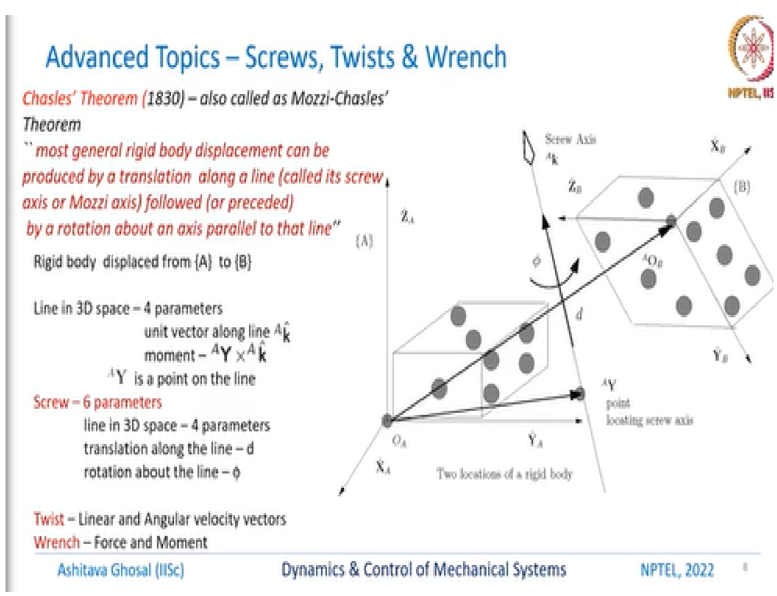
So, this is a video of the combined rotation and translation of this dice. So, it is a rotation about [1 1 - 2] vector by 90 °. So, you have to make it as a unit vector and translation by [-1 – 2 1]. So, initially it is at this location it is at 0 0 0 and then it is going to some making these two one rotation and one translation.

(Video Starts: 09:55)

So, you can see this video, it ends up at the place where we really want it to be and this rotation matrix is given by this.

(Video Ends: 10:22)

(Refer Slide Time: 10:23)



So, just a quick thing about some advanced topics in kinematics there are things called screws, twists, wrenches and so on. It is based on this very famous theorem called Chasles' theorem which was given in 1830. It is sometimes also called Mozzi Chasles theorem. So, what it states is that the most general rigid body displacement can be produced by a translation along a line it is called the screw axis or the Mozzi axis followed or preceded by a rotation about an axis parallel to the line.

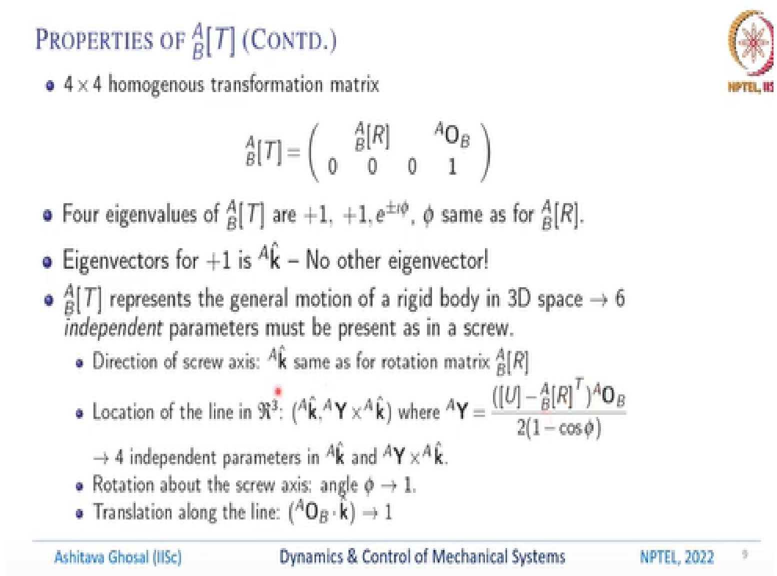
So, I have this dice and I want to take it here. So, according to this theorem what it says is this is can be done by rotation about one line which is this line which is called the screw axis then rotation by an angle ϕ and translation along this line by d. So, a line in 3D space has four parameters so if you want to locate this line in X Y Z you need four parameters so if you think a little bit.

So, you can have $Y = M_1 X + C_1$ and $Z = M_2 X + C_2$ so $M_1 M_2 C_1 C_2$, there are four parameters. A more formal way of saying it a line in 3D space can be given by and unit vector k so there are two parameters here and a moment of this unit vectors about this origin which is AY .k and AY is any point on this line. So, Ak and AY dot cross Ak there are four parameters and the line has four parameters.

But as I said this general displacement consists of not only this line in 3D space but a rotation about this line and a translation about this line. So, you have 4 plus then d and then ϕ so these are 6 parameters. So, recall that general rotation plus translation of a rigid body can be given by 3 parameters from the rotation and 3 from the translation. So, there are total of 6 parameters and in this representation also there are 6 parameters, so 4 from the line and d and ϕ.

In kinematics we also have linear and angular velocity vectors they can be altered represented by these lines and these rotations you know about this line and translation about this line so these are called twists. And then we do not want to get into it now but the if you ever want to do some advanced kinematics you need to worry about these screws, twists and lines and so on.

(Refer Slide Time: 13:44)



So, a very quick recap of this homogeneous transformation matrix. So, you have a rotation part and you have a translation part and the last row is 0 0 0 1. So, similar to the rotation matrix we can try and see what are the Eigen values and Eigen vectors of this transformation matrix. So, it turns out they are 1 1 and $e^{\pm i\phi}$ So, ϕ is same as what we obtained from the rotation matrix.

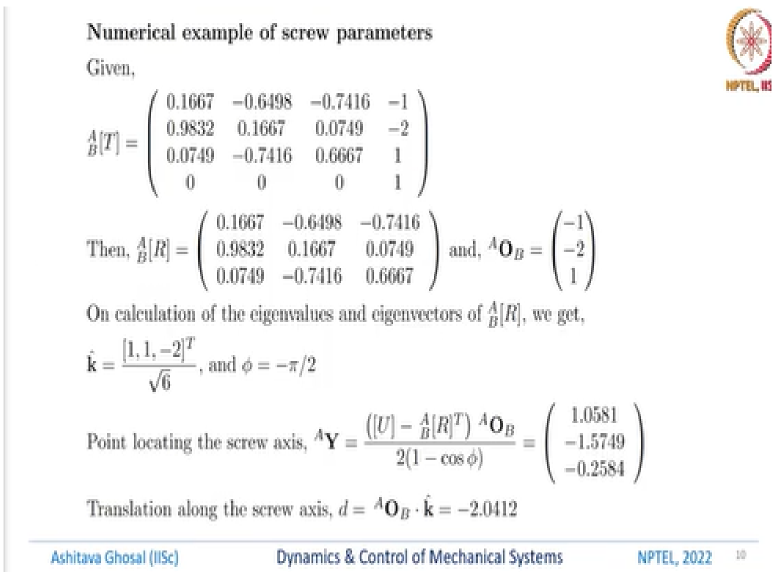
So, there are four Eigen values two of these are repeated they are + 1 and + 1. But the really interesting part of this is there are no two real Eigen vectors there is exactly one real Eigen vector corresponding to one. So, although there is a multiplicity of two for the Eigen values the Eigen vectors are still there is still only one there is no other eigenvector. So, this transformation matrix consisting of rotation and translation represents a general motion of a rigid body in 3D space.

So, there are six independent parameters here and they are sort of related to the screw, screw means a line. And then there are these d and ϕ and here also there are three rotations and three translations. So, how are they related? You can show that the screw axis is same as the obtained from the rotation matrix, so this is the axis about which it rotates. The location of the line can be given in terms of this axis of rotation.

And a point along the line about which it rotates where this AY which locates the screw axis is given in terms of identity matrix transformation rotation matrix and transpose the linear translation and then 2 (1 - cos ϕ). So, this is a formula which was derived some time back and hence given the rotation matrix I can find out k just by the Eigen value of the rotation part given this translation AO_B so basically given A B transpose I can find k and this location of this line in 3D space.

I can find out ϕ again from the rotation matrix remember one of two of the Eigen values were $e^{\pm i\phi}$ and we can find the translation of the line by taking this dot product AO_B . k. So, this is one parameter this is one parameter there are four here and hence it is six and this is consistent with three from the rotation and three from the translation.

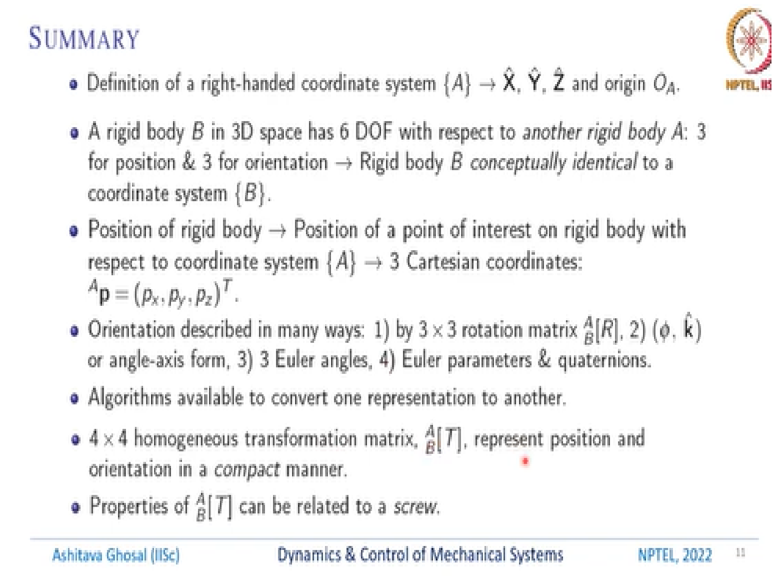
(Refer Slide Time: 16:47)



So, this is a simple numerical example so again the same example which we did before. So, if the transformation matrix is given in this form, then the rotation matrix is this the translation is this and we can calculate k from the eigenvalue and eigenvector corresponding to 1 and then this is the angle which is - ϕ by 2. The location of the screw axis is given by this formula which is given by this and the translation along is given by this AO_B . k.

So, again the basic idea is that if I give you a 4 by 4 homogeneous transformation matrix, I can find out what is the translation along the screw axis? And what is the rotation about the screw axis.

(Refer Slide Time: 17:44)



So, in summary we represent a rigid body in 3D space first we represent first we need a right-handed coordinate system consisting of X axis Y axis Z axis and an origin. A rigid body in 3D space has 6 ° of freedom with respect to another rigid body 3 for position and 3 for orientation and the rigid body B is basically identical to a coordinate system B. So, remember in kinematics we are not really interested in the shape, the size or the mass and inertia of the rigid body.

So, in some sense a rigid body was very similar conceptually similar to a coordinate system. So, we were attaching a rigid coordinate system B to the rigid body B and continuing onwards. The position of a rigid body is basically a position of a point of interest on the rigid body with respect to the reference coordinate system A. So, most of the time we; will use these 3 Cartesian coordinates $p_x \, p_y \, p_z$.

The orientation can be described in many ways the natural first simplest way is this 3 by 3 rotation matrix $BA[R]$ then we then I showed you that you can derive an angle and an axis form. Then we can describe by 3 Euler angles and then we can have this 4 Euler parameters and quaternion. And most importantly we can go from any one to another one and vice versa. Then we had this 4 by 4 homogeneous transformation matrix and $BA[T]$

represents position.

And orientation in a compact manner and these properties of $BA[T]$ can be related to a screw. So, with that we stopped.