Dynamics and Control of Mechanical Systems

Prof. Ashitava Ghosal
Department of Mechanical Engineering
Centre for Product Design and Manufacturing
And
Robert-Bosch Center for Cyber Physical Systems
Indian Institute of Science-Bengaluru

Lecture-22
Solution of State Equations

In the last lecture we derive the state space representation, we obtain the state space equations and the output equations from the differential equations of motion by linearizing the nonlinear differential equations about an equilibrium point. In this lecture we will look at the solution of the state equations.

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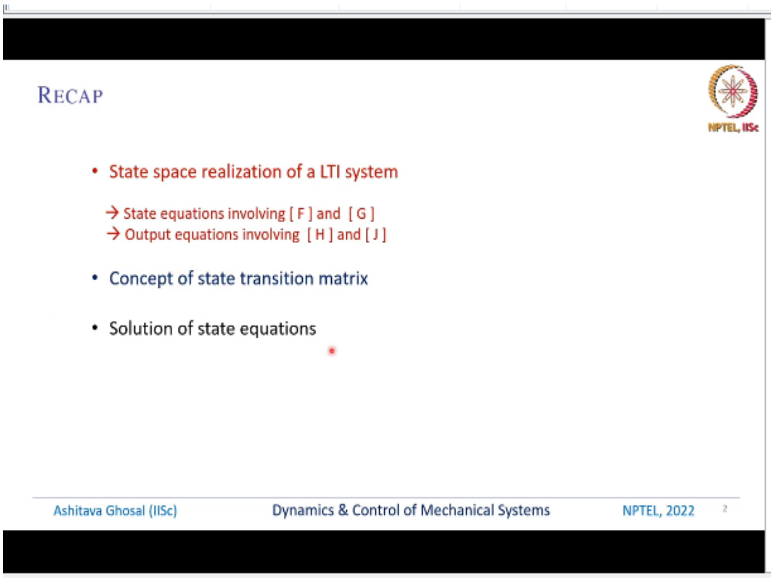

To recap we showed that this state space realization of a linear time invariant system is given by 4 matrices. So, the state equations involving [F] and [G] and the output equations involving [H] and [J]. I also discussed and presented the concept of a state transition matrix and in this lecture we look at the solution of the state equations.

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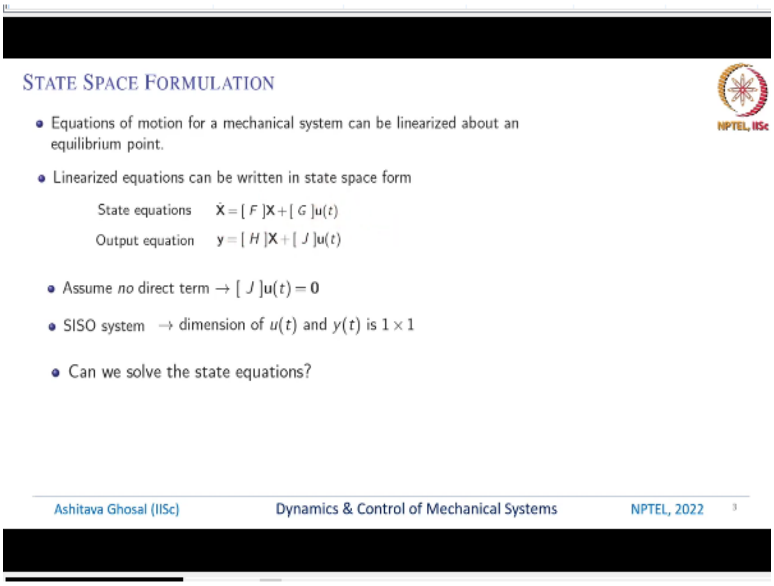

So, again to continue the equations of motion for a mechanical system can be linearized about an equilibrium point. The linearized equations can be written in the state space form. So, the state equations are of the form $\dot{X} = [F]X + [G]u(t)$. So, the dimension of $X = n \times 1$, so hence the dimension of F is $n \times n$ and so the output equations can be written as y = [H] X +[J] u. The dimension of u could be $m \times 1$.

So, hence [G] is $n \times m$. The number of measurements of the outputs could be $p \times 1$. So, hence this [H] matrix is $p \times n$ and the [J] matrix is $p \times m$. So, if you assume that there is no direct term basically the input $u(t)$ is not directly connected to the output $y(t)$ then this [J] into $u(t)$ will be equal to 0. So, $u(t)$ here means that the input u is a function of time. $\dot{X}$ is d/dt of the state variable X and Y here is the output variable.

So, for single input single output system the dimension of $u(t)$ and $y(t)$ is 1×1. So, that is what single input single output implies. So, the question is can we solve these state equations? So, this is a first order differential equation with constant coefficients, however this is a matrix equation. So, there are n of these ordinary differential equations. These are also non-homogeneous, because it is not only $\dot{X} = [F]X$ but there is also a $u(t)$.

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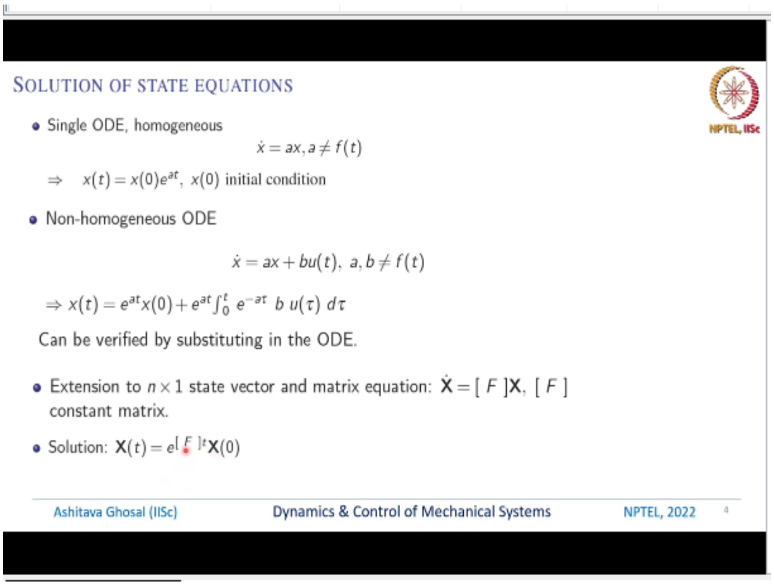

So, let us look at solution of state equations. Before we look at the general form of n matrix equations at the same time we look at a single ODE, if you have a single ODE which is homogeneous it can be written as $\dot{x} = ax$ and remember we are talking about linear time invariant system. Hence, this a is a constant, a is not a function of time. So, this is very, very standard, it is very well known what is the solution.

$x(t)$ is nothing but $x(0)e^{at}$, where $x(0)$ is the initial condition. If you have a non homogeneous ODE meaning that we have $\dot{x} = ax + b\,u(t)$ and again a and b are not functions of time, we can solve this non-homogeneous single ODE and we can write it as $x(t) = e^{at}x(0) + \int_{l/2}^{-l/2} - e^{-at}bu(t)d\tau.$

So, τ is a dummy variable which goes from 0 to t. So, if you do not remember this please go back and see any textbook and calculus or ordinary differential equations which you would have done and you can see that this is indeed the solution of this. You can also substitute this back into this equation and convince yourself that this is indeed a solution to this non homogeneous ordinary differential equation.

So, as I said it can be easily verified that this is a solution to the non-homogeneous ODE by just simply substituting it in this differential equation. The extension to n by 1 state vector and the matrix equation $\dot{X} = [F]X$, where $[F]$ is now a constant matrix can be also now tackled. We can write the solution of this $n \times 1$ matrix equation as $X(t) = e^{[F](t)}\ X(0)$.

So, this is exponential of this matrix. So, remember [F] is an $n \times n$ matrix; X is a $n \times 1$ vector. So, hence $X(t)$ is some e to the power matrix some exponential of a matrix [F] t and into $X(0)$. We will see how we can evaluate or what is meant by $e^{[F](t)}$ very soon.

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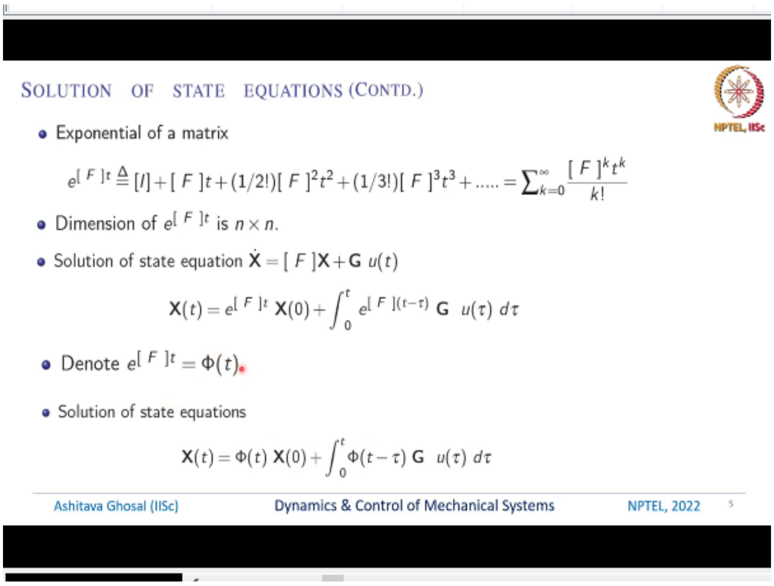

So, the exponential of a matrix just like exponential of a constant e^{ax} is written by $1 + ax + a^2x^2(\frac{1}{2!})$ and so on. So, here $e^{[F]t}$ is given by identity matrix $[F](t) + (\frac{1}{2!}) + [F]^2 t^2$ and so on and we can rewrite it in this form. The next term will be $\frac{1}{3}[F]^3 t^3$. So, the dimension of $e^{[F](t)}$ is $n\times n$.

Why because this is an $n\times n$ identity matrix, this is an $n\times n$ $[F]$ matrix, which we have obtained from the state space realization, $[F]^2$ will also be $n\times n$ and so on. This can be written as $\int_{k=0}^{\infty} \frac{[F]^k t^k}{k!}$. The solution of the state equation $\dot{X}$ = F x + G u can be written similar to what happened in the non-homogeneous case.

So, we will have $X(t) = e^{[F](t)}X(0) + \int_0^t e^{[F](t-\tau)}G\,u(\tau)d\tau$. So, previously in the non homogeneous case e to the power something was outside, but then everything has been taken inside. So, τ is a dummy variable which goes from 0 to t. We will denote $e^{[F](t)} = \phi(t)$.

So, the solution of the state equations can also be written in terms of $\phi\ (t)$which is nothing but $X(t) = \phi(t)X(0) + \int_0^t \phi(t-\tau)G\,u(\tau)d\tau$. So, $\phi(t)$ is an $n\times n$ matrix.

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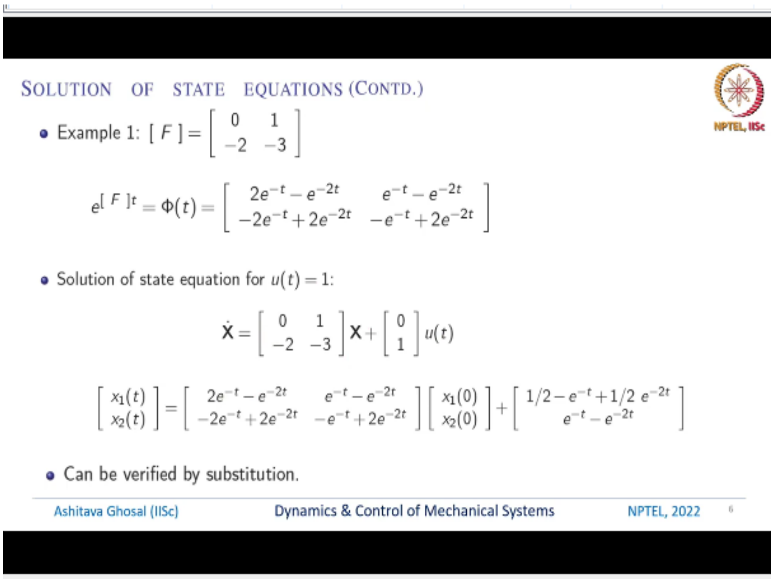

So, let us take some examples, so we will start with a very basic and simple example. Let us assume that this matrix F has element 0 1, -2 -3, it is a 2×2 matrix. We can show that $e^{[F]t}$ which is the $e^{[F](t)}$ which is also denoted by this $\phi(t)$ is given by $2e^{-t} - e^{-2t},\ -2e^{-t} + 2e^{-2t},\ e^{-t} - e^{-2t}, -e^{-t} + 2e^{-2t}$.

So, it might seem like magic, we do not know where we obtain these forms, but at least we can verify that this is indeed correct but just by simple substitution. We will see very soon how we can obtain the exponential of a matrix. So, if you want to find the solution of the non-homogeneous part and let us assume $u\ (t)$ = 1 then we have $\dot{X}$ is again F is [0 1 -2 - 3] X + 0 1 $u\ (t)$ and we are going to assume $u\ (t)$ is a constant = 1

So, once we do this we can find the solution to the non-homogeneous part also, non-homogeneous or ordinary differential equation which is given by same thing as first part is same $2e^{-t} - e^{-2t}, -2e^{-t} + 2e^{-2t}$, this is the first column of this matrix ϕ. The second column is $e^{-t} - e^{-2t}, - e^{-t} + 2e^{-2t}$ multiplied by the initial conditions x_1 (0) x_2 (0) and the particular part which is the non-homogeneous part due to this is $1/2 - e^{-t} + 1/2\, e^{-2t}$.

And the second term or the second element is $e^{-t} - e^{-2t}$ again and it us not yet clear how I got this but at least you can verify that this is indeed the solution by simply substituting it. So, we can take x_1 (t)as this, x_2 (t)as this and then you can go back and substitute $\dot{X}$ and then you can simplify and show that the solution x_1 $(t)$and x_2 (t) indeed satisfies this non-homogeneous linear ordinary differential equation.

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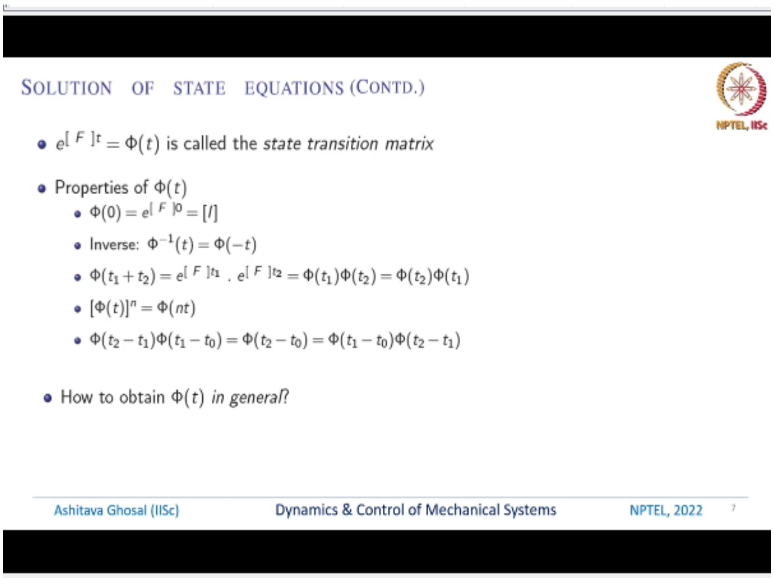

So, before we see how we obtain this $e^{[F]t}$ just quickly let us see some of the properties of this matrix $e^{[F]t}$ which I am also going to call $\phi(t)$. So, first thing is ϕ (0) is nothing but $e^{[F]0}$, so $t = 0$. That is nothing but the identity matrix, which is $\phi^{-1}(t)$ is given by $\phi(-t)$ because $e^{[F](t)}$ and $1/e^{[F](t)}$ will be $e^{-[F](t)}$.

So, it is $\phi^{-1}(t) = \phi(-t)$, we can also show very clearly and easily that $\phi\,(t_1 + t_2)$ is nothing but $e^{[F](t_1)}.e^{[F](t_2)}$ which is $\phi(t_1)\,\phi(t_2)$and we can change the order because if you have $e^a.e^b$ it is same as $e^b.e^a$. So, then we can write it as $\phi(t_2).\ \phi(t_1)$.

Likewise $[\phi(t)]^n\ \phi(nt)$, again it follows from this expression. So, then again you will have so many products of these matrices and you can show that this is $\phi(nt)$,. Lastly you can see that $\phi(t_2 - t_1)\,\phi(t_1 - t_0)$, so I go from $t_2 \rightarrow t_1$ and then $t_1 \rightarrow t_0$. So, $\phi(t_2 - t_1)\,\phi(t_1 - t_0)$ is same as $\phi(t_2 - t_0)$.

Again it follows from the fact that these are exponentials and $e^{(t_2 - t_1)}\ .e^{(t_1 - t_0)}$ will be $e^{(t_2 - t_0)}$ and again we can reverse the order, we can write this as $\phi(t_1 - t_0)$ and $\phi(t_2 - t_1)$. So, $e^{[F](t)}$ which is as I showed you is $\phi(t)$ is also sometimes called as the state transition matrix. This is a very important term in control theory especially when we are using the state space formulation.

And we are analyzing a system using the state space formulation the exponential of this matrix $F\,t$ is a matrix which is $\phi(t)$ and it is called as the state transition matrix. And these are some of the properties of $\phi(t)$ which I just now discussed. So, the question is how to obtain $\phi(t)$ in general? So, I have showed you some examples where I said okay here is the solution.

So, the question is where did I get it from? So, we need to look at what are the ways to obtain this exponential of a matrix or the state transition matrix.

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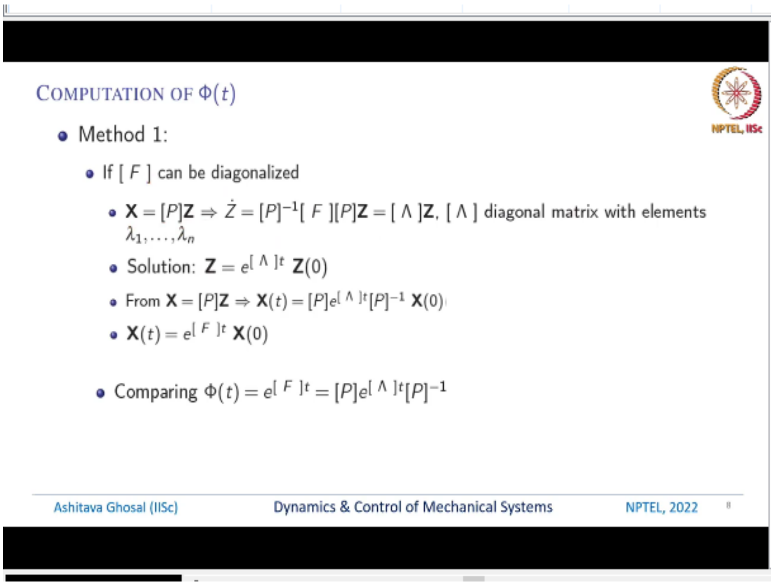

So, let us look at some simple methods first. So, if $[F]$ can be diagonalized, so what do we mean by $[F]$ can be diagonalized, we can convert from $X = [P]Z$ are related by $X = [P]Z$ then $\dot{Z} = [P]^{-1}[F][P]Z$ and as I said $[P]^{-1}[F][P]$ is diagonal. So, then we can write this as a diagonal matrix ΛZ. So, what does this Λ contains?

It is the diagonal matrix with elements $\lambda_1, \ldots, \lambda_n$. So, if you have a $[F]$ which can be diagonalized by a change of state variables, a linear transformation of X to Z. Remember $[P]$ is a constant matrix, it is a full rank and $[P]^{-1}$ exists. So, then $Z = e^{[\Lambda]t}Z(0)$, because what do we have $\dot{Z} = \Lambda Z$.

So, the solution will be $e^{[\Lambda]t}\,Z(0)$, each term is like $Z(1)$is $e^{[\Lambda_1]t}$ into Z 1 0, second term $Z(2)$ is this because this is a diagonal matrix now. So, hence let us continue, so from $X = [P]Z$ we can rewrite X (t)= $[P]\ e^{[\Lambda]t}\ [P]^{-1}X(0)$. So, where did I get this from? So, remember $X = [P]Z$, Z is $e^{[\Lambda]t}\,Z(0)$.

So, $X = [P]\,e^{[\Lambda]t}, Z(0)$ can be written as $[P]^{-1}X(0)$, $Z\,(0)$ is $[P]^{-1}X(0)$. So, if you substitute all those things you will get $[P]\ e^{[\Lambda]t}\ [P]^{-1}Z(0)$. The solution of this $X\,(t)$ can also be written as $e^{[F]t}$ into $X(0)$, because what did we have? We have $\dot{X}$ = F times X. So, X t is also $e^{[F]t}\,X(0)$.

So, if you compare these two expressions what you can see is $e^{[F]t}$ is nothing but $[P]$ $e^{[\Lambda]t}$ $[P]^{-1}$ and what was $e^{[F]t}$? This was the state transition matrix. So, very simply if $[F]$ can be diagonalized all we need to do is find out the diagonal Eigen values of this $[F]$ matrix, all the n Eigen values and then we do $e^{[\Lambda]t}$ pre-multiplied by some the matrix $[P]$ and post multiply by $[P]^{-1}$. $[P]$ and $[P]^{-1}$are the transformation which converts $[F]$ into a diagonal form.

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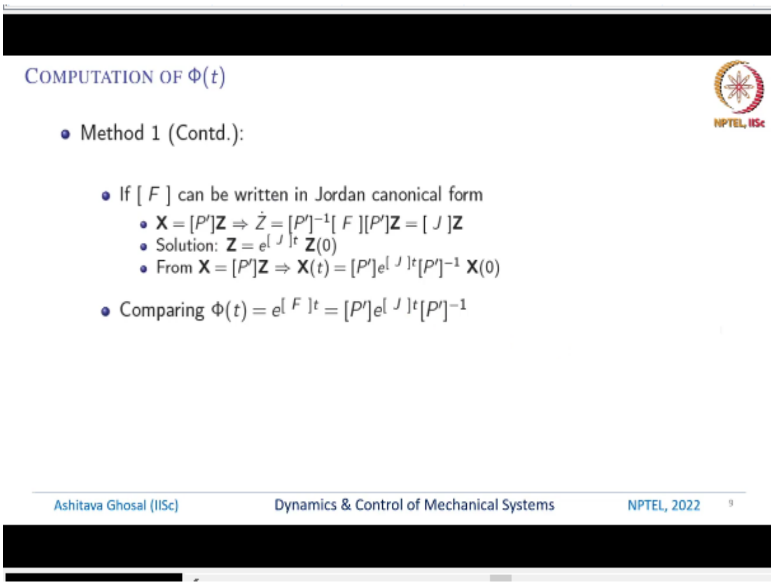

If $[P]$ can be written in the Jordan canonical form meaning that some of the Eigen values are repeated then again we can find out what is $\phi(t)$ which is the state transition matrix and the procedure is as follows. So, we write again $X = [P']Z$. So, remember when you have a Jordan canonical form the $[P]$ which was used for with distinct roots is not the same when you have repeated roots.

So, that is why I am using $[P']$ here. So, we can have $X = [P']Z$. So, hence $[P']^{-1}$ $[F][P']Z$. So, $[P']^{-1}$ $[F][P']$ is this Jordan canonical form into Z. So, the solution to this is Z is $e^{[J]t}$ which is that matrix which we get in the Jordan canonical form into $Z(0)$ and hence if you go back and see what in terms of X. So, $X = [P']Z$.

So, hence $X(t) = [P']e^{[J]t}$ $[P']^{-1}$ $X(0)$ and again comparing this with this we can see that the $\phi(t)$ which is $e^{[F]t}$= $[P']e^{[J]t}$ $[P']^{-1}$. So, if the$[F]$ matrix can be either converted into a pure diagonal form or into a Jordan canonical form we can find $[F]$ state transition matrix by doing $e^{[J]t}$ or $e^{[\Lambda]t}$ and pre-multiply by a transformation matrix $[P']$ and post multiplied by a transformation matrix $[P']^{-1}$.

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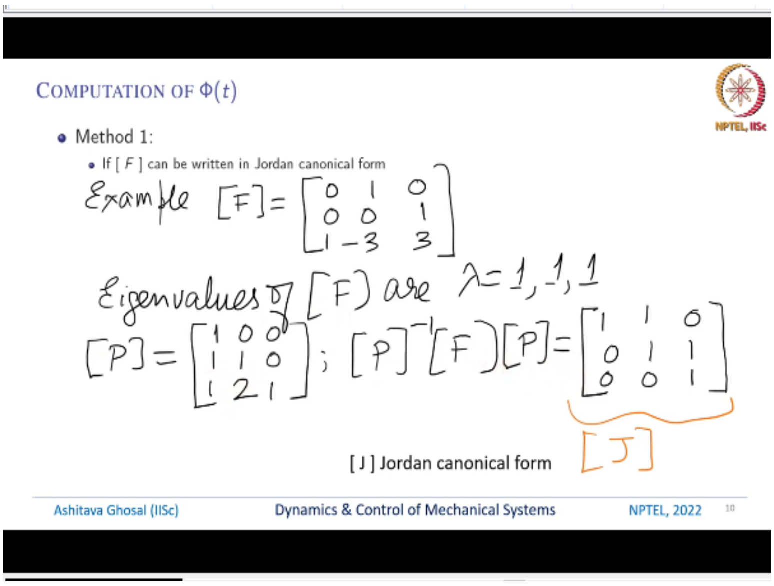

So, let us see an example. So, if $[F]$ can be written in Jordan canonical form. So, for example this $[F]$ has element 0 1 0, 0 0 1, 1 -3 3. This is picked up from somewhere that arbitrary numbers. So, the Eigen values of this matrix are all equal which is 1 1 and 1. So, hence this $[P]$ the matrix which will convert it into the Jordan canonical form can be written as 1 1 1, 0 1 2, 0 0 1.

Where did I get this? If you go back and see your notes, you can see if the roots are repeated once then we have the second column is the derivative of the first column. So, here it is repeated twice. So, this is what you will get as $[P]$ and $[P]^{-1}[F][P]$ is now given in this form. So, it is not a diagonal. So, the diagonal elements are 1, but then you have two of diagonal elements 1 and 1. So, this is the Jordan canonical form after we do this $[P]^{-1}[F][P]$.

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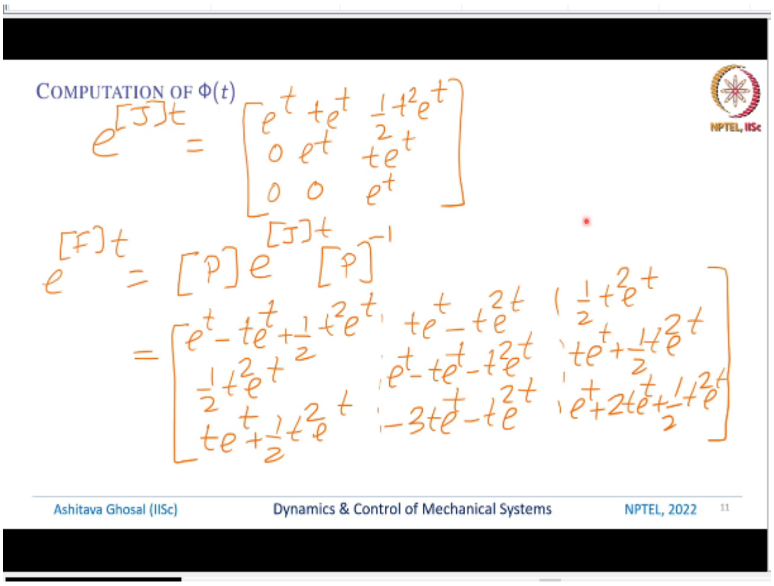

So, $e^{[J]t}$, $[J]$ was the Jordan canonical form can be written as $e^{t}0\ 0$ then $t\,e^{t}\ e^{t}\ 0$ and $\frac{1}{2}t^{2}\ e^{t}\ \ te^{t}\ and\ e^{t}$. So, this is basically in some sense similar to that we have repeated roots, e^{t}. The next one is repeated, so we will have $t\,e^{t}$ then it is repeated again it is $\frac{1}{2}t^{2}\ e^{t}$ and so on.

So, e^{t} will become $t\,e^{t}$ and then finally we have only one term three, three element which is e^{t}. So, $e^{[F]t}$ is this $[P]\ e^{[J]t}\ [P]^{-1}$and remembers we found out what was $[P]$ in the previous slide. So, if you substitute $[P]\ \ e^{[J]t}\ \ [P]^{-1}$. All of these are existing, you will get this reasonably complicated expression for the state transition matrix or the exponential of this matrix.

So, I do not want to go through each and every term, but the first term is $e^{t} -\ t\,e^{t} + \frac{1}{2}t^{2}\ e^{t}$ which is straightforward multiplication of $[P]\ e^{[J]t}\ [P]^{-1}$ and then you find the 1, 1 term. The 2, 1 term is $\frac{1}{2}t^{2}\ e^{t}$. The last term in the first column is $t\,e^{t}\ +\ \frac{1}{2}t^{2}\ e^{t}$.

Likewise if you see the 3, 3 term it is $e^{t} +\ 2t\,e^{t}\ + \frac{1}{2}t^{2}\ e^{t}$. So, how are these obtained? You can use any computed algebra system or a tool to multiply all these expressions. So, we have a $[P]$ which is a constant matrix $e^{[J]t}$ is this matrix containing exponential with time and then

$[P]^{-1}$ which was also known and then you can multiply simplify and you will get this expressions for $e^{[F]t}$. So, $e^{[F]t}$, where the matrix $[F]$ had three repeated Eigen values of 1.

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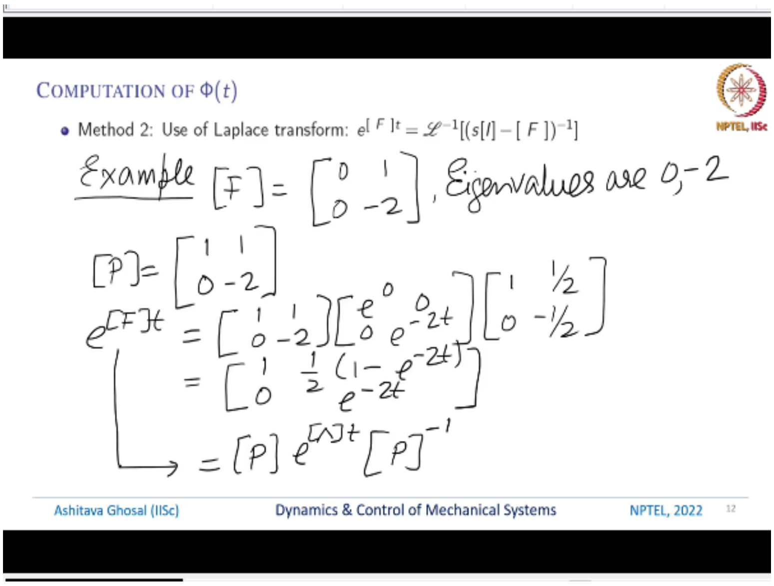

Let us look at another method. So, this is the use of Laplace transforms. So, remember we showed you that we can write the state equations using the Laplace transform form. So, $e^{[F]t}$ was $L^{-1}[\,(s\,[I\,] - [F])^{-1}\,]$. So, please go back and see the block diagram representation of the state in state equation written using Laplace transforms. This is what was mentioned there.

So, if I want to find the $e^{[F]t}$ using this Laplace transform approach then I have to find $(s\,[I\,] - [F])^{-1}$ and then take the inverse Laplace transform of this. So, let us take an example. So, this matrix $[F]$ is given by 0 1, 0 -2. So, the Eigen values are 0 and -2. So, here we can find out $[P]$ which is 1 0, 1 -2 because remember the first element is 1 λ_1.

Second element is 1 λ_2. So, we can find out the $[P]$. So, $e^{[F]t}$ we can go back and use the expression because these are distinct Eigen values. So, it will be $[P]$ some $e^{[\Lambda]t}$, λ are the two Eigen values 0 and -2. So, e^{0} and e^{-2t} into $[P]^{-1}$. So, if you multiply this out and simplify you will get 1 0 , $1/2\,(1 - e^{-2t})$ and e^{-2t}. So, this is obtained from as I said $[P]$ $e^{[\Lambda]t}$ $[P]^{-1}$. Let us do the same thing using Laplace transform.

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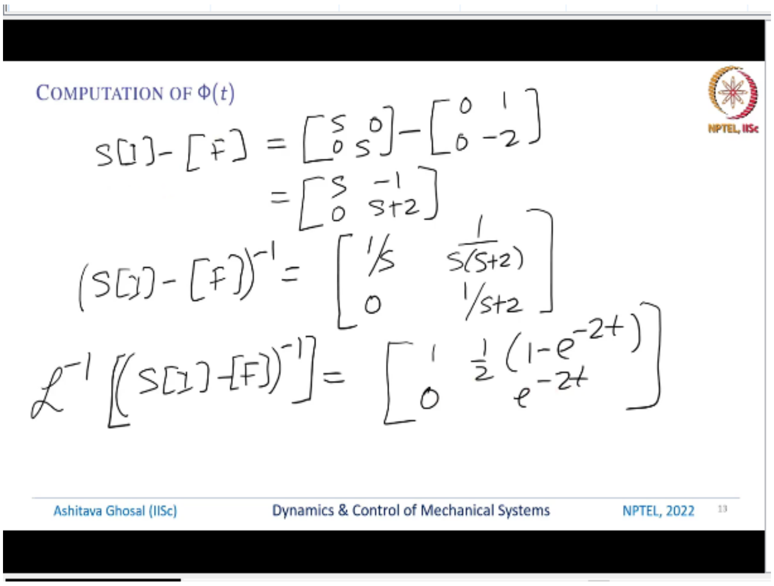

So, what is $S[\ I] - [F]$? So, S into $[I]$ is $S\ S$ and 0. So, $[I]$ is the identity matrix - $[F]$ is 0 0 1 -2. So, if you simplify this you will get $S\ 0\ -\ 1\ S\ +\ 2$. So, $(S[\ I] - [F])^{-1}$ will be $1/S$, this $1\ /S\ (S\ +\ 2)$ and $1\ /\ (S\ +\ 2)$. In this case it is very simple, but at 2 by 2 matrix we can find very easily the inverse in symbolic form. So, this at standard methods in linear algebra. So, we have the inverse of this matrix $(S[I] - [F])^{-1}$ which is this.

And the Laplace inverse of this matrix what is the inverse of $1/\ S$? That is unity. So, remember I had given you a set of common Laplace transforms and their inverse. So, what about $1\ /S\ (S\ +\ 2)$.. This you can also go back and see it is $\frac{1}{2}(1\ -\ e^{-2t})$. So, $S\ +\ 2$ will give you this e^{-2t}. So, this is what is shown here, it is e^{-2t} and when you have $1/\ S(S\ +\ 2)$ you can do partial fractions and then you will get this term.

So, as you can see the expression for Laplace inverse of this is the same as what we got using $[P]e^{[\Lambda]t}\ [P]^{-1}$. So, we can find the state transition matrix also using Laplace inverse.

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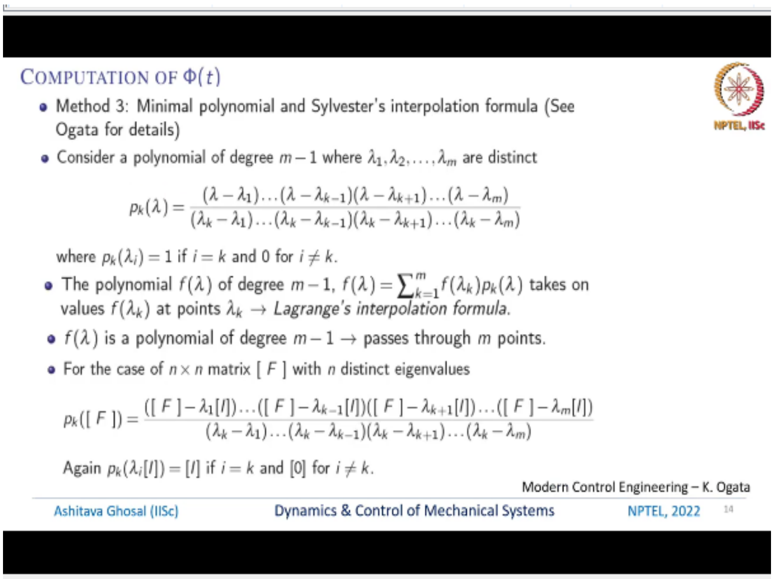

The third method is much more complicated and I will go through it in brief if anybody is interested they can go and see this modern control engineering by k Ogata. This is a little bit more complex. So, these uses two ideas, one is something called as a minimal polynomial and something called as a Sylvester's interpolation formula. So, if you consider a polynomial of degree $m - 1$ where λ_1, λ_2 dot all these Eigen values lambdas are distinct.

Then I can define a polynomial $p_k(\lambda)$ which is in this form. So, which is $(\lambda - \lambda_1) \ldots (\lambda - \lambda_{k-1})(\lambda - \lambda_{k+1}) \ldots (\lambda - \lambda_m)/(\lambda_k - \lambda_1) \ldots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \ldots (\lambda_k - \lambda_m)$. So, both of these numerator and denominator $(\lambda - \lambda_k)$or $(\lambda_k - \lambda_k)$will be 0 is missing.

So, hence this $p_k(\lambda_i)$ this polynomial of any lambda $i = 1$, if $i = k$ and 0 for i not equal to k. So, you can see that the polynomial $f(\lambda)$ of degree $m - 1$ can be defined in terms of this $p_k(\lambda)$ and that is $\sum_{k=1}^{m} f(\lambda_k)p_k(\lambda)$ and what does this do? This polynomial takes on values of $f(\lambda_k)$ at points λ_k. This is well known; it is called Lagrange's interpolation formula.

So, what is f of lambda? This is a polynomial of degree m – 1. So, hence it passes through m points. So, this is little bit of background that we are going to use this notion of this Lagrange's interpolation formula in a special way to obtain the state transition matrix or $e^{[F]t}$. So, for the case of $n \times n$ matrix F with n distinct eigenvalues. For the moment let us assume that we have n distinct eigenvalues.

So, instead of $p_k(\lambda_i)$, we can find Pk of this matrix. So, this is very similar concept that we can substitute in any characteristic equation the matrix which gave the characteristic equations. So, Pk of F can be written as $([F] - \lambda_1[I]) \dots ([F] - \lambda_{k-1}[I])([F] - \lambda_{k+1}[I]) \dots ([F] - \lambda_m[I]) / (\lambda_k - \lambda_1) \dots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \dots (\lambda_k -$
. So, basically here instead of lambda we will put this matrix $[F]$.

So, some of you may have heard of this something called as the Kelly Hamilton theorem which states that the matrix satisfies its characteristic polynomial. So, basic idea is something similar there that whatever is the characteristic polynomial, instead of the eigenvalues if you substitute the matrix that also satisfies that. So, here I can get a polynomial in which the elements are the matrices f and again $p_k(\lambda_i)$ $[I]$ if $i = k$ and 0 if i not equal to k.

It follows more or less from whatever was happening to this $p_k(\lambda)$ and whatever is $p_k([F])$, instead of λ we now have the matrix $[F]$ which is in the state equations.

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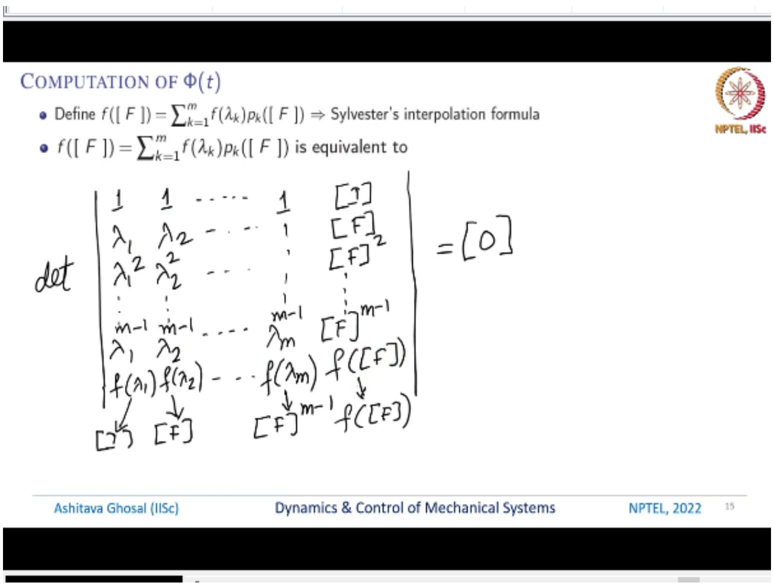

So, let us let us now define a $f([F])$which is given in this form which is $\sum_{k=1}^{m} f(\lambda_k) p_k([F])$ which we defined earlier and $p_k([F])$. This is called Sylvester's interpolation formula. This is some sense is an extension of the Lagrange's interpolation formula and this f function of this matrix $[F]$ can be written in this form which is this form here.

And it is equivalent to this complicated looking expression which is what this is the determinant of some matrix. The first row of the matrix is 1 1 1 and this is identity, second row is λ_1 λ_2 all the way then $[F]$. Then $[F]^2$ and all the way $f(\lambda_1)$ $f(\lambda_2)$ $f(\lambda_m)$ and then f of function of this matrix. So, this function is equivalent to determinant of this matrix = 0.

So, if you are interested in the proof of this please go and see some advanced linear algebra book. So, what do we have? I want to find out $e^{[F]}$, you know I have done so much math and so much background, but eventually I have a way to find exponential of a matrix because this function $f[F]$, f of this matrix I can choose anything.

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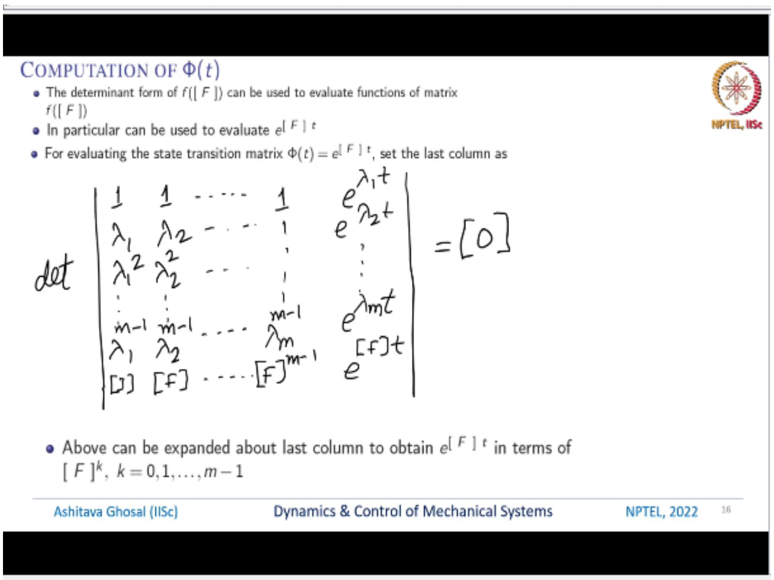

So, the determinant form of $[F]$ of this matrix can be used to evaluate functions of a matrix. So, any function $f[F]$, in particular we can use this determinant form to evaluate $e^{[F]t}$. So, for evaluating the state transition matrix $\phi(t)$which is $e^{[F](t)}$we set the last column in this form. So, everything here remains the same, but this is $e^{[\lambda_1](t)}$ $e^{[\lambda_2](t)}$ $e^{[\lambda_m](t)}$ and this is $e^{[F]t}$.

And the last row is $[I]$ $[F]$ $[F]^{m-1}$ and this determinant will be equal to 0. So, what have we done and we are looking at the case when the eigenvalues of this matrix $[F]$ are distinct. So, now I have a way to calculate this determinant I have an expression which says determinant of something is equal to 0 and what I can do is I can evaluate this determinant by **by** expanding about the last column.

So, basically I can say that I can expand about this last column and then obtain $e^{[F]t}$ into something is equal to something and hence using this form we can obtain what is $e^{[F](t)}$, because this is an equation which involves many terms but we can expand about this last element and then we will get $e^{[F]t}$ into something will be equal to something.

And then we can solve for $e^{[F]t}$. It turns out we will get $e^{[F]t}$ in terms of $[F]^k$ basically these terms here, where k = 0, 1 through $m - 1$. So, this is a very complex way but very general purpose way to find the exponential of a matrix.

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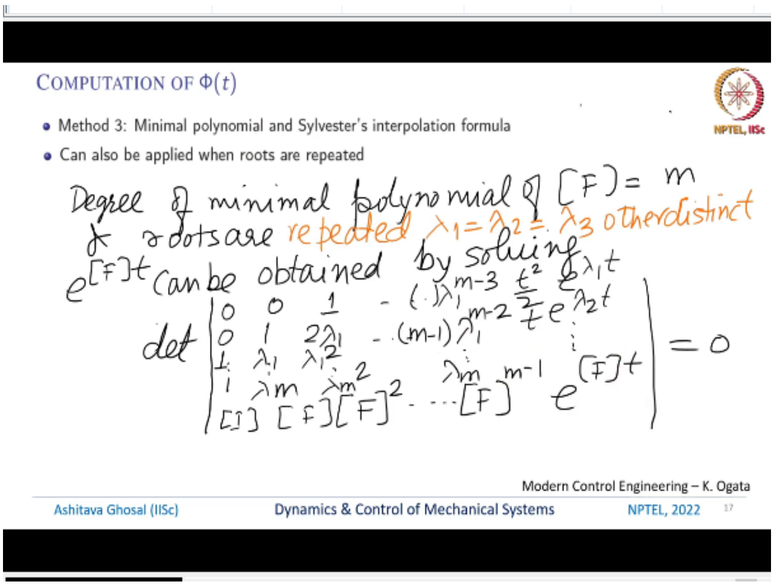

This method of minimal polynomial and Sylvester's interpolation formula can also be applied when the roots are repeated. Again, I do not want to go into this. If anybody is interested, they can look at this modern control engineering by Ogata or they can look at some advanced linear algebra book. So, just as an example if the three of the roots are repeated let us say $\lambda_1 = \lambda_2$ is λ_3.

All others are distinct then $e^{[F]t}$ can be obtained by this determinant equation which is determinant 0 0 1 and then some other terms then 0 1 λ_1 then 1 $2\lambda_1$ λ_1^2. So, where did I get this from? This is because $\lambda_1 = \lambda_2 = \lambda_3$, it is repeated. Rest of the term is very similar. The term on the last column is different now which is some $\frac{t^2}{2} e^{\lambda_1 t}$ because it is repeated twice.

Then the next one is $t\,e^{\lambda_2 t}$ and all the way to $e^{[F]t}$. Again, we have determinant of something very complex many terms are there is equal to 0 and again by expanding about $e^{[F](t)}$ this element we can obtain $e^{[F]t}$ into something will be equal to something and we can obtain this state transition matrix.

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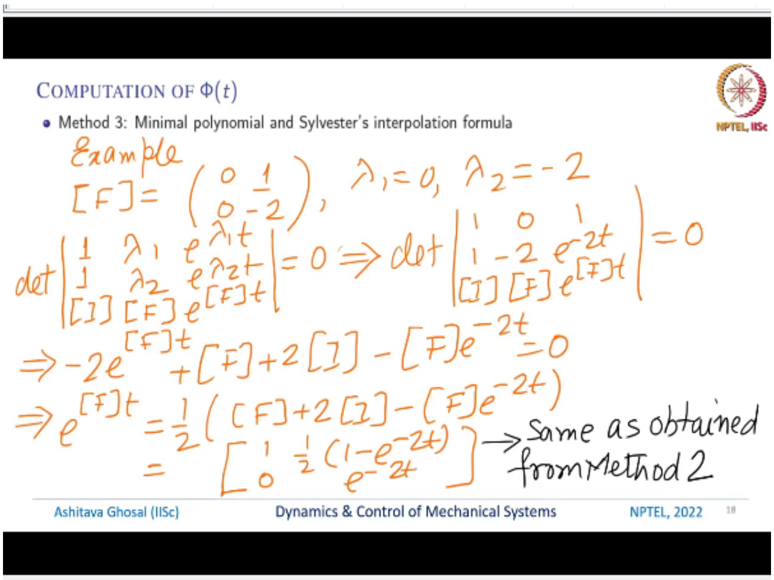

To illustrate the method of minimal polynomial and Sylvester's interpolation formula we take an example which we have seen earlier. The matrix $[F]$ is given by 0 1 0 -2, the eigenvalues are 0 and -2. To follow this method of minimal polynomial and Sylvester's interpolation formula we obtained this determinant which has these elements. So, the first column is 1 1 $[I]$, the second column is λ_1 λ_2 and this$[F]$ matrix.

The third column is $e^{\lambda_1 t}$, $e^{\lambda_2 t}$ and $e^{[F]t}$. So, the determinant of this equal to 0 can be simplified by substituting what is λ_1 and λ_2. So, we have 1 1 $[I]$, 0 -2 $[F]$, 1 e^{-2t} - $e^{[F]t}$ = 0. So, if you expand this we will get $-\ 2e^{[F]t}\ +\ [F]\ +\ 2\,[I] -\ [F]\,e^{-2t}\ =\ 0$ which will further simplify as follows.

So, we can take $e^{[F]t}$ outside and put everything on the other side we will get o $\frac{1}{2}(F\ +\ 2[I] -\ [F]\ e^{-2t})$ and this if you substitute what is $[F]$ then you can obtain as 1 0 the 1 2 element is $1/2(1\ -\ e^{-2t})$ and the 2 2 element is e^{-2t}. So, this is the exponential of this given matrix and you can see and compare yourself that this is the same as obtained from method 2.

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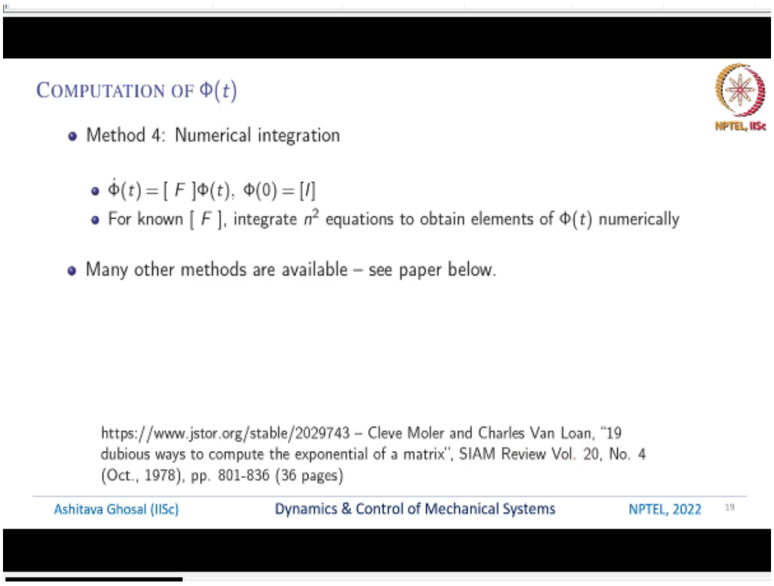

The last one which I am going to discuss is a purely numerical approach to obtain the state transition matrix. This is by numerical integration. So, the reasoning is the following we know $e^{[F]t}$ is ϕ. So, now if you take the derivative of ϕ, so $\dot{\phi}$ what is the derivative of this matrix? Derivative of each term in the matrix you will get $[F]\phi(t)$and what is the initial value $\phi(0)$ is $[I]$.

So, hence if I know what is $[F]$, I have n^2 differential equations. So, remember ϕ is $n\times n$, so we have $\dot{\phi}_{11}$ is equal to some $[F]\phi_{11}$. Likewise we have n^2 differential equations which you can integrate and we can obtain the elements of $\phi(t)$numerically. So, this can be integrated numerically using some code like in Matlab. So, this is a way to obtain $e^{[F]t}$ which is $\phi(t)$.

By solving this n square differential equations and these are ordinary differential equations, linear because $[F]$ is constant. There are many other methods which are available which I do not want to go into this, but there is a paper which is available in this link which is written by some Cleve Moler and Charles Van Loan, he is a very well-known linear algebra person. So, Golub and Van Loan; he is that same Van Loan and this is a very interesting way of saying how to obtain the exponential of matrix.

So, he says that there are 19 dubious ways to compute the exponential of a matrix and the SIAM review is a professional very well known journal. So, this came out long, long time back and it is 36 pages. So, anybody who is interested in other methods other than this 4 please go and get hold of this paper and take a look at it.

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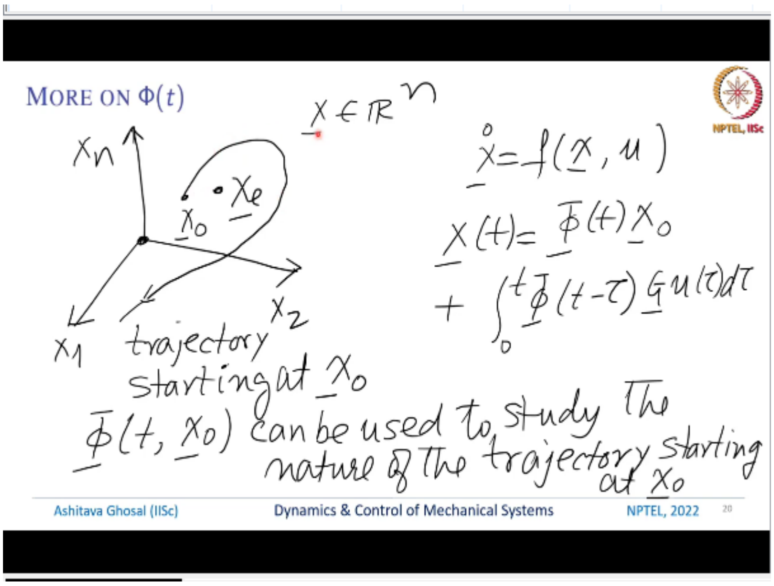

So, what can we do once we have this state transition matrix? So, basically what do we have our state space is X_1 X_2 all the way till X_n, it is an n dimensional. So, X is an element of R^n. So, what we can see is if I start from a point near to this equilibrium point we remember we took the non-linear differential equations and we linearized it about an equilibrium point.

So, if you start from a point which is $X(0)$ initially which is maybe near to this equilibrium point, then the trajectory(t) his X is governed by this differential equation $\dot{X} = f(X, u)$and what is the solution to this differential equation which is X (t) = $\phi(t)$ X_0 + $\int_0^t \phi(t - \tau)Gu(\tau)d\tau$. So, this is also sometimes called a convolution.

So, what is this schematically telling you? That if I can solve this equation which I can because I know how to compute $\phi(t)$. So, then I should be able to do all of these and then I can show you what is the trajectory starting at $X(0)$. So, what is the X $(t)(t)$ as we go along, as we increase in time? So, this ϕ (t, X_0) can be used to study the nature of the trajectory starting at $X(0)$.

And this is conceptually important that later on we will see that we will look at these trajectories to look at things called stability. So, the state transition matrix tells us what is the trajectory$y(t)$

he state variables or what is X(t)as we start from some initial $X(0)$ near to an equilibrium point and where it is going.

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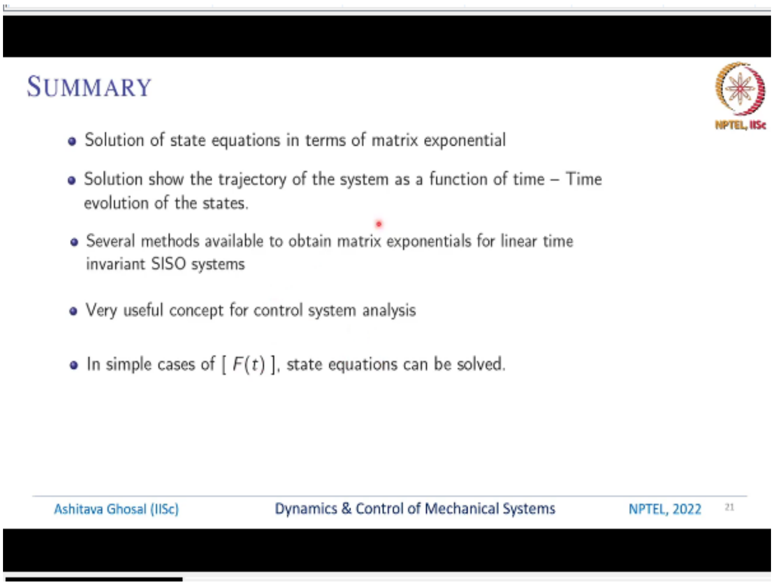

So, in summary the solution of state equations can be given in terms of matrix exponential. The solution shows the trajectory of the system $function\ of\ time$. So, basically it also tells you how the states are evolving in time. So, $X\ (t)$ and X are the states of the system. There are several methods to obtain matrix exponential for linear time invariant single input single output systems.

I have shown you four, I have indicated to you a link to a very well-known paper where there are many other methods. This is a very useful concept for control system analysis. So, this is the formal way of looking at what is happening to my system as time progresses. So, given states given F given G what is happening X of t? So, it will tell you what is how the system is evolving in time.

In simple cases where $[F(t)]$ very, very simple cases, so $[F]$ need not be constant it would be some simple constant functions of time, we can still solve the state equations, I am not going to go into this in this lectures but you can look at some control systems book, advanced control theory books where they might be having examples where this $[F(t)]$. So, this is for some simple linear time varying systems I can still get a solution in terms of this $\phi(t)$.