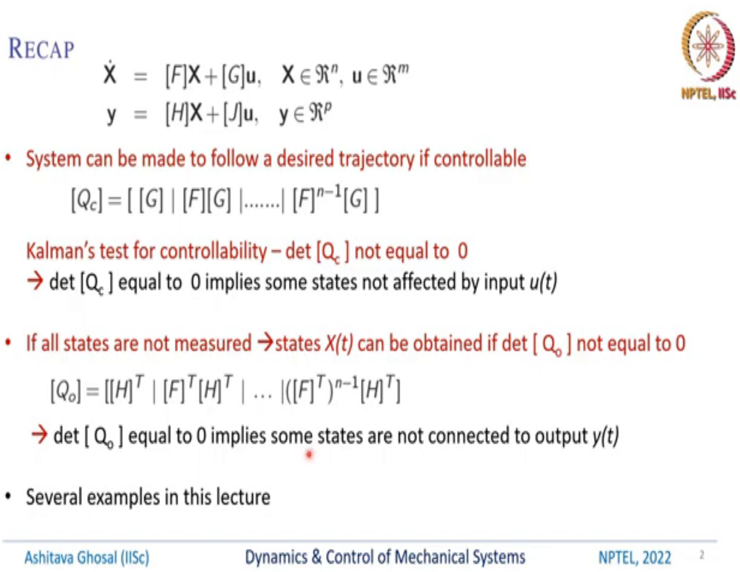
Dynamics and Control of Mechanical Systems
Prof. Ashitava Ghosal
Department of Mechanical Engineering
Indian Institute of Science –Bengaluru

Lecture – 25
Examples of Controllability and Observability

In this last lecture of this week, we will look at several examples of controllable, not controllable, observable and not observable systems.

(Refer Slide Time: 00:37)



To recap in the last lecture, we had looked at a state space representation of a linear time invariant system given by the state equations which is $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$, $\mathbf{X}$ is an element of R^n, $\mathbf{u}$ is an element of possibly R^m and output equation $\boldsymbol{y} = [H]\mathbf{X} + [J]\mathbf{u}$, where $\boldsymbol{y}$ could be a p x 1 vector. Most of the examples we looked at where u was single input and the output y was also one output - so, it was single input, single output systems.

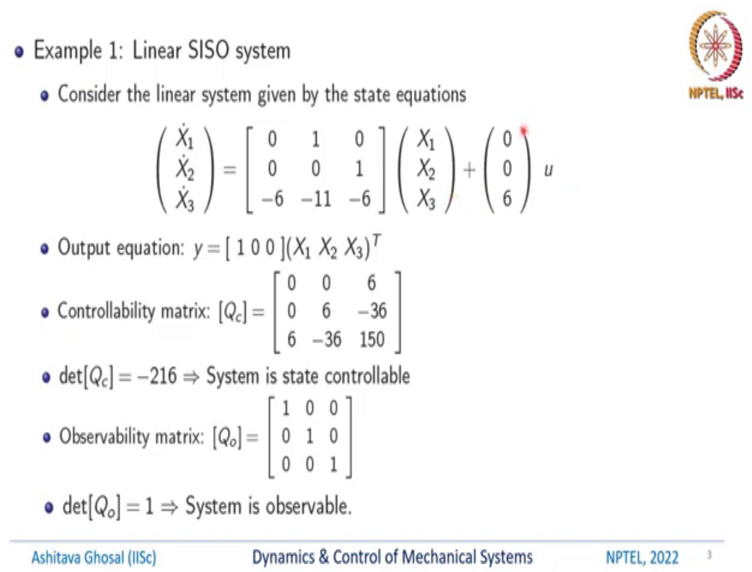
So, we showed you that the system can be made to follow a desired trajectory if the system is controllable. And to test for controllability, we obtain this matrix $[Q_c]$ which has $[G]$ as it is first column, $[F][G]$ as the second and so on all the way up, till $[F]^{n-1}[G]$. So, we get this matrix. $[Q_c]$ and Kalman's test for controllability was the following -- that if the determinant of $[Q_c]$ is not equal to 0 then the system given by $\dot{X} = [F]\mathbf{X} + [G]\mathbf{u}$ and $\boldsymbol{y} = [H]\mathbf{X} + [J]\mathbf{u}$ is controllable. If determinant of $[Q_c] = 0$ this implies some states are not affected by the input

$u(t)$ and hence we cannot take those states from some initial state to another arbitrarily desired state in finite time by application of $u\ (t)$.

We also discussed observability if all states are not measured and states **X** (t) can be obtained if determinant of $[Q_o]$ is not equal to 0 and this $[Q_o]$ matrix is again derived from $[F]$ and $[H]$. So, the first column is $[H]^T$ then the second column is $[F]^T[H]^T$ and so on, the last column is $([F]^T)^{n-1}[H]^T$ transpose. And,again Kalman showed that if this determinant of this matrix $[Q_o]$ = 0, it implies some states are not connected to the output $y\ (t)$. If the determinant is not equal to 0, then the states **X** (t) can be obtained from measurements $y\ (t)$.

In this lecture, I will show you several examples of systems which are controllable, observable, not controllable, not observable.

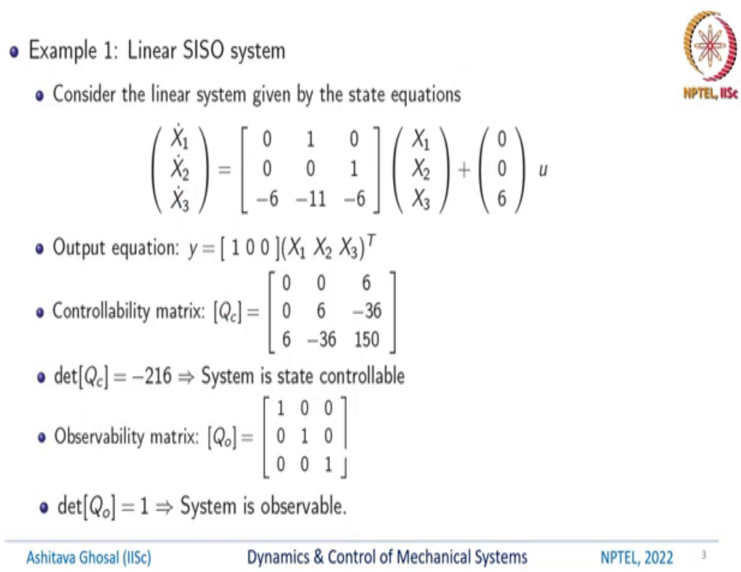
(Refer Slide Time: 03:47)



We start with a very simple example of a linear SISO system, single input single output. So, the linear system is given by $\dot{X} = [F]\mathbf{X} + [G]u$, $[F]$ is 3 by 3 so, the states are X_1, X_2, X_3 and $[F]$ is given by [0 1 0, 0 0 1, –6 –11 –6]. And the $[G]$ vector is (0 0 6). The output equation $y = [H]\ \mathbf{X}$, $[H]$ is [1 0 0], so, essentially y is X_1. We can compute the controllability matrix $[Q_c]$ which is given by first column, is $[G]$ second column is $[F][G]$, the third column is $[F]^2[G]$, and we will get (0 0 6) from here then (0 6 –36) then (6 –36 150). So, if you compute the determinant of $[Q_c]$, we will get this –216. Hence the system given here is state controllable. We can also compute the observability matrix $[Q_o]$ which is, as I showed you, it is $[H]^T$ then

$[F]^T$ $[H]^T$ and then $([F]^T)^2[H]^T$, and in this case, we will get an identity matrix and the determinant of $[Q_o]$ is 1. So, it is not equal to 0, hence the system is observable.

(Refer Slide Time: 05:49)



Next, we consider a simple linear time invariant system. So, in this case again we have $\dot{X} =$ $[F]\mathbf{X} + [G]$u, $[F]$ again is a 3 by 3 matrix which is [0 1 0, 0 0 1, 0 –2 –3], **X** is X_1, X_2, X_3 and the $[G]$ vector is (0 0 1). In this case, the output equation is not only 1. So, it depends on X_1, X_2, X_3. So, y is (3 4 1) into X_1, X_2, X_3. It is still a single output but it is not as simple as the last case where y was X_1.

Again we can compute the controllability matrix which in this case is [0 0 1, 0 1 –3, 1 –3 7]. It is $[G]$, then $[F][G]$ and then $[F]^2[G]$ and we can show or we can compute the determinant of this $[Q_c]$ matrix it is –1 which implies that this system is state controllable. This is from Kalman's test for controllability. We can also obtain the observability matrix -- in this case. we have to do $[H]$ which is [3 4 1], sorry $[H]^T$, then $[F]^T$ $[H]^T$ and then $([F]^T)^2[H]^T$, and we will get [3 4 1, 0 1 1, 0 –2 –2]. Here you can see that the determinant of $[Q_o]$ or Q observable is 0 -- the second column and the last column is multiplication by –2. So, the system is not observable. So, we have a system, first one which was both controllable and observable, now, we have a system which is controllable but not observable.

(Refer Slide Time: 08:10)

- Example 2: LTI system (Contd.)
  - Eigenvalues of $[F]$ are 0, -1 and -2 $\Rightarrow \mathbf{X} = [P]\mathbf{Z}$
  - In terms of $\mathbf{Z}$, output equation: $y = [3\ 0\ -1]\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$
  - State variable Z_2 is not connected to y and hence *not* observable.

Ashitava Ghosal (IISc) Dynamics & Control of Mechanical Systems NPTEL, 2022 5

Let us continue with the previous example of that LTI system and let us find out why it is not observable. So, the eigenvalues of $[F]$, we can compute, are 0 –1 and –2. So, we make a transformation which is $\mathbf{X}$ equals $[P]$ $\mathbf{Z}$ and then we rewrite the state equations and the output equation in terms of $\mathbf{Z}$. The output equation in terms of $\mathbf{Z}$ will be $y = [3\ 0\ –1]\ (Z_1, Z_2, Z_3)$. So, it will be $y = [H]$ $\mathbf{X}$ so, it will be $[H]$ $[P]$ $\mathbf{Z}$ and $[H]$ $[P]$ will give you (3 0 –1). So, what you can see is Z_2 is not connected to y. So, y will be some 3 Z_1 + 0 Z_2 - 1 Z_3. So, whatever you do to Z_2, it will not change y. Hence the system is not observable because Z_2 is not connected to y.

(Refer Slide Time: 09:24)

- Example 3: LTI system
  - Consider the linear system given by the state equations

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} 0.8 \\ 1 \end{pmatrix} u$$

  - Output equation: $y = [\,0\ 1\,](X_1\ X_2)^T$
  - Controllability matrix: $[Q_c] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$
  - $\det[Q_c] = 0 \Rightarrow$ System is *not* state controllable
  - Observability matrix: $[Q_o] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$
  - $\det[Q_o] = -1 \Rightarrow$ System is observable.

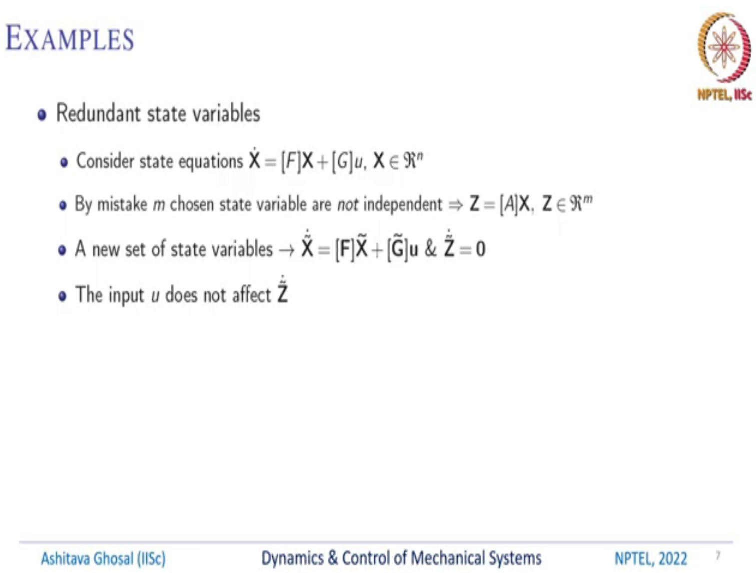
Ashitava Ghosal (IISc) Dynamics & Control of Mechanical Systems NPTEL, 2022 6

Let us continue with another example -- this is a very simple linear system. Again, we have $\mathbf{X}$ as X_1, X_2, the state variables as X_1 and X_2. And we have $\dot{X} = [F]\mathbf{X} + [G]$u in this case $[F]$ is a

2 x 2 matrix because **X** was only X_1, X_2 and the elements of [F] are [0 –0.4, 1 –1.3] and the elements of [G] are 0.8 and 1. And we will consider an output equation which is [0 1] (X_1, X_2). So, basically, we are only measuring X_2.

The controllability matrix in this case is again [G] and then [F][G] , so, (0.8 1) and [F] [G] is (–0.4 –0.5). You can see that the determinant of this is 0. If you multiply this by –2 you will get this -- so, the two columns are not independent, so, the determinant is 0. Hence the system is not state controllable. Let us look at the observability matrix which is $[H]^T$ = (0 1) and then $[F]^T[H]^T$ which is (1 –1.3). So, the determinant of [Q_o] is –1 which says that the system is observable. Here is an example where the system is not controllable but it is observable.

(Refer Slide Time: 11:10)

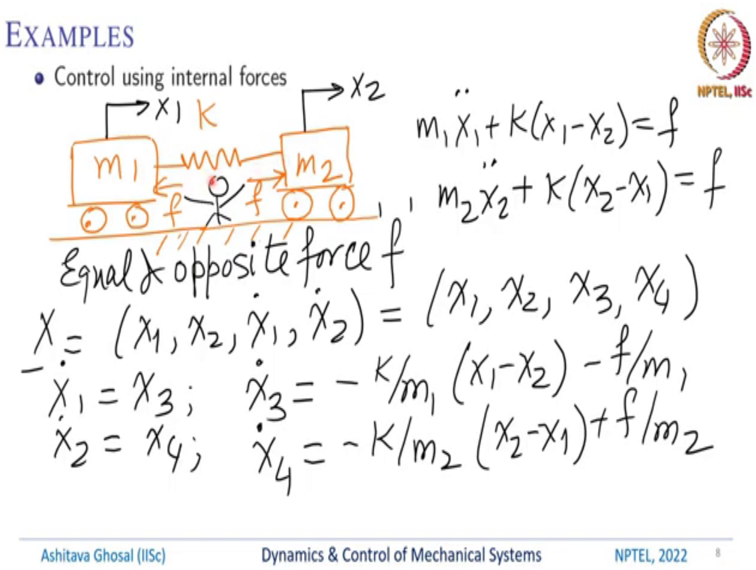


So, apart from these mathematical, and sort of you can say cooked up examples, let us go and see some real-life examples for systems which are not controllable or not observable. So, one very interesting example is that of a system which has been model using redundant state variables. So, what do we mean by redundant state variables so, by the system is actually, n-dimensional **X** should be n x 1 but by mistake you think that there are many more variables. So, while modelling, you have chosen some extra state variables which are actually, not independent. This is by mistake, and this can happen if there is a huge plant and you have many, many state variable. Let us say you have 20 state variables containing all kinds of variables and then you have chosen some state variable which are actually related to the other previously chosen state variables.

Let us consider state equations $\dot{X} = [F]\mathbf{X} + [G]u$, $\mathbf{X}$ is an n-dimensional vector. By mistake m chosen state variables are not independent, so, these m Z of these state variables, denoted by **Z,** are related to the **X** by $[A]$ **X** and **Z** is m dimensional. So, we can now convert these two sets of equations, as some get $\dot{\tilde{X}} = [F]\tilde{X} + [G]u$ and $\dot{\tilde{X}}\dot{\tilde{Z}} = 0$. So, we can do some linear combination and choose a new set of variables state variables $\tilde{X}$ such that this is what we get and $\dot{\tilde{X}}\dot{\tilde{Z}} = 0$.

Essentially, what happens is the state variable **Z** are not affected by u because you can see, $\dot{\tilde{X}}$ $\dot{\tilde{Z}} = 0$. So, some of these state variables actually, $\tilde{Z}$ these ones are not affected by the u and hence the system is not controllable. So, this is a not a realistic example in the sense that why would you make such a mistake? But mistakes do happen. While you are modelling a system, you can choose many more state variables than what is really, really required.

(Refer Slide Time: 13:59)

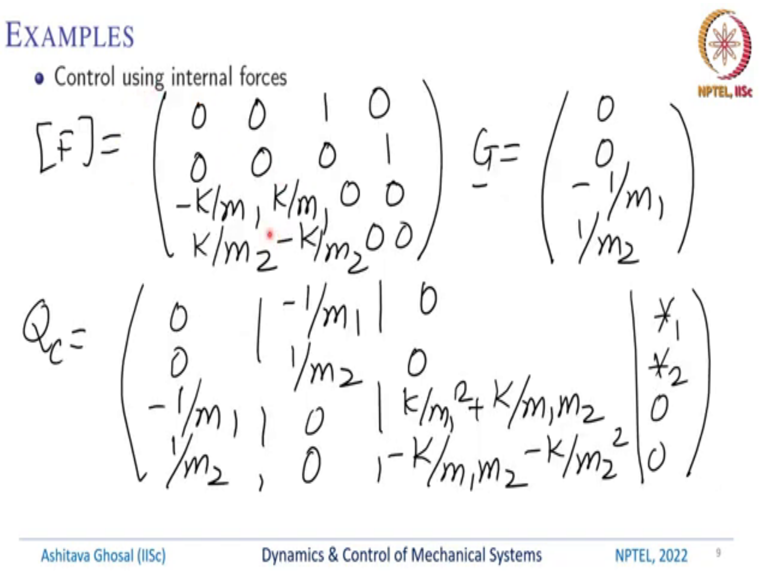


Let us look at another example which is a mechanics example, so, this is an example of a cart, there is an m_1 which is rolling on some flat ground. There is another cart which is m_2 and then both of these are connected by a spring K. The motion of this first one is given by X_1. The motion of the second one is given by X_2, this is the direction, and then there is a person sitting here in between the two carts and they are trying to push these two carts by applying equal and opposite force f.

He is pushing this cart by f and this is pushing this other cart by f, and both these forces are equal and opposite. Now, instead of pushing you could have probably some kind of a jet some fluid which is coming in from somewhere here and then it is pushing this cut by some f and this is pushing to the right this cart by f and to the left this cart by f. So, it need not be a person, you could have a mechanical system which is applying equal and opposite forces.

For such a system I can write the equations of motion. This is a very straight forward undergraduate set of equations where you have a mass and a spring. So, we can write $m_1\ddot{X}_1 + K\left(X_1 - X_2\right) = f$ and $m_2\ddot{X}_2 + K\left(X_2 - X_1\right) = f$. So, in this case it is $X_1 - X_2$ and in this case, it is $X_2 - X_1$. So, the state variables are X_1, X_2, $\dot{X}_1$, $\dot{X}_2$. I can write $\dot{X}_1$ as X_3, $\dot{X}_2$ as X_4 and $\dot{X}_3$ (I can use these 2 equations) and $\dot{X}_4$ I can use these 2 equations of motion to write $\dot{X}_3$ is $-K/m_1(X_1 - X_2) - f/m_1$, $\dot{X}_4$ is $-K/m_2(X_2 - X_1) + f/m_2$ So, this is very straightforward. Anybody who can sit down and write the equations of a motion of a mass which is two masses connected by a spring and some forces f, one is acting to the left and one is acting to the right for these two masses.

(Refer Slide Time: 16:45)

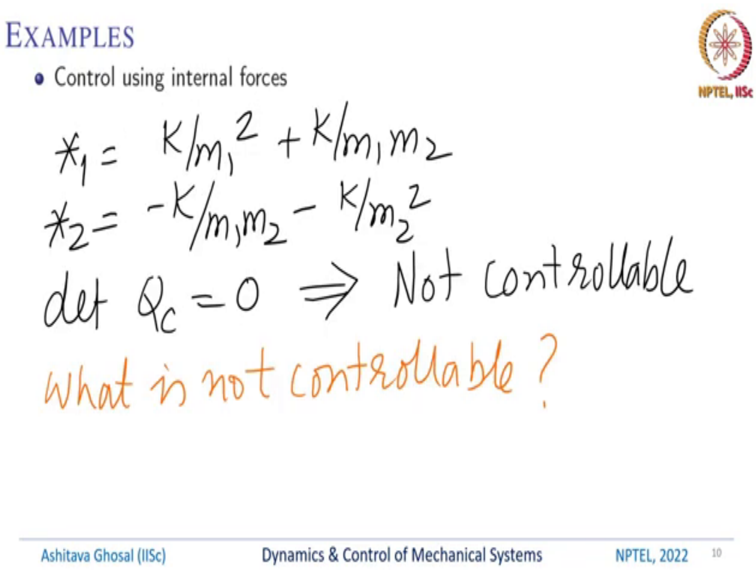


So, we can rewrite in the state space form. We have $\dot{X} = [F]\mathbf{X} + [G]u$, $[F]$ will be this matrix 4 x 4 matrix, so, we will have (0 0 1 0, 0 0 0 1). This element is $-K/m_1$, second element is $+K/m_1$. This is K/m_2 and this is $-K/m_2$. The $[G]$ vector will be 0 0 $-1/m_1$ and then $+1/m_2$.

And from $[F]$ and $[G]$ we can derive the controllability matrix. So, $[Q_c]$ is given by (0 0 $-1/m_1$ $1/m_2$), so that is $[G]$. Then the second column will be $[F][G]$. The third column will be $[F]^2[G]$ and the fourth column will be $[F]^3[G]$. $[F][G]$ will be ($-1/m_1$ $1/m_2$ 0 0), $[F]^2[G]$ is

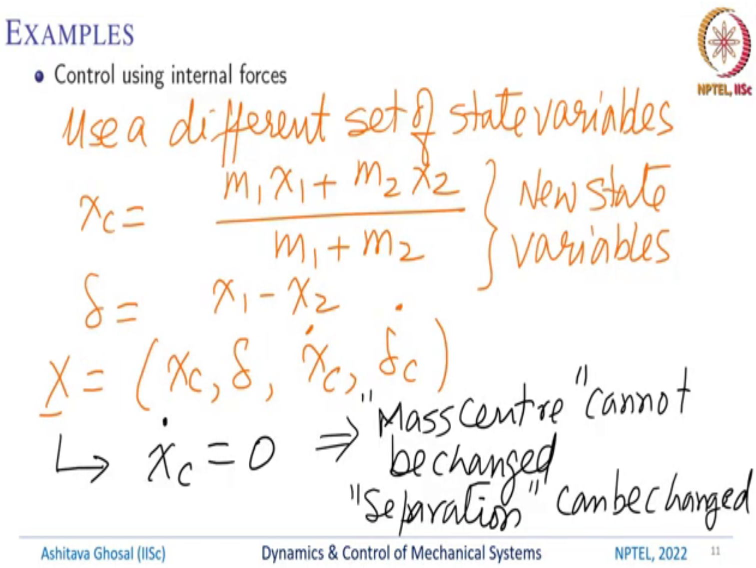
(0 0 $K/m_1^2 + K/m_1m_2$). This is ($-K/m_1m_2 - K/m_2^2$)and these two expressions will be from $[F]^3[G]$ and I will show you what these expressions are in the next slide.

(Refer Slide Time: 18:05)



So, the first expression *1 is $K/m_1^2 + K/m_1m_2$ and the second term, this is the (2, 4) term is $-K/m_1m_2 - K/m_2^2$. So, we have all the elements of the $[Q_c]$ matrix and we can compute the determinant. You can sit down and do it by hand on a sheet of paper or you can go ahead and use Maple. We have discussed this computer algebra system called Maple earlier and we can use this or in Matlab. There are these other, simpler computer algebra systems and this is a very, very simple application in computer algebra system. So, we can compute the determinant and determinant of $[Q_c]$ will be 0. What it means is that this system $\dot{X} = [F]\mathbf{X} + [G]u$ is not controllable. The national question is why is it not controllable?

(Refer Slide Time: 19:16)

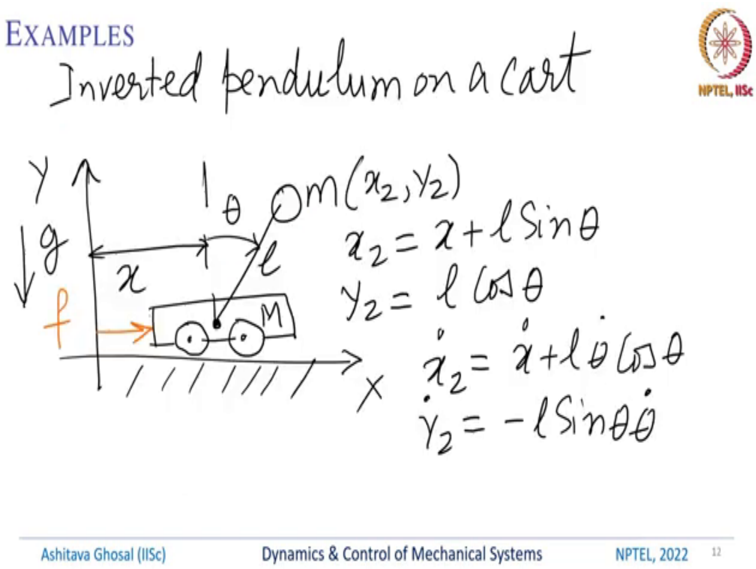


So, in order to find out why it is not controllable? We use a different set of state variables, so, we had used X_1 which was the motion of the first mass to the right X_2 which was the motion of the second mass, also to the right. Instead of that we will use X_C which is given by $(m_1 X_1 + m_2 X_2)/(m_1 + m_2)$. So, this is the change and δ which is $X_1 - X_2$. This is X_C and δ which can be obtained by some change of coordinates of X_1 and X_2. So, we basically do some linear transformation from X_1, X_2 to X_C and δ. So, delta you can see, is just $X_1 - X_2$, and what is X_C? It is some combination of mass into X_1 + mass into X_2 divided by the mass. My new state variables are X_C, δ, and $\dot{X}_C$ and $\dot{\delta}$.

Once if you convert from X_1, X_2, $\dot{X}_1$, $\dot{X}_2$ to X_C, δ, $\dot{X}_C$, $\dot{\delta}$, you will see that one of the differential equation is $\dot{X}_C = 0$. What is $\dot{X}_C$? This is basic mechanics. What you can see is X_C is nothing but the centre of mass of these two and what is δ -- it is the distance between two mass. This is the mass centre and this is the separation of the two carts or the two objects on wheels. What this is telling you is that I cannot change the mass centre. So, $\dot{X}_C$ will remain same whatever equal and opposite force that you apply f in both the opposite directions to the mass, the centre of mass remains at the same place. I cannot change the centre of mass. So, if I have a desired trajectory which says centre of mass needs to go from here to here then using f, which is u in this case, which is the control input in this case, I cannot change the centre of mass.

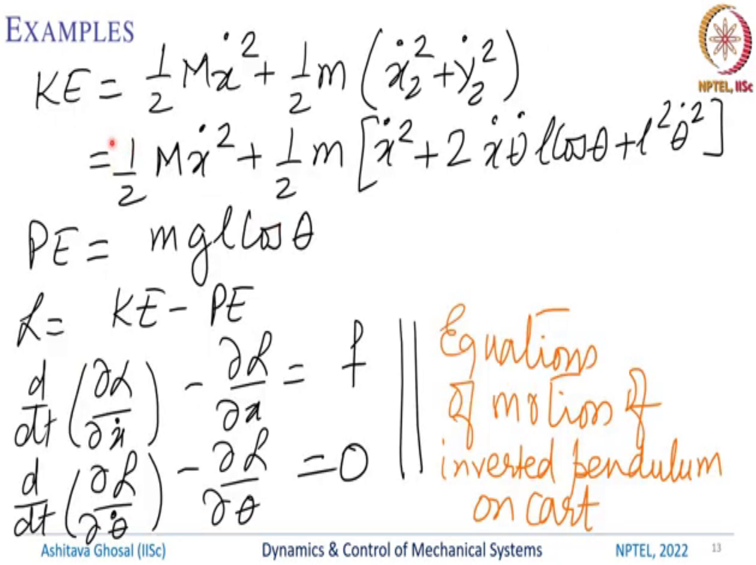
Whereas the separation can be changed because only $\dot{X}_C$ will turn out to be 0. So, physically what is happening is that we have f which is equal and opposite. You can think of it as a internal force in your system. the system consisting of two masses and a spring and this f is an internal force. So, you cannot control a system using internal force. Some state variables cannot be taken from some location to another desired **X** (t) in finite time.

(Refer Slide Time: 22:21)



Let us look at another interesting example. This is a very common and a very well-known example of a pendulum on a cart. We have this inverted pendulum on a cart. This cart can move along the X direction. So, it is like wheels so, this is a sketch of this inverted pendulum on a cart So, we have X and a Y axis gravity is acting this way. So, this pendulum has length l it has a mass m and I am going to denote the centre of this pivot where the pendulum is by a variable x. The angle of the pendulum, this tilting angle is θ. Hence, the mass coordinates x_2, y_2 can be written as $x_2 = x + l \sin\theta$ and y_2 is $l \cos\theta$. So, from here we can obtain the derivatives which is $\dot{x}_2$, $\dot{y}_2$ -- this is $\dot{x} + l\,\dot{\theta}\cos\theta$, $\dot{y}_2$ is $-l \sin\theta\,\dot{\theta}$.

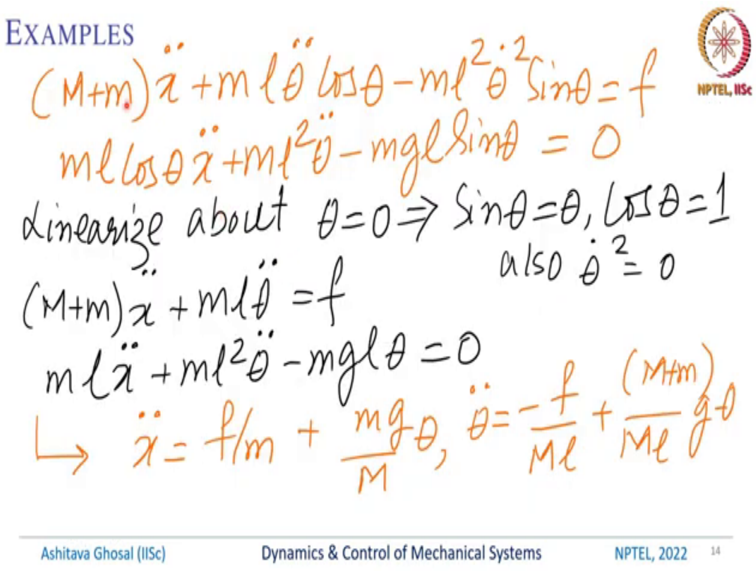
(Refer Slide Time: 23:34)



We can find the kinetic energy which is (1/2)M -- M is the mass of the cart and small m is the mass of the pendulum bob -- so, (1/2) $M\dot{x}^2$ + (1/2) m ($\dot{x}_2^2 + \dot{y}_2^2$), and you can expand this and we will get (1/2) $M\dot{x}^2$ + (1/2) m ($\dot{x}^2 + 2\,\dot{x}\,\dot{\theta}\,l\cos\theta + l^2\,\dot{\theta}^2$). This is very, very straight forward.

We have done enough equations of motion in a previous module and we can easily derive the kinetic energy of this very, very simple system. The potential energy is given by $mg\,l\cos\theta$. We can find the Lagrangian which is $KE - PE$ and then following the Lagrangian formulation, we take partial of Lagrangian with $\dot{x}$, partial of Lagrangian with $\dot{\theta}$, partial of Lagrangian with x, partial of Lagrangian with θ. And then the time derivative of partial of Lagrange with $\dot{x}$ and time derivative of partial of Lagrangian with $\dot{\theta}$, and assembled it together. The only force that is acting is along the X direction which is this force? There is no force in the θ for the generalized coordinate θ. The equation of motion of this inverted pendulum on cart can be derived because I know the kinetic energy, I know the potential energy and we can do all these partial derivatives and assemble the equations of motion.

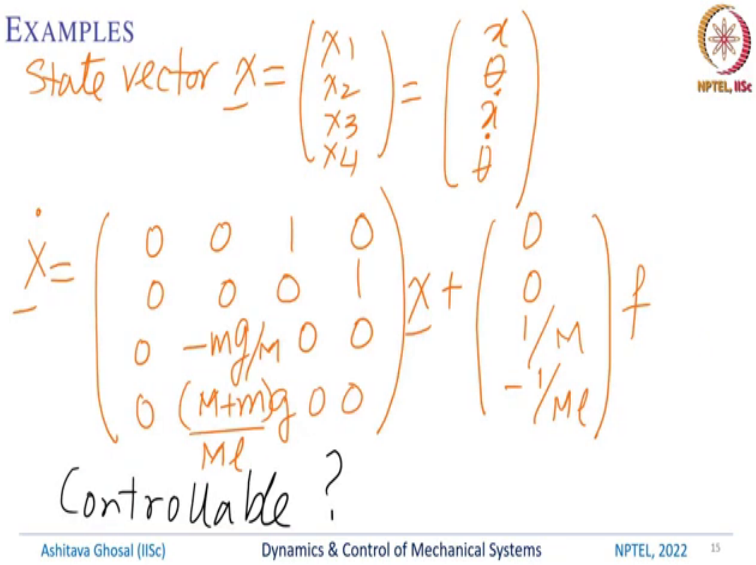
(Refer Slide Time: 25:23)



So, the equation of motion looks like this. It is $(M + m)\ddot{x} + m\,l\,\ddot{\theta}\cos\theta - m\,l^2\,\dot{\theta}^2\sin\theta = f$ and then $m\,l\cos\theta\,\ddot{x} + m\,l^2\,\ddot{\theta} - m\,g\,l\sin\theta = 0$. These are two non-linear equations -- you can see that there is a $\dot{\theta}^2$ term, there is a $\sin\theta$ - so, it is a nonlinear- and $l\sin\theta$ is also there.

These are non-linear equations, and we can linearize about θ equals 0 which basically means substitute $\sin\theta$ is θ, $\cos\theta$ is 1 and also $\dot{\theta}^2 = 0$. We linearize these equations, and we get this linear ordinary differential equations. We can easily see from the first equation. We will get $(M + m)\ddot{x} + m\,l\,\ddot{\theta}$ -- $\cos\theta$ will be 1 and $\sin\theta$ will be θ, but $\dot{\theta}^2$ will go to 0. And the second equation is $m\,l\,\ddot{x} + m\,l^2\,\ddot{\theta} - m\,g\,l\,\theta = 0$. This equation can be written as $\ddot{x} = f/m$ (the first one) $+ m\,g\,\theta/M$ and $\ddot{\theta} = -f/M\,l + ((M + m)/M\,l)\,g\,\theta$. These are two linearized differential equations -- second order ordinary differential equations obtained from the full non-linear OD is for the pendulum on a cart for an inverted pendulum on a cart.

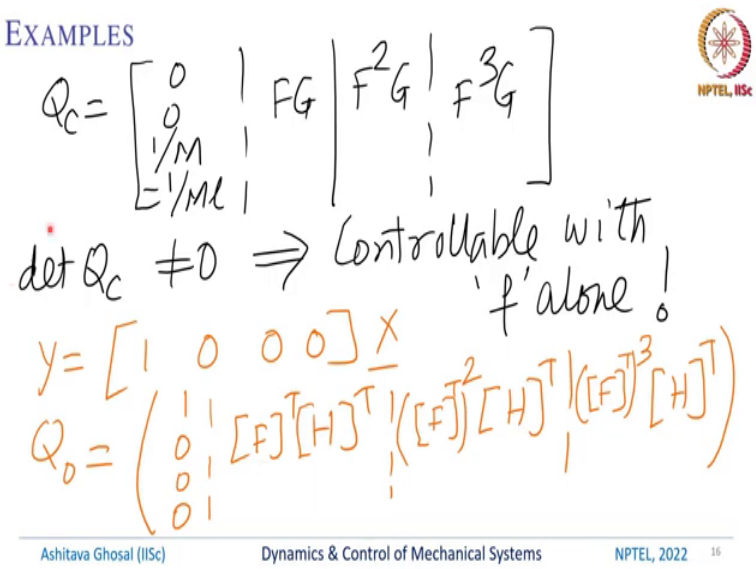
(Refer Slide Time: 27:38)



We can write this in the state space form and the state vector is X_1, X_2, X_3, X_4. There are four of them and these are nothing but X_1 is x, X_2 is θ, X_3 is $\dot{x}$, X_4 is $\dot{\theta}$, and we can write it in the form $\dot{X} = [F]\mathbf{X} + [G]$u. What is u? Here u is the force which you are applying on the cart and $[F]$ is a 4 by 4 matrix which is [0 0 1 0, 0 0 0 1, 0 $-mg/M$ 0 0]. Then [0 $(M + m)\, g\, /M\, l$ 0 0] and f will be (0 0 $1/M$ $-1/M\, l$)

So, our natural question is this system controllable? Meaning can I take x and θ -- the state variables (x θ $\dot{x}$ $\dot{\theta}$) from some initial state to a final state by applying this force f. So, we want to test for controllability.

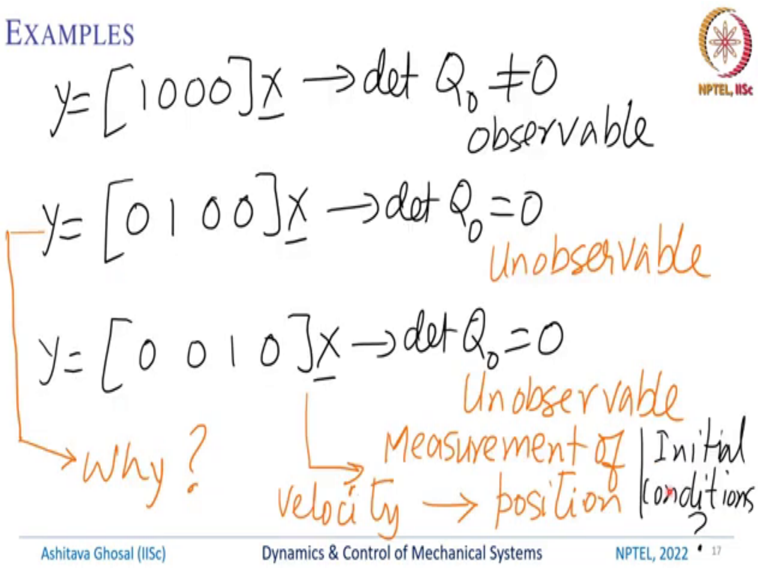
(Refer Slide Time: 28:59)



We can obtain the $[Q_c]$ matrix. Again, the first column will be $[G]$, second is $[F][G]$, third is $[F]^2[G]$ and fourth column $[F]^3[G]$. I am not writing all the terms, but you can easily derive them and you can show that this determinant of $[Q_c]$ is not equal to 0. So, hence it is controllable with f alone. Remember, there are two variables, one is x one is θ. There is only one force f which is sort of acting along this generalized coordinate x, even then it is controllable. Meaning, let us go back to the basic definition of controllability, I can take the system from some initial state to another final state in finite time by applying this force f.

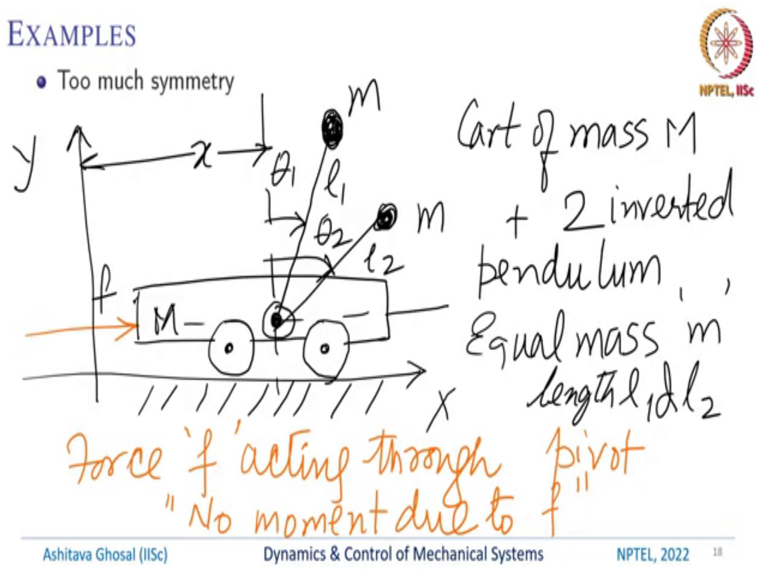
Now, let us look at the observability. We are only going to measure x. So, y, $[H]$ is [1 0 0 0] **X**. So, what is $[Q_o]$ -- $[H]^T$, $[F]^T[H]^T$, $([F]^T)^2[H]^T$, $([F]^T)^3[H]^T$. So, again we can obtain, we know what is $[F]$, we know what is $[H]$. We can obtain this matrix in symbolic form, using some computer algebra system like Maple.

(Refer Slide Time: 30:31)



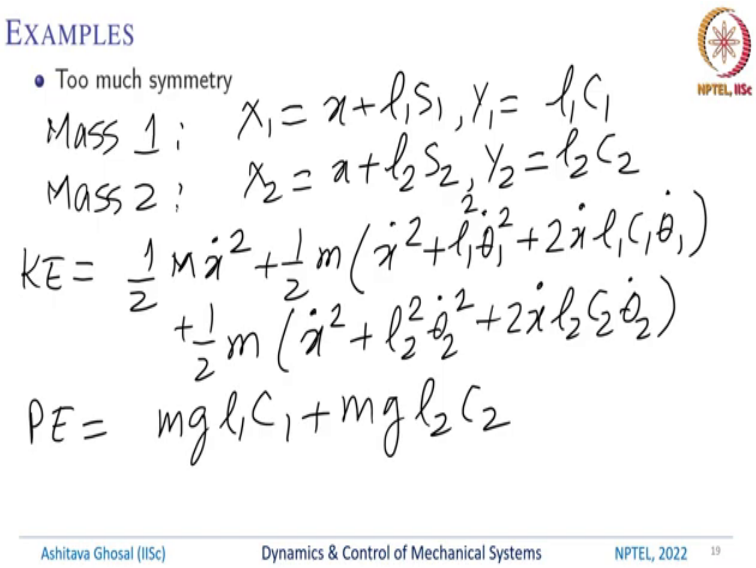
If you choose y as x, then determinant is not equal to 0. Hence the system is observable. So, what were the state vectors? $(x\ \theta\ \dot{x}\ \ \dot{\theta})$ so, if I just measure the position of the cart then the system is observable. If I measure only θ but the force is applied on the cart - so, 0 1 0 0 then determinant of $[Q_o]$ is 0. Hence it is unobservable, meaning that if I just were to measure θ then you cannot do it. We can also choose to measure only $\dot{x}$ which is $[H]$ is [0 0 1 0] **X**, again, it is not observable. Basically what it means is, if you were to measure the velocity, this is what you are doing -- you are measuring $\dot{x}$ -- you are measuring the velocity, I cannot obtain all the states **X**. Basically, it is sort of intuitively correct. If you measure the velocity, you can say that we can integrate to obtain the position of that cart. However, that is not true because we do not have the initial conditions -- we do not know what is the initial condition. Hence, by measuring velocity which and by integration we will still not be able to find the position of the cart. This is more interesting. We are measuring θ (not x) but it is still not observable. So, think about it why the system is not observable when you measure θ.

(Refer Slide Time: 32:25)



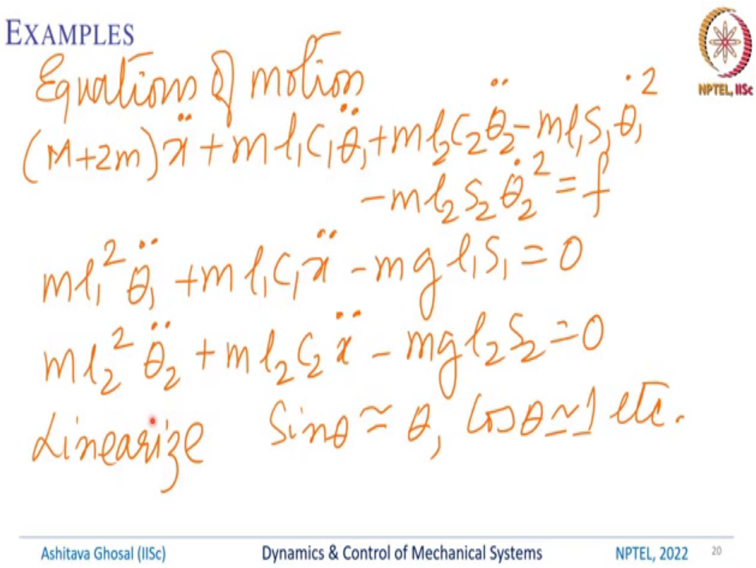
Let us take one more example which is a little bit interesting example. It is an extension of this inverted pendulum on the cart but now we have two inverted pendulums. This is the same cart - we have a force which is applied, the mass of the cart is M and the location of the hinge of both the pendulum is given by x. The first pendulum is rotating by θ_1, it has m and l_1. The second pendulum is rotating by θ_2, it is m and l_2 -- both the masses are equal but the lengths are different -- this is l_1 and l_2 and the force is acting in the cart which is along the hinge. Basically, there is no moment due to this force, it is just a linear force.

(Refer Slide Time: 33:26)



Again we can derive the equations of motion of this system. Before we do that, we find the position of the first bob pendulum bob which is X_1 which is $x + l_1 s_1$. The Y coordinate is $l_1 c_1$. For the mass 2, X_2 is $x + l_2 s_2$ and Y coordinate is $l_2 c_2$. I can find the kinetic energy of the cart and the kinetic energy of the mass -- which is (1/2) $M \dot{x}^2$ + (1/2) $m \dot{X}_1^2$ and $\dot{X}_1^2$ and $\dot{Y}_1^2$. And similarly, for mass 2, I can find (1/2) m ($\dot{x}^2 + l^2 \dot{\theta}_2^2 + 2x l_2 c_2 \dot{\theta}_2$). Again, we have seen this kind of derivation earlier, even in the last example, but we have seen similar derivations when we get the dynamics for a single pendulum. The potential energy of the system is $mg\ l_1 c_1$ for the first mass and the second mass is $mg\ l_2 c_2$. So, we have the kinetic energy, we have the potential energy, again we can use the Lagrangian formulation and derive the equations of motion.

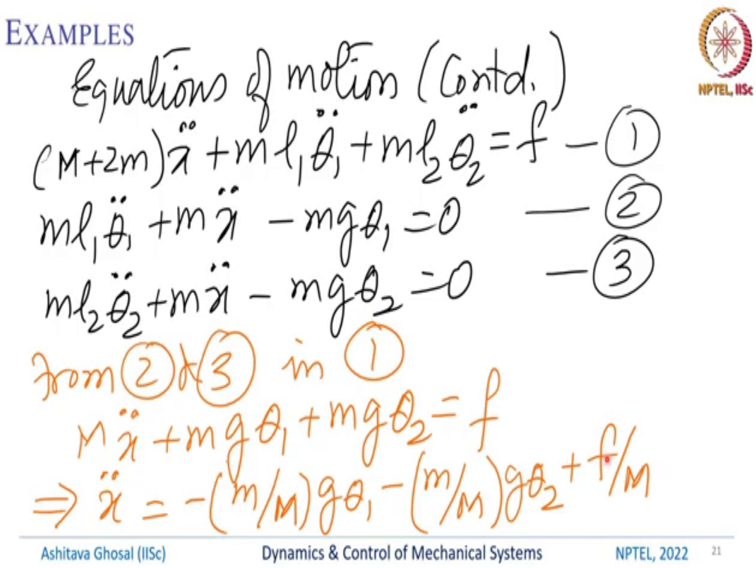
(Refer Slide Time: 34:53)



The equations of motion will be three of them because there is one cart and then there are these two pendulums. So, if you do some simple calculations of finding all the partial derivatives and all the time derivatives and then we organize it together, which I am not going to do because by now you should be able to do it very easily, you will get three such equations of motion.

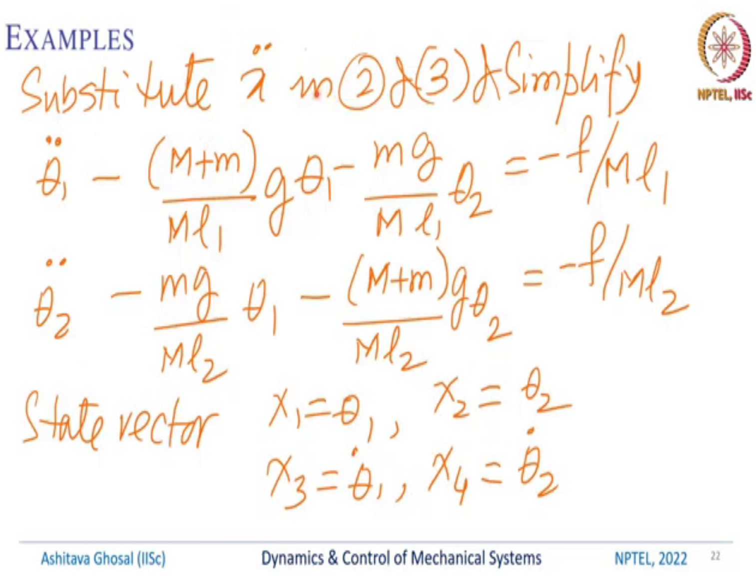
And these are $(M + 2m)\ddot{x} + m\, l_1 c_1\, \ddot{\theta}_1 + m\, l_2 c_2\, \ddot{\theta}_2 - m\, l_1 s_1\, \dot{\theta}_1^{\,2} - m\, l_2 s_2\, \dot{\theta}_2^{\,2} = f$. And for the analyse coordinates θ_1 and θ_2, we will get $m\, l_1^2\, \ddot{\theta}_1 + m\, l_1 c_1\, \ddot{x} - m\, g\, l_1\, s_1 = 0$ and exactly very similar for the second pendulum which is $m\, l_2^2\, \ddot{\theta}_2 + m\, l_2 c_2\, \ddot{x} - m\, g\, l_2\, s_2 = 0$. So, we have three non-linear ODE's we can again linearize by substituting sin θ equals to θ,
cos θ is 1 and so on.

(Refer Slide Time: 36:21)



And then we will get three linearized equations of motion for this cart with two pendulum. The first equation is $(M + 2m)\ddot{x} + m l_1 \ddot{\theta}_1 + m l_2 \ddot{\theta}_2 = f$. This is equation (1). $m l_1 \ddot{\theta}_1 + m\ddot{x} - m g \theta_1 = 0$, and similarly, for the second bob, second pendulum, which is $m l_2 \ddot{\theta}_2 + m\ddot{x} - m g \theta_2 = 0$. We have three equations in three unknowns, θ_1, θ_2 and x. From (2) and (3) -- you substitute (2) and (3) in (1) -- you will get $M \ddot{x} + m g \theta_1 + m g \theta_2 = f$. You can see that this is some $M \ddot{x} + m l_1 \ddot{\theta}_1 = m g \theta_1$. Similarly, this is $m g \theta_2$. We can get rid of these $m l_1 \ddot{\theta}_1$ and $m l_2 \ddot{\theta}_2$ here from these two equations and you will get this. And hence we can solve for $\ddot{x} = -(m/M) g \theta_1 - (m/M) g \theta_2 + f / m$. In this example, we can get rid of, we can write $\ddot{x}$ directly in terms of θ_1 and θ_2.

(Refer Slide Time: 38:15)



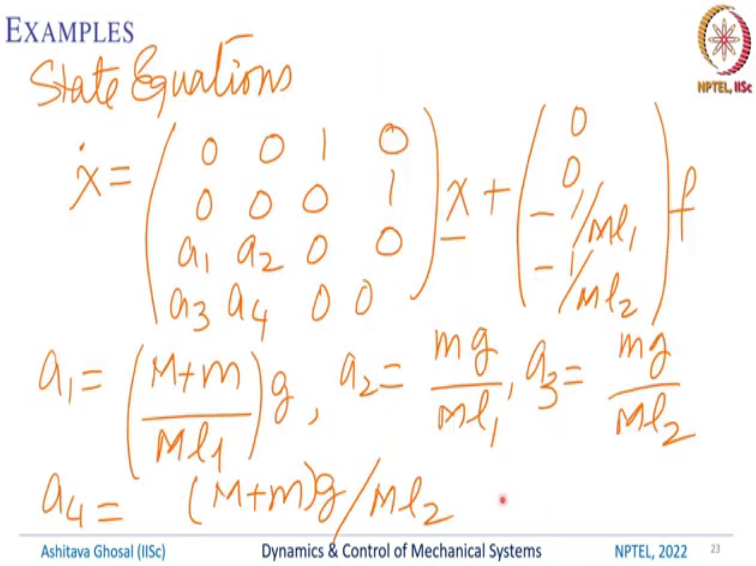
So, we can substitute $\ddot{x}$ back into (2) and (3) and simplify. Basically, what have we done? We have gotten rid of $\ddot{x}$ -- from those three equations in three unknowns, we have two equations in two unknowns. The first equation is

$\ddot{\theta}_1 - [(M+m)/M\, l_1]\, g\, \theta_1 - [mg/\, M\, l_1]\, \theta_2 = -f/\, M\, l_1$ and likewise the second equation.

Second equation is $\ddot{\theta}_2 - [m\, g/M\, l_2]\ \theta_1 - [(M+m)/M\, l_2]\, g\, \theta_2 = -f/\, M\, l_2$.

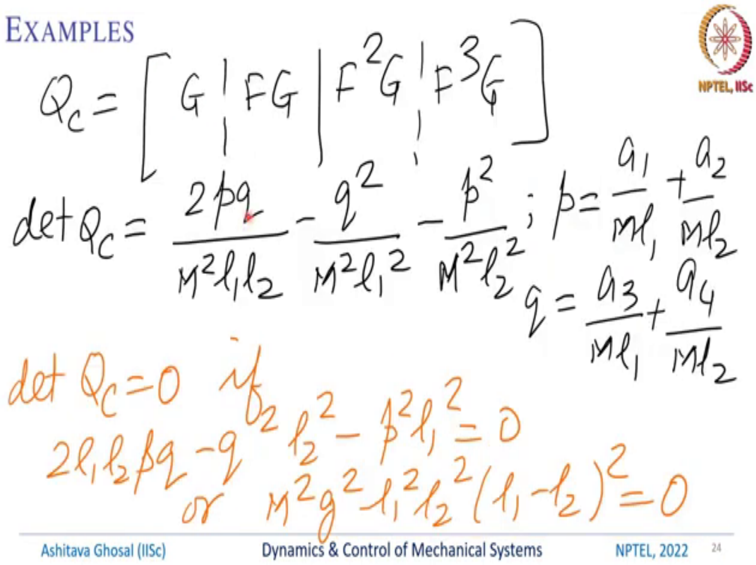
These are two second order linear ordinary differential equations and we can write it in state space form and the state vector in this case is X_1 is θ_1, X_2 is θ_2, X_3 is $\dot{\theta}_1$, X_4 is $\dot{\theta}_2$. Now, we have done lot of work but eventually this is what I want to get to.

(Refer Slide Time: 39:33)



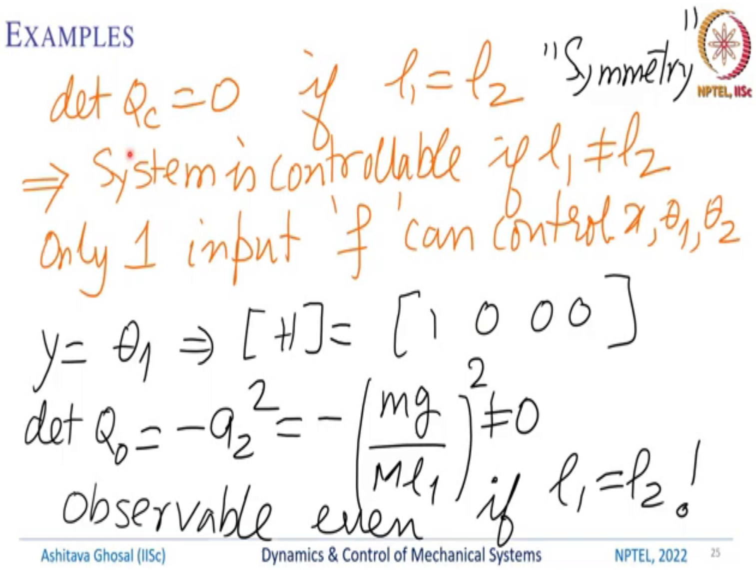
So, we want to write the state equations as $\dot{X} = [F]\mathbf{X} + [G]u$. In this case u is f which is the force and $[G]$ will be $(0\ 0\ -1/M\,l_1\ \ -1/M\,l_2)$. $[F]$ is a little bit more complicated. The first two rows are straight forward which we have seen happening many times it is [0 0 1 0] and [0 0 0 1] and these terms, a_1, a_2, a_3 and a_4 are slightly more complex terms -- not very complex but in order to save space, I am writing a_1 here, a_2 here, a_3 and a_4. So, a_1 is $[(M+m)/M\,l_1]\,g$, a_2 is $[m\,g/M\,l_1]$, a_3 is $[m\,g/M\,l_2]$ and a_4 is $[(M+m)g/M\,l_2]$. You can work this out or you can again use the computer algebra system to derive the state equations. It is not very hard, and this is a reasonably simple system, as far as computer algebra is concerned (it is not). It is a nice system as far as control system is concerned and state space representation of a mechanical system is concerned. But all these simplifications and all these writing, in the state space form can be very easily done using a computer algebra system.

(Refer Slide Time: 41:14)



Let us now find out what is the controllability matrix $[Q_c]$. As usual, the first column is $[G]$, second column is $[F][G]$, third column is $[F]^2[G]$ and fourth column is $[F]^3[G]$. So, we know what is $[G]$, $[F]$, hence you can obtain this controllability matrix. It turns out that the determinant of this $[Q_c]$ is given by this complicated expression. It is $2pq/(M^2\,l_1\,l_2) - q^2/M^2\,l_1^2 - p^2/M^2\,l_2^2$, where p can be written in terms of a_1, a_2 and q can be written in terms of a_3 and a_4, like this. p is $a_1/M\,l_1 + a_2/M\,l_2$, q is $a_3/M\,l_1 + a_4/M\,l_2$. It turns out that this determinant of $[Q_c]$ can be 0 if some expression involving l_1, l_2, p, q is 0. So, $2\,l_1\,l_2\,p\,q - q^2\,l_2^2 - p^2\,l_1^2$ is 0. This can be simplified as capital $M^2\,l_1^2\,l_2^2\,(l_1 - l_2)^2 = 0$. This is what I wanted to do right from the beginning -- I wanted to find out under what conditions that cart with two pendulum is not controllable. When is the determinant of the controllability matrix for this system is 0. And you can see very clearly it will be 0 when l_1 will be equal to l_2 because then this term will go to 0.

(Refer Slide Time: 43:22)

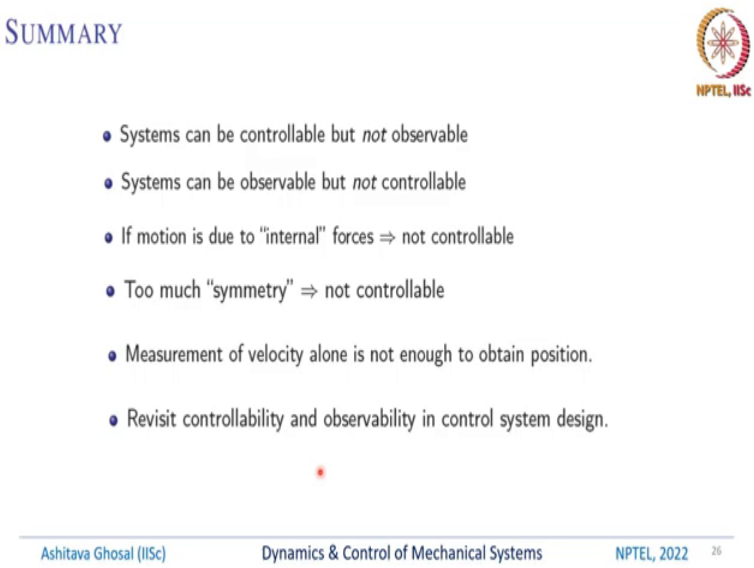


So, hence determinant of $[Q_c] = 0$, if $l_1 = l_2$. So, this is what I mean by saying that there is too much symmetry -- we have a mass which is going translating and that there are these two pendulums with equal masses but the lengths are also equal. So, it is in some sense, very special and very symmetrical. So, this system is not controllable if $l_1 = l_2$ and is controllable if $l_1 \neq l_2$ -- then you could see that the determinant of $[Q_c]$ is not equal to 0.

So, it is in the sense it is quite interesting. We have only one input, remember f and if l_1 is not equal to l_2, I can control three variables x, θ_1, θ_2 -- whole system is controllable. Meaning I can take this two pendulum system on a cart and I can go from one state to another state in finite time by applying one single input which is f. If, on the other hand $l_1 = l_2$, then it is not controllable, and I cannot transfer from some initial state to another final state by applying f in finite time.

Let us look at observability of this system. Let us assume that we are only going to measure one of the rotation of the pendulum which is θ_1 -- this is θ_1, θ_2 and x but we are only going to measure θ_1. So, $[H]$ is what (1 0 0 0) -- because remember, we have two differential equations in $\ddot{\theta}_1$ and $\ddot{\theta}_2$ and the state variables are θ_1, θ_2, $\dot{\theta}_1$, $\dot{\theta}_2$. In this final form after I have gotten rid of x. So, $[H]$ is [1 0 0 0] and we can again find the determinant of this $[Q_o]$ matrix. $[Q_o]$ is $H]^T$, $[F]^T[H]^T$ and so on and we will get it as $-a_2^2$ which is $-(mg/M\,l_1)^2$. This is never going to be equal to 0, g is positive, m is positive. None of the elements are 0. So, the really interesting part of the story is the following if l_1 were to be equal to l_2, it is not controllable but it is observable -- because we do not have anything, any condition, under which this observability matrix is singular or the determinant of the observability matrix is 0. With this, we are going to stop examples and let us continue.

(Refer Slide Time: 46:38)



In summary, a system can be controllable but not observable.

A system can be observable but not controllable. I can cook up lots of $[F]$ matrix and $[G]$ matrix and we can get all these possibilities.

In a physical system, in an actual mechanical system, and not just cooked up $[F]$ and $[G]$ matrices and $[H]$ matrices, if the motion is due to internal forces then the system is not controllable -- internal means equal and opposite like the example which I showed you about, these two carts connected by a spring and two forces being applied in the opposite direction -- equal and opposite.

If there is too much symmetry, this setup is very special, like that example which I gave you $l_1 = l_2$, then it is not controllable.

Measurement of velocity alone is not enough to obtain position. We can see that if I just measure the velocity and then hope to integrate and obtain the position, which is also one of the states, it is not possible because we do not know what are the initial conditions for the integration.

We will revisit controllability and observability in control system design in the next module. Both these $[Q_c]$ and $[Q_o]$ and all these are very, very useful concepts not only to study or analyse control system but also to design control systems.