Dynamics and Control of Mechanical Systems
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Lecture – 23
Stability of Dynamical Systems

Welcome to this NPTEL lectures on Dynamics and Control of Mechanical Systems. My name is Ashitava Ghosal, I am a professor in the Department of Mechanical Engineering in the centre for product design and manufacturing and also in the Robert Bosch centre for cyber physical systems, Indian Institute of Science, Bangalore. So, in this lecture we will look at Stability, Controllability and Observability of Systems.

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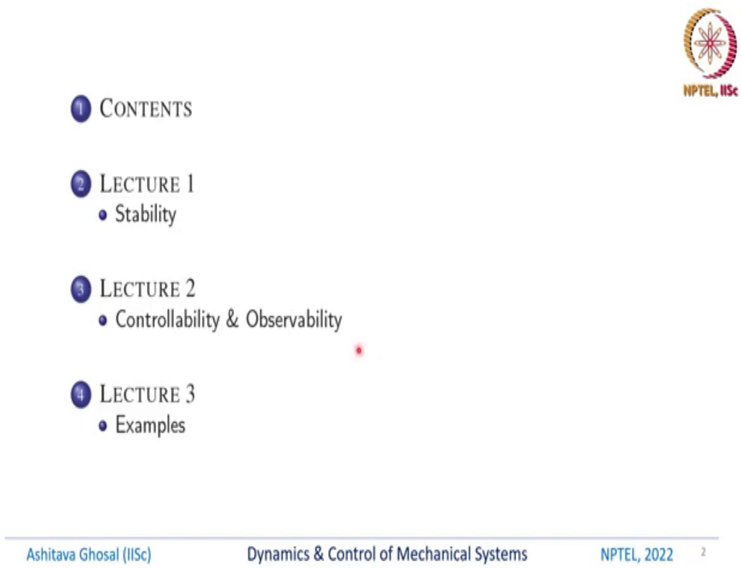

In this module there will be three lectures, in the first lecture we will look at stability of systems and mechanical systems in particular. In the second lecture we will look at controllability and observability of most of the time linear time in variant systems. And in the third lecture, I will show you lots of examples on stability, controllability and observability.

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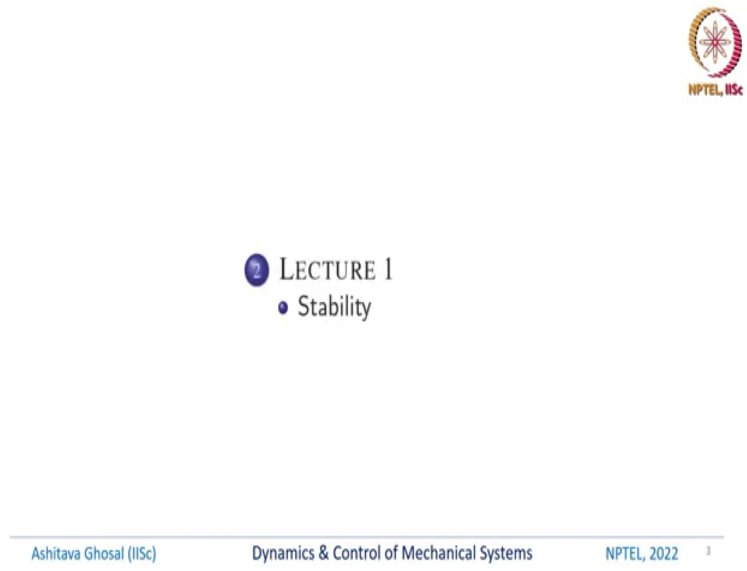

So, let us start, the first lecture is on stability.

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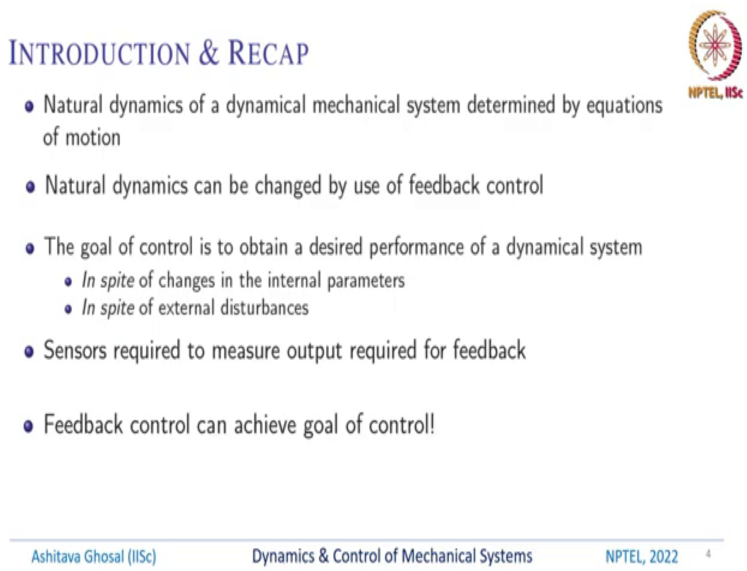

Quick introduction and recap, we have seen that the natural dynamics of dynamical mechanical systems is determined by the equations of motion. So, we could derive the equations of motion using Newton Euler-Lagrangian approaches. And then we could solve the equations of motion and then we could see that for an external force the system parameters given.

And the generalized co-ordinates $q(t)$ would evolve in time according to the equations of motion so, this is the natural dynamics of a mechanical system. And I have also showed you at least for two examples that the natural dynamics could be changed by use of feedback control. So, remember we had a mass in which force was acting so, it was $F = m\,a$, we could integrate the equations of motion for a given F and m.

And I showed you what $X(t)$ as a function of time would look like, it is basically a parabolic curve. And I showed you that if you could measure the velocity or you could measure the position and feed it back. And then change the force with which the mass is subjected to then, we could change the nature of the output which was the velocity of the mass. So, I showed you that the natural dynamics which in this case is $X(t)$ as a parabola, could be altered by means of feedback control.

So, the goal of control is to obtain a desired performance of a dynamical system and in spite of changes in the internal parameters and in spite of external disturbances. So, in the last week, I showed you this example of a single link being driven by a DC motor and then if the friction or if some parameters of the system changes. Then with feedback, the effect of these changes in the internal parameters will not be seen in the output of the system.

Same thing I showed you that if there are external disturbances acting and if you could choose the controller gains in some particular way the effect of the external disturbances would not be seen in the output. And as a result of feedback both of these could be achieved. We also saw that sensors are required to measure the output which in turn could be fed back. So, feedback control can achieve the goal of control that is what I showed you.

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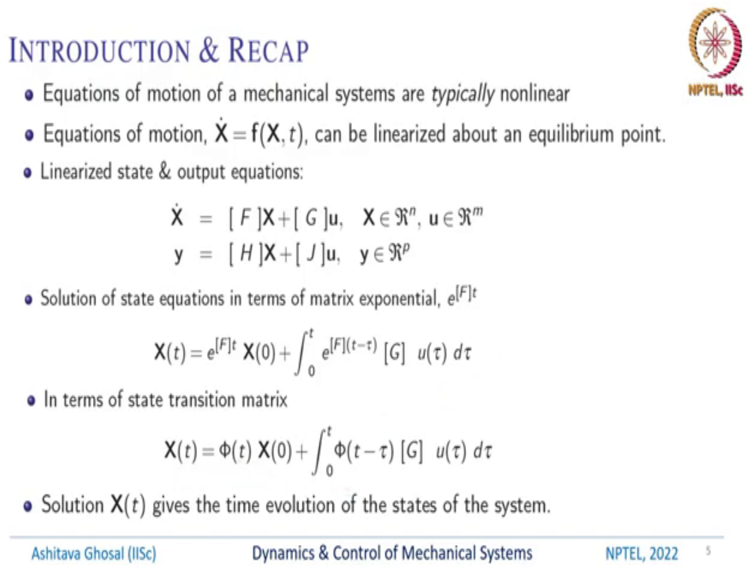

Let us continue, the equation of motion of a mechanical systems are typically non-linear. You could write the equations of motion as $\dot{X} = f(X, t)$ so, this is the state space form of a typically second order differential equation which we get when we apply Newton's law or when we apply Euler's equation or when we use the Lagrangian formulation. So, these equations of motion could also be linearized about an equilibrium point.

And if you recall the equilibrium points are all X such that $f(X, t) = 0$ for all t. So, we would have to solve $f(X) = 0$ and we could get more than one equilibrium point depending on the nature of that function $f(X)$. So, once we linearize about an equilibrium point, we will get these two linearized state and output equations. So, we had $\dot{X} = [F]X + [G]u$, X is n dimensional, $u \, is \, m$ dimensional and $[F]$ is a matrix of constants which is $n\times n$ matrix, $[G]$ is a matrix of $n\times m$ dimension so, this is the state equation.

We also could have the output equation which is basically what is the output of the system which is denoted by y? And that could be given by $[H]X + [J]\, u$. And there could be only p of the states or which we could measure so, y could be dimension of $p \, times$ 1. Then I also showed you, what is the nature of the solution of the state equations in terms of a matrix and exponential $e^{[F]t}$?

So, $X(t) = e^{[F]t}X(0)$. And then this $\int_0^t t\, e^{[F]t} - \tau\, [G]u(\tau)\, d\tau$. So, this is like a convolution and then in terms of a state transition matrix which is $\phi(t)$, I showed you that $X(t)$ could be written as $\phi(t)$ into $X(0)$ and again $\int_0^t \phi(t - \tau)\, [G]u(\tau)\, d\tau$. So, $\phi(t)$ is nothing but $e^{[F]t}$.

So, this is the $n\times n$ matrix and I showed you how to find this $e^{[F]t}$? There are several methods which we discussed last week. So, let us assume we can find out this $\phi(t)$. So, what does this $X(t)$ tell you? It gives you the time evolution of the states of the system. So, if I give you what is $u(t)$ which is the input in some sense like a force or an external input? Then it tells you the solution of this equation tells you how $X(t)$ will change with time.

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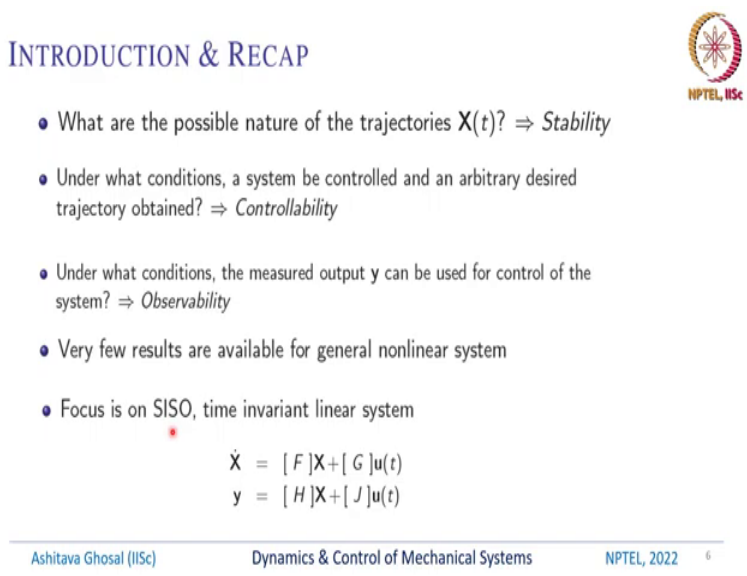

So, the question is what are the possible nature of the trajectories $X(t)$? So, this is the topic under stability. So, we will look at the trajectories $X(t)$ and that this is intimately related to whether the system is stable or not. We can also look at under what conditions systems can be controlled and an arbitrary desired trajectory obtained. So, remember one of the goal of feedback is to achieve a desired trajectory.

I can change the natural dynamics of a system by means of feedback and by application of a proportional or a controller gain and then we could show you that I could achieve the desired trajectory. So, the natural question is can we do that all the time? And this is intimately connected to this notion of controllability. The third important concept is under what conditions the measured output y can be used for control of the system?

So, most of the time we will not be measuring all the states. So, if the number of states is n we could be only measuring p of them. So, under what conditions these p measurements can be used to control the system? So, there are very few results for general nonlinear systems, both for stability, controllability and observability. So, in this module and in fact most of this course, the focus is on SISO system.

So, what is SISO? Again, we have discussed this earlier, single input, single output system and also we will look at only time invariant linear systems. So, again what is time invariant linear SISO system? We have $\dot{X} = [F]X + [G]u(t)$ and $y = [H]X + [J]\, u(t)$, u will be of single dimension. So, u is 1×1 similarly, y would be 1×1 for SISO system.

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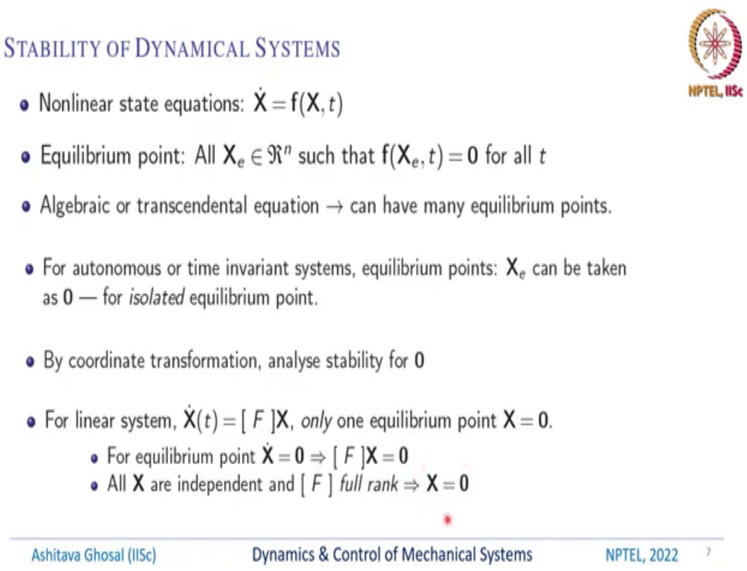

So, let us continue with this topic of stability of dynamical systems. So, we start with the general nonlinear state equations which is $\dot{X} = f(X, t)$ we find the equilibrium points as I mentioned all X_e such that $f(X_e, t) = 0$ is 0 for all t. So, if this are n state equations then X_e is of dimension $n\times 1$. And since this right-hand side could be an algebraic or a transcendental equation, we can have many equilibrium points.

So, for autonomous or time invariant system, the equilibrium point X_e can be taken to be 0 for isolated equilibrium points. So, for example if X_e were to be say let us say some number 3, 5. I could always do a coordinate transformation and analyse the stability about 0, 0. This is possible if you have what are called as isolated equilibrium points, roughly speaking what is an isolated equilibrium point? It is a single element.

So, the equilibrium is not along a line or in a plane. So, for example if you have a ball which is on a flat surface, the every point on this flat surface is roughly speaking like an equilibrium point. So, we are not interested in those kinds of system, where the equilibrium point is like a line or a surface. But if it is an isolated equilibrium point then we can transfer that equilibrium point to the origin and we can look at the stability about the origin.

So, for linear systems if you have a equation which is $\dot{X}(t) = [F]X$ so, F is now a constant matrix, there can be only one equilibrium point which is $X = 0$, a very simple proof. So, for equilibrium point we have to set $\dot{X} = 0$ which implies $[F]X = 0$. Now, we have to assume that we have chosen all the state variables are independent or $[F]$ is a full rank.

So, hence if you have a matrix equation $[F]X = 0$ and determinant of $[F]$ is $\neq 0$. So then $X = 0$ is the only possible solution. So, for linear systems our equilibrium point is always the origin of the state space.

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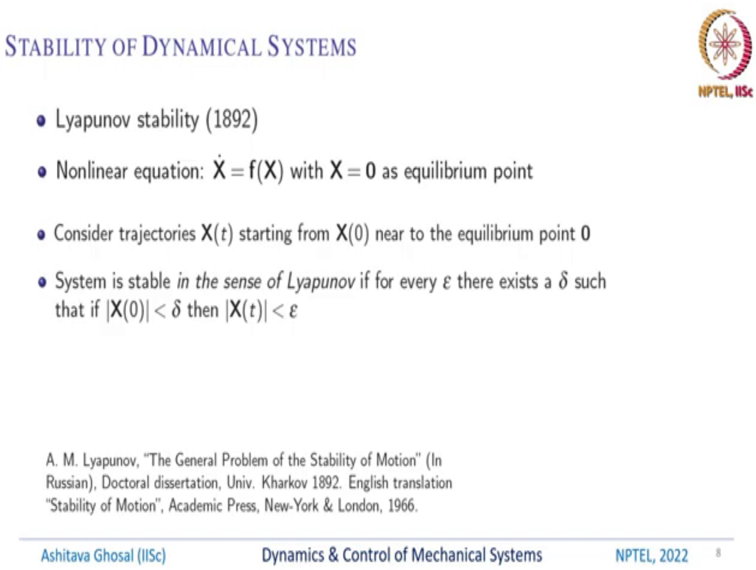

So, this in the stability of dynamical systems we have this very famous person called Lyapunov. So and we even now follow Lyapunov stability so, Lyapunov in 1892 much more than 100 years back, he wrote this paper, the general problem of stability of motion this is his doctoral dissertation and then he wrote some papers also. So, this is available as an English translation in stability of motion academic press 1966.

So, in 1892 he studied this notion or the problem of stability of dynamical systems. So, he came up with the following so, if you have a non-linear equation which is $\dot{X}(t) = [F]X$ and $X = 0$ is the equilibrium point. We consider trajectories $X(t)$ starting from $X(0)$ near the equilibrium point 0, I will show you pictures in a few moments. The system is stable in the sense of Lyapunov or according to Lyapunov if for every ϵ there exists a δ such that if $|X(0)| < \delta$ then $|X(t)| < \epsilon$.

So, this is a very abstract definition but basically what Lyapunov is trying to say is? That we have an equilibrium point which is $X(0)$in this case and I start near to the equilibrium point and let the trajectories evolve in time. So, if it is starting near to this equilibrium point which is basically that it is at most at a distance δ from the equilibrium point. And if the trajectory does not go outside, the region which is this ϵ then the system is stable.

So, what is δ and ϵ? So, this is a very abstract definition so, we cannot really say what is ϵ and δ.

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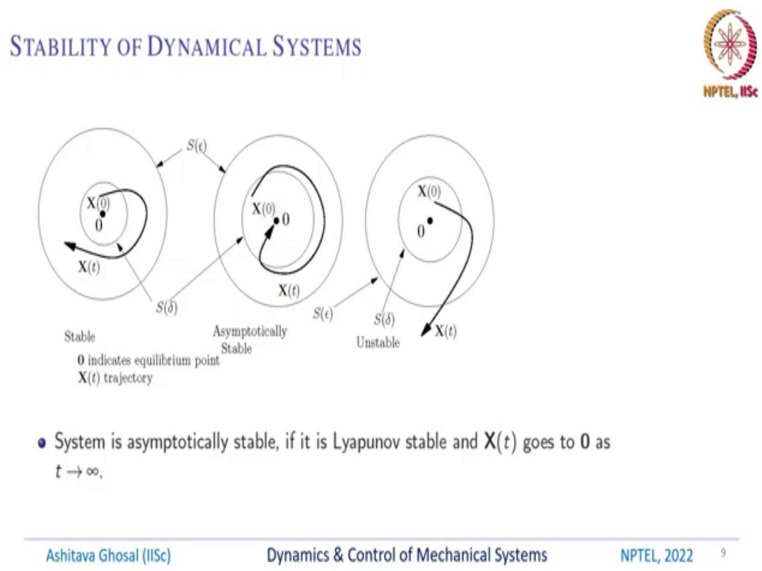

But nevertheless, here are the pictorial ways of trying to say what is Lyapunov stability. So, the leftmost picture here shows 0 is the equilibrium point and I start from a point which is $X(0)$ which is very close or in some region around this equilibrium point. And what is this region S (δ)? So, it is a ball or a sphere of radius δ around this equilibrium point 0 so, as the system evolves in time so, due to the action of $u(t)$ or external forces.

So, this is the trajectory $X(t)$ so, if this trajectory does not go outside another sphere which is S ϵ then according to Lyapunov or in his formal definition of Lyapunov stability this is stable. Another possibility is that we start from $X(0)$ which is inside this ball of radius δ and then after some time eventually the trajectories go to the equilibrium point. So, see the difference here it is $X(0)$ and it is staying inside another ball.

Whereas here it is starting from some point which is close to the equilibrium point and then it comes back to equilibrium point as which time so, this according to Lyapunov is called asymptotically stable. And the final case is that we start from some $X(0)$ near the equilibrium point 0 then if the trajectories goes outside this $S(\epsilon)$ then it is unstable. So, this is a nice and formal way to look at Lyapunov stability.

So, the system is asymptotically stable if it is Lyapunov stable so, it must be stable firstly and $X(t)$ goes to 0 as $t \to \infty$. So, the trajectories come back to the equilibrium point 0.

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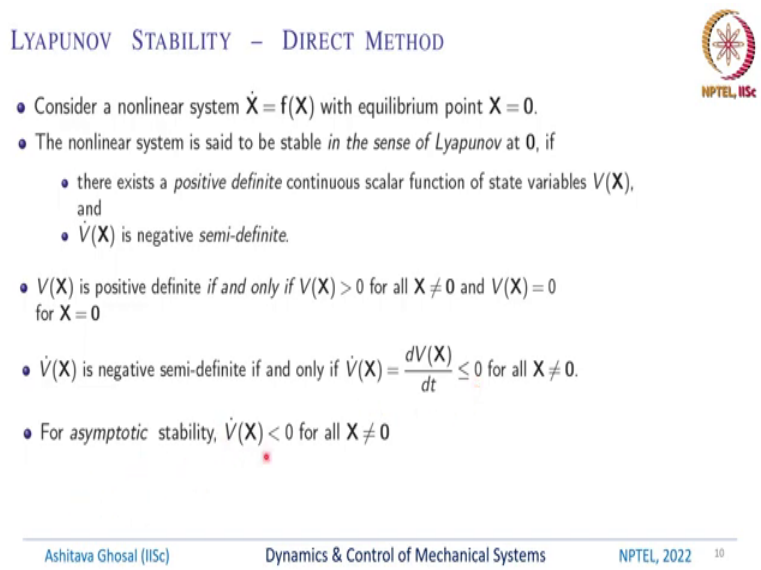

So, that is just like a definition of what is stability and Lyapunov stability in particular? How do you actually evaluate if a system is stable or not? So, Lyapunov gave this method which is called as the Lyapunov's direct method or the second method. So, he said the following, considerably non-linear system $\dot{X}(t) = f(X)$ with equilibrium point $X = 0$. The non-linear system is said to be stable in the sense of Lyapunov at 0.

If there exists a positive definite continuous scalar function of the state variables denoted by $V(X)$ and the $\dot{V}(X)$, $\dot{V}(X)$ is negative semi-definite. So, let us go over this once more so, Lyapunov is now giving us a way to test whether the system is stable, what he is telling is? That consider this non-linear system $\dot{X}(t) = f(X)$. And we are trying to investigate the stability of this non-linear system at the equilibrium point or around the equilibrium point $X = 0$.

So, according to Lyapunov, a system is stable this nonlinear system is stable if there exists a positive definite continuous scalar function of the state variables $V(X)$. So, this is a scalar function, whose derivative is negative semi-definite at least. So, what is $V(X)$ means positive definite. So, $V(X)$ is positive definite if and only if $V(X) > 0$ for all $X \neq 0$ and $V(X) = 0$ for X = 0.

So, this is a very well-known kind of function from linear algebra. So, if I give you a function which is always greater than 0 for all $X \neq 0$ but is also equal to 0 for X = 0. And $\dot{V}(X)$ is negative definite if and only if the derivative of $V(X)$ which is $\frac{dV(X)}{dt} < 0$ for all $X \neq 0$. And for asymptotic stability, Lyapunov said that this $\dot{V}(X)$ should be even stronger it should be negative definite, not semi definite means what?

$\dot{V}(X) < 0$ for all $X \neq 0$. So, stability is semi-definite means that this derivative could be < 0 but for asymptotic stability $\dot{V}(X) < 0$.

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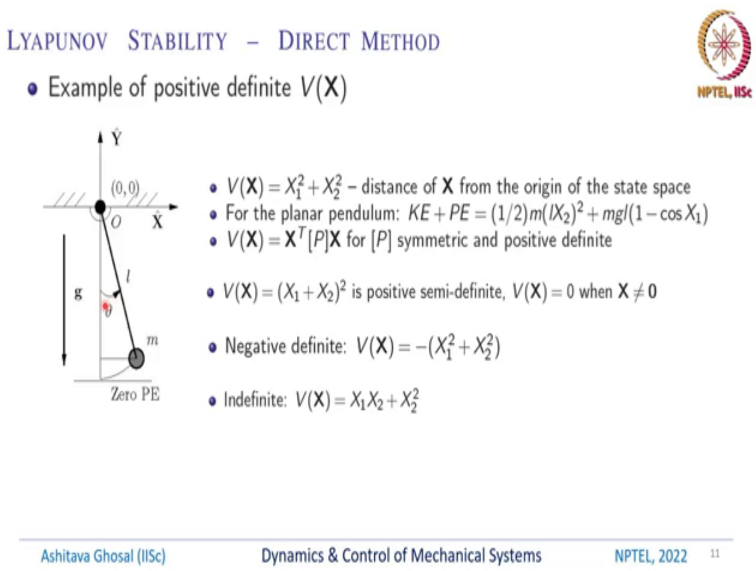

So, let us take an example of Lyapunov stability and also of what we can use as a positive definite $V(X)$? So, this is an example of a pendulum so, we have this origin 0, 0 and this is a pendulum of length l and a mass m, there is a gravity acting this way. And we are going to measure the rotation of this line or this rod massless rod by theta from the vertical. And the 0 PE is here with the lowest point of this pendulum.

So, we can use $V(X) = X_1^2 + X_2^2$ so, what is X_1 here? $X_1 = \theta$, some state variable θ which is this and $X_2 = \dot{\theta}$. So $X_1^2 + X_2^2$ is the distance of X from any point from the origin of the state space. So, for the planar pendulum, the kinetic plus potential energy can be written as $\frac{1}{2}mlX_2^2$. So, $X_2 = \dot{\theta}$, $mgl(1 - cos\,\theta_1)$ so, θ is X_1.

So, my state variables are θ and $\dot{\theta}$ and the kinetic plus potential energy can be written in this form so, this we have seen earlier. So, it is nothing but this is the kinetic energy which is $\frac{1}{2}mV^2$ and this is the potential energy which is like mgh. So, for $V(X)$ this Lyapunov function, we can choose some $X^T[P]X$ where $[P]$ is symmetric and positive definite.

In particular if $[P]$ were an $[I]$ matrix then $V(X)$ is $X_1^2 + X_2^2$, we could have also choose chosen $V(X)$ as $\left(X_1 + X_2\right)^2$. So, is this positive definite? No, it is positive semi-definite because although $V(X)$ is always greater than 0 it is ≠ 0, when X = 0. So, this is example not of a positive definite $V(X)$ because I could have written X_1 as 1, X_2 as –1 so then $V(X)$ will be 0.

So, $V(X) \neq 0$, when X is identically equal to 0, X_1 and X_2 are 0. How about negative definite? $V(X)$ is – $(X_1^2 + X_2^2)$. So, this follows the definition that $V(X) < 0$ and = 0, when X_1 and X_2 are both 0. How about if $V(X)$ is $X_1 X_2 + X_2^2$? We do not know. So, these are called indefinite functions. So, in this case of a pendulum this KE + PE is a positive definite $V(X)$.

So, you can see that $\frac{1}{2}mlX_2^2 + mgl\,(1 - cos\,X_1)$>0 and is equal to 0, when X_1 which is θ_1 is 0 and X_2 and $\dot{\theta}$ is 0.

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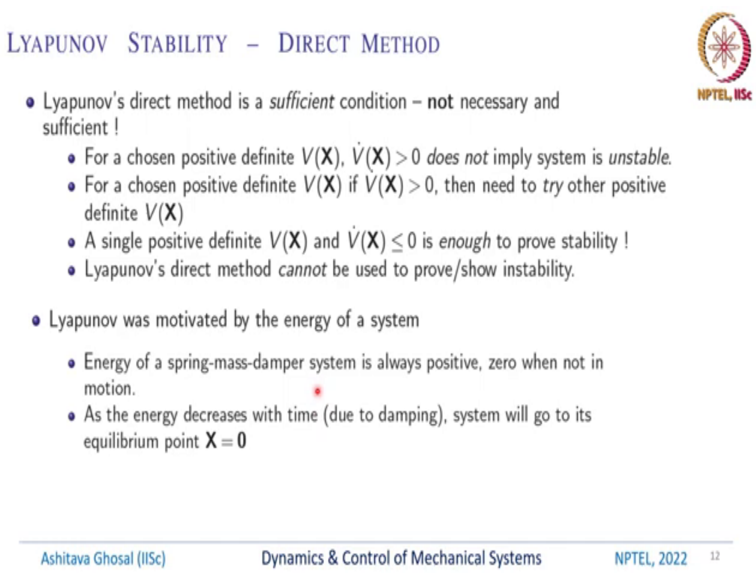

So, let us continue with this Lyapunov stability. So, Lyapnov's direct method is a sufficient condition not necessary and sufficient. Remember in many theorems in math so, for example, if you want to find the maxim optimization if you are interested in optimization we say a function is maximum when the first derivative is 0 and it is sufficient condition is when the second derivative is < 0.

So, typically we do theorems and we find conditions which are both if and only if Lyapunov's direct method is only a sufficient condition. So, what does it mean? By saying it is a sufficient condition and not a necessary condition. So, what it means is? For a chosen positive definite $V(X)$, $\dot{V}(X)$ does not imply it is unstable. So, $\dot{V}(X) < 0$ implies it is stable but $\dot{V}(X)$ greater than 0 does not imply system is unstable.

What it means is that for a chosen $V(X)$ if $\dot{V}(X) > 0$ then what it means is? You need to try other positive definite $V(X)$. There could be infinite number of positive definite functions $V(X)$, the one which you choose for which $\dot{V}(X) > 0$ is not a good choice. You need to go and try some other ones, I cannot conclude that $V(X) > 0$ and $\dot{V}(X)$ is not < 0, implies that the system is unstable.

However, on the other side of the story if you are able to find a single positive definite $V(X)$ and $\dot{V}(X)$ is $<$ 0 that is enough to prove stability, somebody gave you some function and you are very lucky. So, out of this infinite number of functions, you suddenly managed to obtain a single $V(X)$ which was positive definite and the derivative $\dot{V}(X)$ is $<\ 0$.

I do not need to try anything more that is enough to prove that the system is stable. In other words Lyapunov's direct method cannot be used to prove or show instability. It is a sufficient condition that $V(X)$ greater than 0 and $\dot{V}(X) < 0$, implies stability. Lyapunov was motivated by the energy of a system. So, if you have a spring mass damper system, we know that the energy of this system is always positive and 0, when it is not in motion.

So, this is a very standard spring mass damper system. So, if it is in motion, the kinetic plus potential energy is always greater than 0 and it is 0 when it is not in motion. So, as the energy decreases with time due to damping, the system will go to it is equilibrium point $X\ =\ 0$. So, I have a spring mass damper system, I perturb it from it is equilibrium point in this case $X\ =\ 0$. So, due to damping the oscillations will slowly die down.

And then you can see that it will go to it is equilibrium point. So, he was motivated by this idea that if I can find a function which is similar to energy in a spring mass damper system. And then if the rate of change of energy with time is decreasing all the time then the system must be stable.

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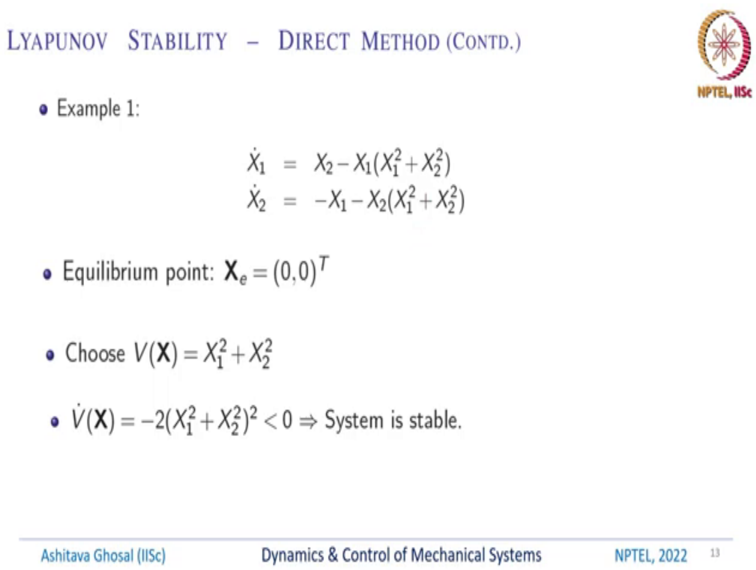

Let us continue some more examples. So, let us say we have a state equation which is $\dot{X}_1 = X_2 - X_1(X_1^2 + X_2^2)$, $\dot{X}_2 = -X_1 - X_2(X_1^2 + X_2^2)$ so, these are some randomly chosen arbitrarily chosen examples. So, what is the equilibrium point? It is 0, 0. So, if you substitute $\dot{X}_1 = 0$ and $\dot{X}_2 = 0$, you have two equations in X_1 and X_2 non-linear equations, however the solutions are only 0, 0.

So, if you choose $V(X)$ as $X_1^2 + X_2^2$, what is $X_1^2 + X_2^2$? It is the distance of the state variables X_1, X_2 from the origin which is the equilibrium point. So, this is clearly positive definite. So, $X_1^2 + X_2^2$ is always greater than 0 and it is equal to 0, only when X_1 and X_2 are both equal to 0. So, if you find the derivative $V(X)$ then what will you get? You will get $2X_1\dot{X}_1 + 2X_2\dot{X}_2$.

And what is $\dot{X}_1$ and $\dot{X}_2$? They are given by the state equations. So, in the derivative we have to substitute back the state equations which is $\dot{X}_1$ is this and $\dot{X}_2$ is this and do some simplification. And you will end up with that $\dot{V}(X) = -2\left(X_1^2 + X_2^2\right)^2$ so, this is clearly negative always. So, hence this system is always stable according to Lyapunov.

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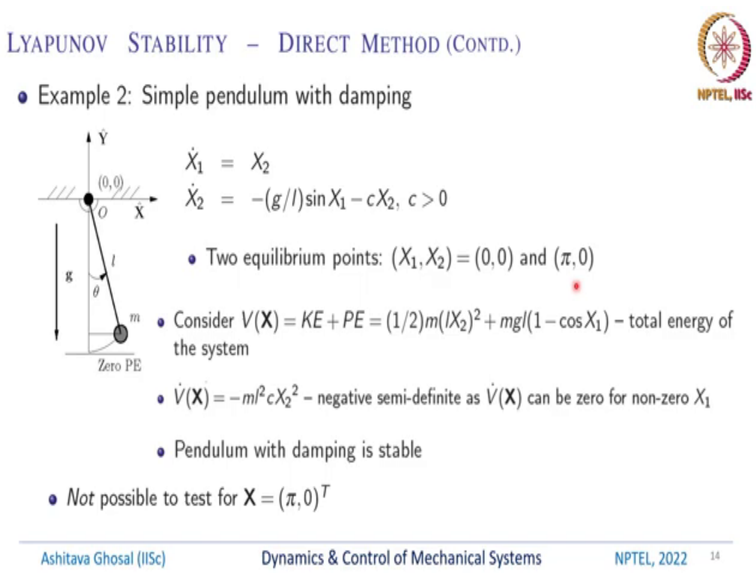

So, let us continue with the simple pendulum with damping. So, for this pendulum with the mass m and length l and gravity acting this way, I can write the state equations as $\dot{X}_1 = X_2$, $\dot{X}_2$

is $-\frac{g}{l} sin\ X_1 - cX_2$. So, this term cX_2 is like damping. So, you call that X_1 is nothing but θ and X_2 is nothing but $\dot{\theta}$. So, this is like $\dot{X}_1$ is X_2 and this is like $\ddot{\theta}$.

So that is $-\frac{g}{l} sin\ \theta - c\dot{\theta}$ and we are going to assume that this c > 0. So, if I want to find what is the equilibrium point for these state equations? So, basically I have to set $\dot{X}_1$ is 0 and $\dot{X}_2$ is 0. So, hence X_2 will always be 0 and then we will have sin X_1 = 0. So, remember X_2 is 0 which means this term will go away $\frac{g}{l}$ is a positive number. So, sin X_1 is 0. So, sin $sin\ \theta$ is 0 at two points.

So, when θ is 0 or when theta is π. So, for this example of a simple pendulum with damping, we have two equilibrium points X_1, X_2 is (0, 0) and (π , 0). So, let us consider a function $V(X)$ which is nothing but the kinetic plus potential energy of this system. So, what is the kinetic energy of the simple pendulum? We have seen this earlier, it is nothing but $\frac{1}{2}m\left(lX_2\right)^2 + mgl(1 - \cos cos\ \theta)$.

In using state variables X_1 and X_2, we can write it as $\frac{1}{2}m\left(lX_2\right)^2 + mgl(1 - \cos cos\ X_1)$ so, this is the total energy of the system. And hence we know that this is always greater than 0 for X ≠ 0 and it is equal to 0, when X_1 and X_2 are 0. So, this is a perfectly nice candidate Lyapunov function $V(X)$. So, let us see whether what happens to $\dot{V}(X)$? Because what are we trying to do?

We are trying to find the $V(X)$ whose time derivative $\dot{V}(X)$ is negative at least semi-definite. So, how do I find $\dot{V}(X)$? You take the derivative of this function. So, when you take the derivative, you will get some 2 X_2 into $\dot{X}_2$ and here you will have some sin X_1 into $\dot{X}_1$ and so on. And we go back and use the state equation $\dot{X}_1$ and $\dot{X}_2$ in the derivatives and again we do some simplification.

And then eventually we will see that $\dot{V}(X) = -ml^2 c \dot{X}_2^2$. So now, let us see what is the nature of the $\dot{V}(X)$? So, what you can see is this is clearly negative semi-definite, it is not negative definite. Why is it not negative definite? Because it can be 0 for any X_1, remember the function is negative definite if it is < 0 for all X ≠ 0 and it is equal to 0, when it is equal to 0.

So, in this case it is ≠ 0, when you have X = X_1 can have any value. So, this is negative semi definite as $\dot{V}(X)$ can be 0 for non-zero X_1. So, what does this mean? That the pendulum with damping is stable, it is not asymptotically stable, it is just simply stable. How about the other equilibrium point which is (π, 0)? It turns out that it is not possible to test for $X = (\pi, 0)$.

Again, from basic mechanics and physics, what is (π, 0)? The pendulum is up like this from 0, it is like this so, this angle is π. So, we know that that configuration of the pendulum is unstable. So, any small perturbation around this that equilibrium point, it will never stay in that region ϵ, any region for that matter. So, it will constantly go outside it will go start swinging all the way around.

So, it is unstable and Lyapunov method is not valid for unstable system, you cannot prove instability, you can only show it is stable. So, what have we done? We have shown this pendulum with damping is stable at 0, 0 and we cannot say anything about the other equilibrium point (π, 0).

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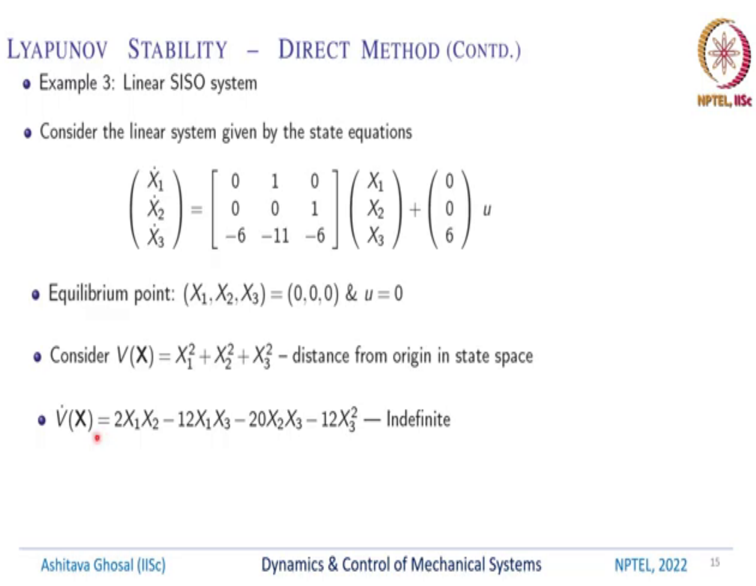

So, let us continue, let us consider a linear SISO system, linear single input single output system. The linear system is given by the state equations which is $\dot{X}_1$, $\dot{X}_2$, $\dot{X}_3$ is equal to some $[F]$ into X + 0 0 6 u. So, the $[F]$ is 0 1 0, 0 0 1, –6 –11 –6 and so, this is a straightforward linear system, this is constant, this is also constant. So, the equilibrium point is when you said the right-hand side is equal to 0 and also u = 0.

So, $[F]$ into X is 0 so which automatically implies that X_1, X_2, X_3 are all 0. So, let us consider a Lyapunov function $X_1^2 + X_2^2 + X_3^2$. So, this is the distance from the origin in state space. So, this is clearly positive definite so, this $V(X)$ is always greater than 0 but it is also equal to 0, when X_1, X_2 and X_3 are all 0. So, what is $\dot{V}(X)$? We can take the derivative of this.

This will be $2X_1 \dot{X}_1 + 2X_2 \dot{X}_2 + 2X_3 \dot{X}_3$. And then we substitute $\dot{X}_1$ is same as X_2, $\dot{X}_2$ is same as X_3, $\dot{X}_3$ is $-6X_1 - 11X_2 - 6X_3$ in that $\dot{V}$ expression. And what you will get is? $\dot{V}(X) = 2X_1 X_2 - 12X_1 X_3 - 20X_2 X_3 - 12X_3^2$. So, can we say anything about this function $\dot{V}(X)$? No, it could be positive.

It could be negative, depending on what is X_1, X_2, X_3 so, this is what is called as a indefinite function. So, we cannot say whether $\dot{V}(X)$ is always positive or negative or even positive definite or negative semi-definite.

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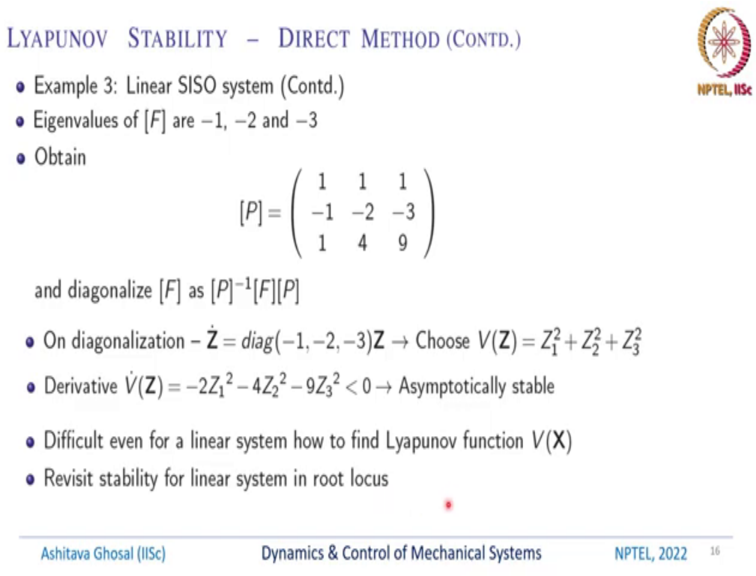

So, let us continue with that same example, we know that the Eigen values of F, are –1, –2 and –3, we have seen these examples earlier. So, once you have that the Eigen values are –1, –2, –3 we can obtain the $[P]$ which is given by 1 –1 1, 1 –2 4, 1 –3 9. Where did we get this from? Remember, there was this Vandermonde matrix in which I can find P as 1 $\lambda_{1,}\lambda_1^2$, 1 $\lambda_{2,}\lambda_2^2$, 1 $\lambda_{3,}\lambda_3^2$.

So, λ_1 is –1 so, this is –1, $\lambda_{1,}\lambda_1^2$is 1. Similarly, if you see the third column it is 1 then –3 and then $(-3)^2$ and you get 9. So and again recall that if you do now, $[P]^{-1}$ $[F]$ $[P]$, we can find the inverse of this matrix easily. We will get a diagonal $[F]$ and that is exactly what happens? So, when you do $[P]^{-1}$ $[F][P]$and we transform it to a linear system with Z is equal to some diagonal into $\dot{Z}$ is equal to diagonal to Z, we will get like this.

So, remember $X = [P]Z$, we make that linear transformation. So, on diagonalization we will have Z that is equal to some diagonal matrix with elements in the diagonal as – 1, – 2, – 3 into Z. So now, for this state equations I can choose $V(Z)$ as $Z_1^2 + Z_2^2 + Z_3^2$. So, again this is nothing but the distance from the origin but now not from the X set of variables but from the Z set of variables.

This is another linear transformation which we have done. So now, The derivative of $\dot{V}(Z)$ you can obtain which will be $2 Z_1 \dot{Z}_1$, $2 Z_2 \dot{Z}_2$, $2 Z_3 \dot{Z}_3$ and $\dot{Z}_1$ can be written as $- Z_1$, $\dot{Z}_2 = - 2 Z_2$ that 3 dot is $- 3\ Z_3$, substitute back in this $\dot{V}(Z)$ expression and you will get $- 2Z_1^2 - 4Z_2^2 - 9Z_3^2$, straight forward $\dot{Z}_1$ is $-2\ Z_1$ and so on.

So, what you can see is? This is strictly < 0 always. So, this is a negative definite function. So, what have we done? We started with $\dot{X} = f(X)$, we could not show stability. Because whatever the $V(X)$ we chose was $\dot{V}(X)$ wasn't indefinite. But we did a transformation, a coordinate transformation basically and we obtain $\dot{Z}$ is some diagonal times matrix into Z.

And then when we choose a candidate Lyapunov function which is $Z_1^2 + Z_2^2 + Z_3^2$, I showed you that $\dot{V}(Z)$ is < 0. So, hence $\dot{Z}$ is equal to diagonal matrix into Z is asymptotically stable. So, it is basically nothing but the same system but all we did was a change of variables, a linear transformation $X = [\,P]\,Z$. So, what you the purpose of this example is?

To show that even for a linear system it is not very obvious to show that a system is stable in the sense of Lyapunov by choosing $V(X)$ and then finding $\dot{V}(X)$. Because in this example I knew that the Eigen values were –1 –2 –3 and I did this clever transformation $X = [\,P]\,Z$. And then I have transformed it into $\dot{Z}$ is equal to something into Z, diagonal matrix into Z and I showed that this system is asymptotically stable.

So, it is not very easy to find a $V(X)$, even for a linear system and we will revisit the stability for linear systems later on next week when we look at root locus.

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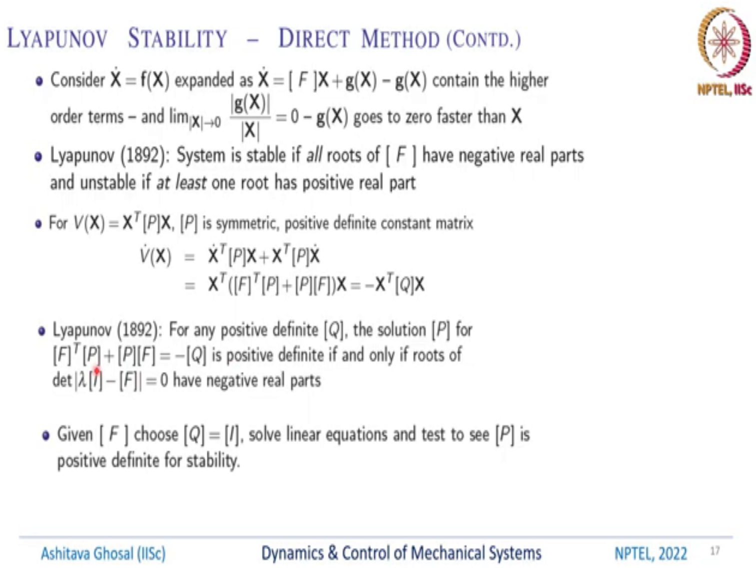

So, the Lyapunov stability theorem of finding a $V(X)$ which is positive definite and $\dot{V}(X)$ which is negative definite is sort of not so easy to apply. Because we have to keep on trying for different $V(X)$. So, Lyapunov in 1892 went on further, he proposed the following or he showed the following. So, let us consider a non-linear system which is given by $\dot{X}(t) = [F]X$.

And we assume that this non-linear equations can be expanded as $\dot{X} = [F]X + g(X)$. So, $g(X)$ contains all the higher order terms and he or we also assume that this $\frac{g(X)}{|X|}_{|X|\to 0}$ divided by X. So, magnitude of $\frac{g(X)}{|X|}$, as X tends to 0 is 0. So, what it means is? This non-linear term $g(X)$ goes to 0 faster than X.

So, we start with this kind of non-linear system which can be expanded as a linear term $[F]$ into X and nonlinear term $g(X)$ and $g(X)$ goes to 0 faster than X. So, Lyapunov went on to show that such a system is stable if all roots of this $[F]$ have negative real parts and unstable if at least one root has positive real part. So, what is meant by roots of a $[F]$? We know that every matrix has a characteristic polynomial.

It is some determinant of $|\lambda [I]-[F]| = 0$. So, we can solve the characteristic polynomial and if all the roots of the characteristic polynomial have negative real parts then the system is stable. If any of the root of the characteristic polynomial has positive real part it is unstable.

So, let us continue so, let us assume a function $V(X)$ Lyapunov of function $V(X)$ which is $X^T[P]X$.

And we start with the assumption that P is symmetric positive definite and a constant matrix. So, one can find the derivative of this $V(X)$ so, using chain rule we get $\dot{X}^T[P]X + X^T[P]\dot{X}$, remember $[P]$ is a constant matrix. So, once we have this right hand side we can substitute on the right hand side $\dot{X}$ equals $[F]X$ and what we will get is $X^T([F]^T[P] + [P]\,[F])\,X$.

So, Lyapunov said that let us assume that this $\dot{V}(X)$ is – $X^T[Q]X$. So, what it, what does it mean? In 1892 Lyapunov showed that for any positive definite $[Q]$ the solutions $[P]$ for this set of linear equations $[F]^T[P] + [P]\,[F]$ is $-[Q]$ is positive definite. If and only if the roots of determinant of $|\lambda\,[I]-[F]| = 0$ have negative real parts. So, what does this determinant of $|\lambda\,[I]-[F]| = 0$?

So, they are the roots of the characteristic polynomial of this constant $[F]$. So, what have we done? We have said that we have assumed a $[Q]$ which is positive definite. So, hence – $X^T[Q]X$ will be negative definite. So then if I can solve for $[P]$ from this set of linear equations $[F]^T[P] + [P]\,[F] =- [Q]$. And it turns out that $[P]$ is positive definite then we have found our candidate Lyapunov function.

Why? Because $V(X)$ remember is $X^T[P]X$ and we have a found of $[P]$ which satisfies $V(X)$, $X^T[P]X$. So, hence if $[P]$ is positive definite $V(X)$ is positive definite > 0. And then I have used this equation $[F]^T[P] + [P]\,[F] =- [Q]$. with $[Q]$ is positive definite. So, $- [Q]$. is negative definite so, hence $\dot{V}(X)$ is negative definite.

So, what he said is? Such a linear set of equations, the solutions of this set of linear equations. The solutions means solutions $[P]$ is positive definite if and only if the roots of determinant of $|\lambda\,[I]-[F]|$. So, this is the relationship between roots of this constant $[F]$ and the existence of a Lyapunov of function $V(X)$ which > 0 and $\dot{V}(X)$ is < 0.

So, this is a much more constructive approach to see whether a set of linear equation $\dot{X}(t) = [F]X$ is stable or not or represents a stable system or not. So, what do we do? We can choose Q as $[I]$ a simplest possible positive definite matrix $[Q]$, we know what is $[F]$. And then we solve for this set of linear equations and test to see if $[P]$ is positive definite for stability. So, if we can find a positive definite $[P]$ then we know that the system is stable.

So, what is the usefulness of this concept? Basically we do not have to keep on searching for a positive definite $V(X)$. The algorithm is quite simple, you say $[Q]$ is $[I]$ which is so – $X^T[Q]X$ is definitely negative definite. And then we solve these linear equations for $[P]$and if $[P]$ is positive definite the system is stable.

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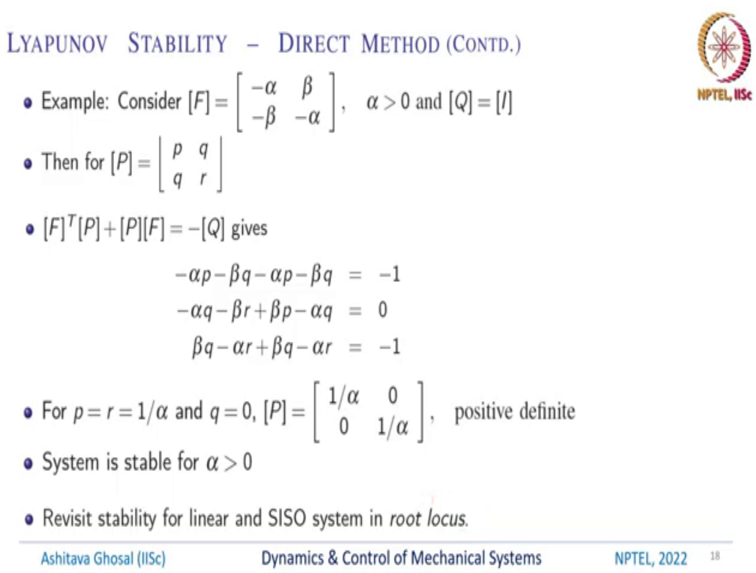

Example so, let us take a simple example of F which is a 2 by 2 matrix it is – α β ,– β – α and α > 0. So, we choose $[Q]$ as $[I]$ and the elements of $p\ are\ p\ q,\ q\ r$ it is a symmetric matrix. So, if you solve $[F]^T[P] + [P] =- [Q]$, you will get these three equations in α β $p\ q\ and\ r$. So, you will get $-\alpha\ p - \beta\ q - \alpha\ p - \beta\ q\ as\ -1$ and so on.

So, if you choose $p = r = 1/\alpha$ and q = 0 if that is the solution then $[P]$ is 1 by α 0 0 1 by α and this is clearly positive definite because α > 0. So, hence what we can conclude is? That this set of equations with F given in this form is stable for α greater than 0. So, in a definite number of steps which is basically solution of three equations.

I can find the condition for which a system given by $[F]$ which is $-\alpha\ \beta,\ -\beta\ \alpha$ is stable. Again this is a very simple example and it is to illustrate this approach of how to use $[F]^T[P] + [P] =- [Q]$ to test for stability. And as and we will come back to this important aspect of stability for linear systems, this is after all a linear system and linear SISO system, again next module when we look at root locus.

That is a much simpler way to handle the question of stability for SISO systems using root locus.

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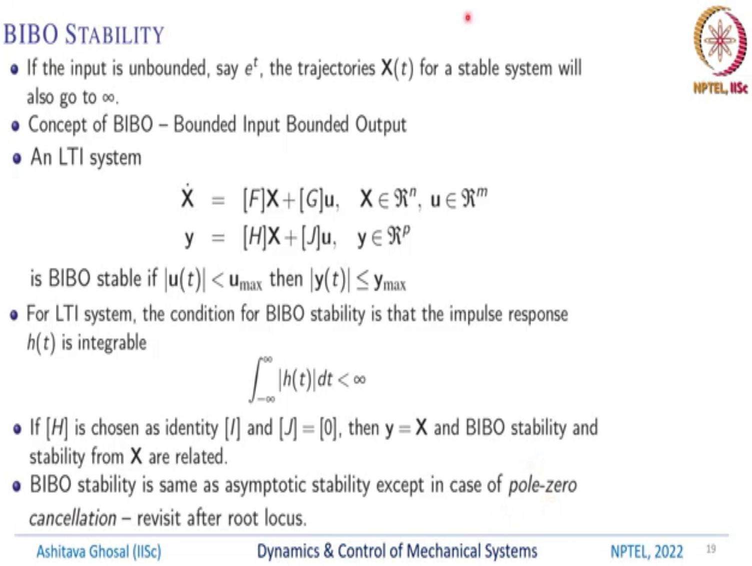

So, till now we looked at stability according to Lyapunov. So, there are other ways or other definitions of stability, one such definition is called BIBO stability. So, BIBO stands for bounded input bounded output and the basic idea is the following. If the input is unbounded say e^t, e^t the function goes to infinity as t tends to infinity. And if the input is unbounded, the trajectory $X(t)$ for the stable system will also go to infinity.

So, if you give infinite inputs the output will clearly be infinite, however if I give a bounded input, meaning that the input is not unbounded, the input is < some number. And then if the output is also < some number then the system is stable. So, this is the concept of BIBO stability so, for a bounded input if the output is bounded then the system is said to be BIBO stable.

So, if you have a linear time invariant system $\dot{X} = [F]X + [G]u$ and again X is n dimensional, u is m dimensional and $y = [H]X + [J]u$. This system is said to be BIBO stable if $u(t)$, the magnitude of the input is $< u_{max}$. Then magnitude of the output is $<$ some y_{max} . So, this is another way of looking at stability it is slightly different from Lyapunov stability.

Lyapunov stability considered the state equations $\dot{X}(t) = [F]X$ + [G]u. Here we have to also find out what is the output equation which is $y = [H]X + [J]u$? So, for linear time invariant system the condition for BIBO stability is that the impulse response $h(t)$ is integrable or $\int_{-\infty}^{\infty} |h(t)|dt < \infty$.

So, I do not want to prove this if anybody is interested they can look into some advanced control book. But the concept of BIBO or stability is also very, very useful. Let us continue if $[H]$ is $[I]$ and $[J = 0$ then we have $y = X$. and So, BIBO stability deals with magnitude of the output which is y of t and we are going to we want to insist that the magnitude of the output is $< y_{max}$, when the input is $< u_{max}$.

But if $[H]$ =$[I]$ then $y = X$ and so, hence in some sense the BIBO stability is related to the stability of X. And stability of from the point of view of state equations was what Lyapunov considered. So, BIBO stability is a little bit like a generalization where we also look at what is happening to the output y? We are not restricted to what is happening to the states.

BIBO stability is same as asymptotic stability, except in case of pole-zero cancellation. So, we will come to this notion of pole-zero cancellations later when we look at root locus, for linear systems we will find out what are the poles and zeros and then if there is some something called pole-zero cancellation then of course, BIBO and asymptotic stability are different but otherwise they are same.

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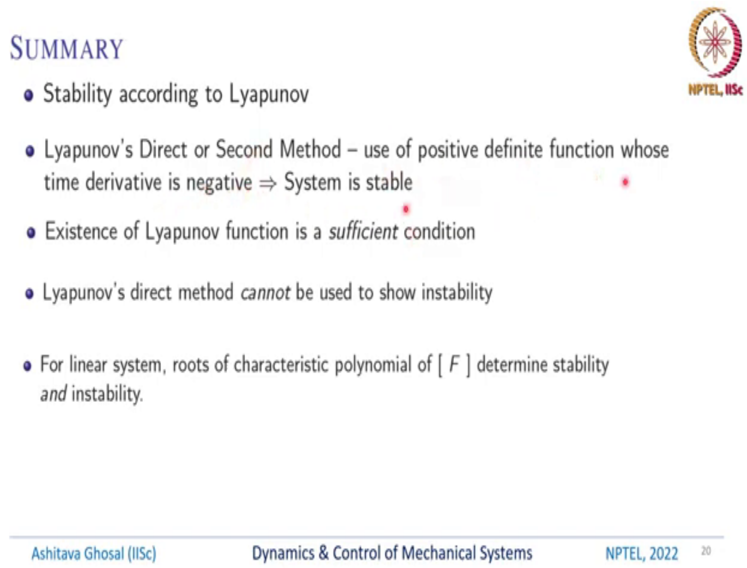

So, in summary, we looked at stability according to Lyapunov. So, basically we start near the equilibrium point and if the trajectory $X(t)$ does not go outside another region then it is stable. If the trajectories comes back to the equilibrium point it is asymptotically stable. And Lyapunov gave his direct or second method which is based on use of positive definite functions, whose time derivative is negative so then the system is stable.

This existence of this Lyapunov function is a sufficient condition, it is not a necessary condition and Lyapunov direct method cannot be used to show instability, it is only a test of stability. For linear systems, the roots of the characteristic polynomial of $[F]$ determine stability and instability. So, if the roots of the characteristic polynomial have negative real parts then the system is stable if even one root has positive, real part then the system is unstable.