Dynamics and Control of Mechanical Systems
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Lecture-21
State Space Formulation

In this lecture we will look at the state space formulation.

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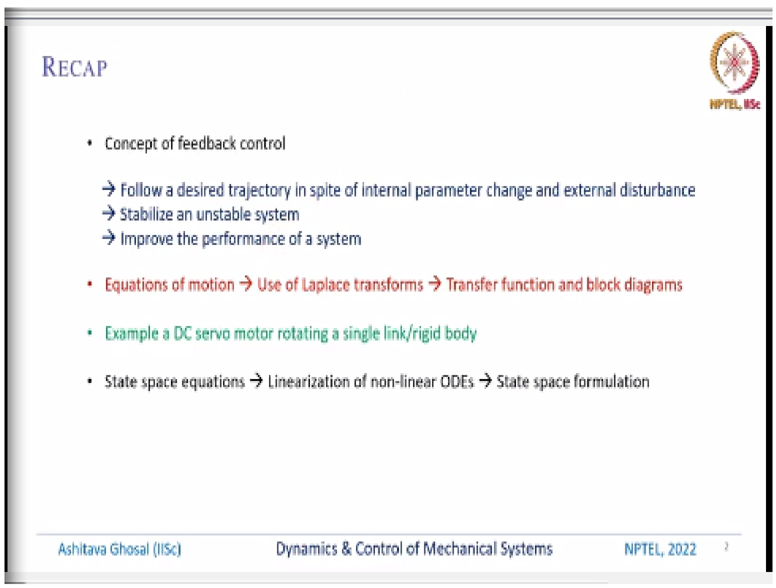

So, in the last lecture we looked at the concept of feedback control. So, before that in the portion of dynamics I had showed you that any rigid body or a multi-rigid body system will have a natural dynamic. So, whenever we apply some external forces or moments it would behave according to Newton's law or Euler's equation. In the concept of feedback control I showed you that we can achieve a desired trajectory in spite of internal parameter change and external disturbances.

So, basically even though the natural dynamics says something, using the concept of feedback control I could make it follow a desired trajectory which is not necessarily the same as the natural dynamics. I also showed you that we could use feedback control to stabilize an unstable system, or we could use feedback control to improve the performance of a system. So, we started with equations of motion, linear equation of motion using an example of a DC servo-motor rotating a link.

Then I showed you that we could use Laplace transform to convert it into s domain. So, basically ordinary differential equations became algebraic equations in s and then we could define the concept of a transfer function which was the ratio of the output to the input. And we could represent these transfer functions in a pictorial from, in a geometrical form using block diagrams. And with this example of this DC servo-motor rotating a single link, I showed you that we could ensure that the rotating link follows a constant speed or some other desired trajectory.

In spite of changes in internal parameter changes or some external disturbance. So, the output in a closed loop feedback control system would be very close to what we want as a desired trajectory. So, in this lecture we will go back to the state space equations which are obtained from the dynamics of the rigid body or a multi-rigid body system. We will show you that we can linearize these nonlinear ODEs and then obtain state space formulation.

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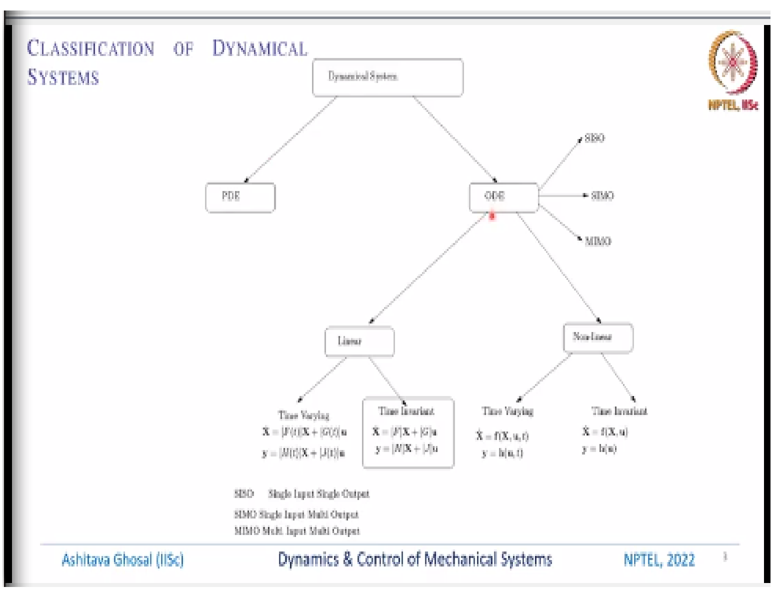

So, let us look at a broad classification of dynamical systems. So, dynamical systems basically systems which changes with time can be described by a partial differential equation or it could be described by ordinary differential equations. So, in our case for multi-rigid body system or even a single rigid body we had ordinary differential equations. The ordinary differential equation can be of 3 kinds, one is called SISO which stands for single input single output systems.

We could also have SIMO single input multi-output systems and then the most general case of multi-input multi-output system. The ordinary differential equation also could be linear or non-linear. So, if they are non-linear we could have time varying or time invariant, if it is time varying then the state equations can be written as $\dot{X} = f(X,\ u,\ t)$, u is an external input, t is explicitly time.

So, sometimes these equations have time explicitly in them. We could also have what is called as an output equation which is $y = h\ (u,\ t)$. If it is a time invariant non-linear system then $\dot{X}$ is a function of X and u and y is a function of u alone, there is no explicit dependence of time in these equations. As I said we could also have ODE which represents a linear system and again in linear system we can have time varying or time invariant.

So, if it is a time varying linear system then we could represent the state equations in the form of $\dot{X} = [F(t)]X + [G(t)]u$, so this matrix contains functions which are functions of time. And the output equation will be $y = [H(t)]\ X\ +\ [J(t)]u$. So, basically what I am trying to show here is that these coefficients which multiply X and multiply u and multiplying again X in the output equation, they could be themselves functions of time.

If you have a time invariant linear system these are also called LTI systems. So, then we could describe this LTI systems by this kind of differential equation which is $\dot{X} = [F]X + [G]u$ and the output equation $y = [H]\ X\ +\ [J]u$. So, in this case $F,\ G,\ H$ and J are constants, they are not functions of time. So, as I mentioned SISO means single input single output system, SIMO means single input multi-output system, MIMO means multi-input multi-output system.

So, in this lecture and in fact in most of the course we will primarily be concentrating on LTI systems, linear time invariant systems. If you want to study linear time varying system or general non-linear systems there are advanced courses in control which you need to look at.

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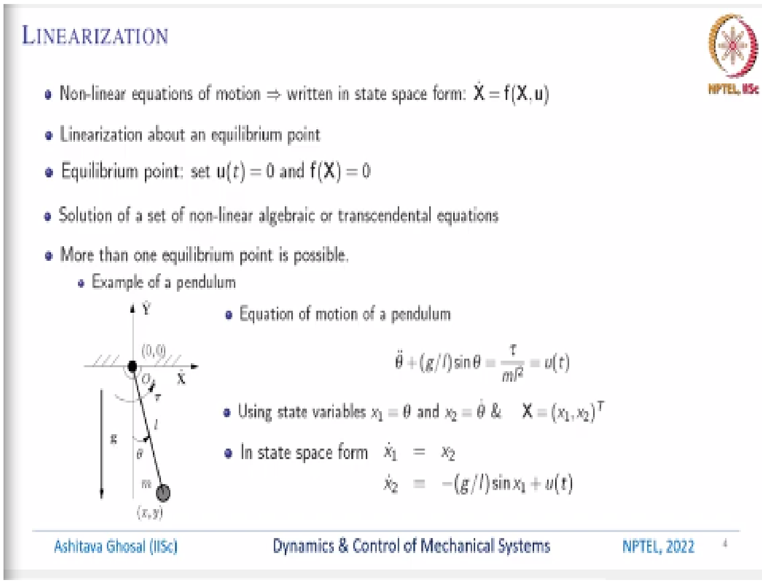

So, let us look at an example of linearization. So, we have this simple example of a pendulum, so we have this rod a mass less rod with the mass at the end and it is rotating about this z-axis, so the rotation angle is θ from the vertical. There is also gravity which is acting here and we will also assume that there is a torque which is acting at this joint. So, there is a motor which is applying this torque.

So, the non-linear equations of motion for this pendulum if you write it in the state space form we can write it as $\dot{X} = f(X, u)$. And if you want to linearize these non-linear first order differential equations we have to linearize about a point. And which point do we choose? Most of the time we will choose or the standard approach is to use linearization about an equilibrium point. And how do we find the equilibrium point?

The equilibrium point is obtained by settin$[G]u$ which is in this case torque = 0, $u(t)$ 0 and then we solve $f(X) = 0$. So, this is the nonlinear algebraic equation. So, the solution of a set of nonlinear algebraic or transcendental equations will give me X and these are the equilibrium points. So, this could be any algebraic or transcendental equation at least conceptually we can solve them and we find all X such that $f(X) = 0$ and we can have more than 1 equilibrium point is possible.

So, again the example of the pendulum. So, the equation of motion of the pendulum is $\ddot{\theta} + \left(\frac{g}{l}\right)\sin sin\,\theta = \frac{\tau}{ml^2} = u(t)$and we will call this as $u(t)$. So, if you use state variables x_1 and x_2, x_1 is θ , x_2 is $\dot{\theta}$, so my state variables X is x_1 , x_2 it is a 2 by 1 vector. So, in state space form we can have $\dot{x}_1 = x_2$, $\dot{x}_2$ is $-\left(\frac{g}{l}\right)\sin sin\,x_1 + u(t)$
. So, this we have done earlier, and I am just repeating it that we can derive the first order equations from the second order equation by using x_1 as θ and x_2 as $\dot{\theta}$.

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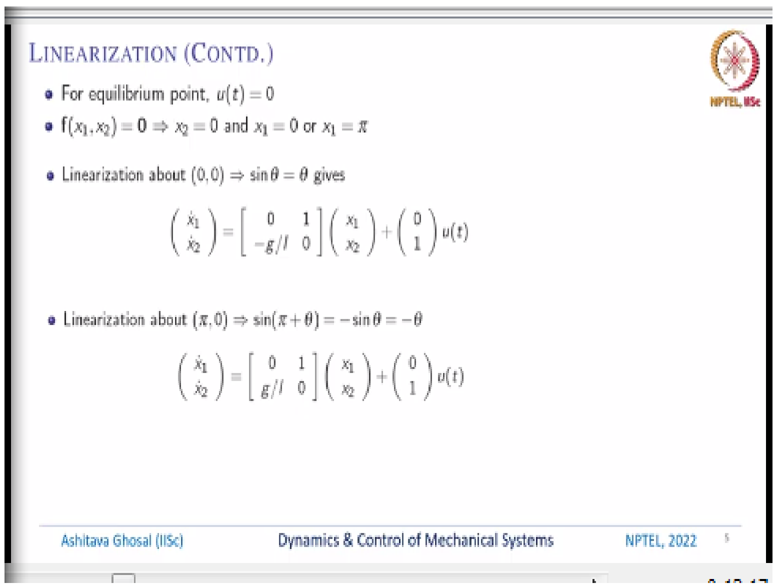

So, continuing with that example of a simple pendulum to find the equilibrium point for those state equations. So, hence the first step is we assume or set $u(t) = 0$, so then the function on the right hand side will be equal to 0 when $x_2 = 0$ and $x_1 = 0$ or π. So, if you linearize about x_1, x_2 of 0 comma 0 basically we have to substitute $\sin sin\,\theta = \theta$. And then the 2 first order differential equations can be written in this form $\dot{x}_1 = x_2$, remember it was written in the previous case, previous slide.

And $\dot{x}_2$ will now become $-\ g / l\,.x_1 + u(t)$, it was g / l . sin sin θ but then θ is same as x_1 so -g / l x_1 into x_2 and this is 0 comma $u(t)$. So, this is the F matrix and this is the G vector in this case. If you want to linearize about (π , 0), remember x_1 could also be π and π is what? It is standing vertically upwards. What is x_1 = 0? The pendulum is hanging down.

So, in that case if you linearize about (π , 0) then what you will get is $sin(\pi + \theta)$ which is $-$ sin $sin\,\theta$ in you will get -θ, so for small θ, sin $sin\,\theta$ is same as θ. So, then the first order differential equations or the state equations are $\dot{x}_1 = x_2$ that one does not change and $\dot{x}_2$ = +(g / l) $x_1 + u(t)$

. Previously when the pendulum is hanging down and when θ is 0 and you are linearizing about the hanging down position then it is – (g / l) x_1, in this case it is + (g / l) x_1.

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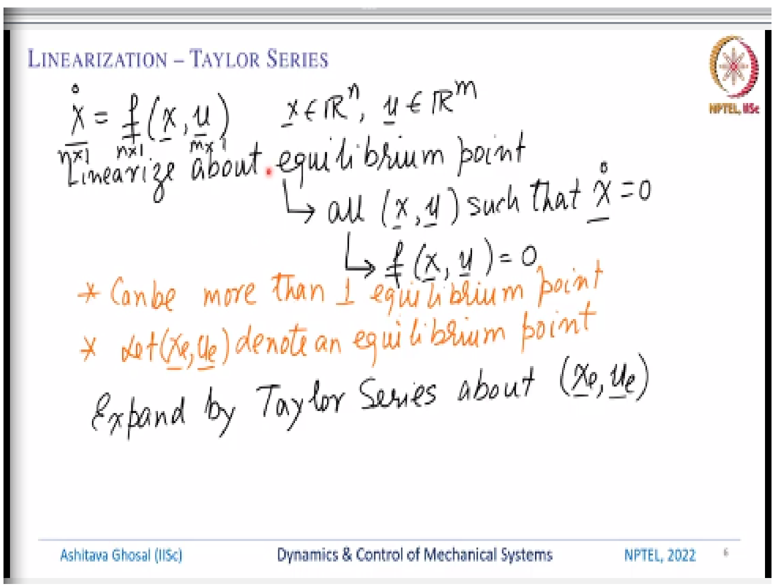

So, in general what can we do? We can use Taylor series. So, instead of simply saying sin $sin\,\theta$ is same as θ and doing this simple thing we can formally linearize any differential equation using non-linear differential equation using Taylor series and again we linearize about an equilibrium point. So, let us assume that we have a set of non-linear differential equation $\dot{X} = f(x, u)$, where x now is an $n\times1$ vector, u is an $m\times1$ vector.

So, the number of inputs could be less than the number of states, so x is an element of R^n and u is an element of R^m. So, we are going to linearize about an equilibrium point and what is the equilibrium point? All $(x,\ u)$ such that $\dot{X}\ =\ 0$ or which implies all $(x,\ u)$ which satisfies this vector non-linear set of equations $f(x,\ u)\ =\ 0$. And again these are non-linear algebraic or transcendental equations, so you can have more than 1 solution for $f(x,\ u)\ =\ 0$.

So, basically we can have more than 1 equilibrium point and so we need to linearize about a chosen equilibrium point. So, let $(X_e,\ u_e)$ denote an equilibrium point we are interested in, so we can expand this right hand side using Taylor series about $\left(X_e,\ u_e\right)$.

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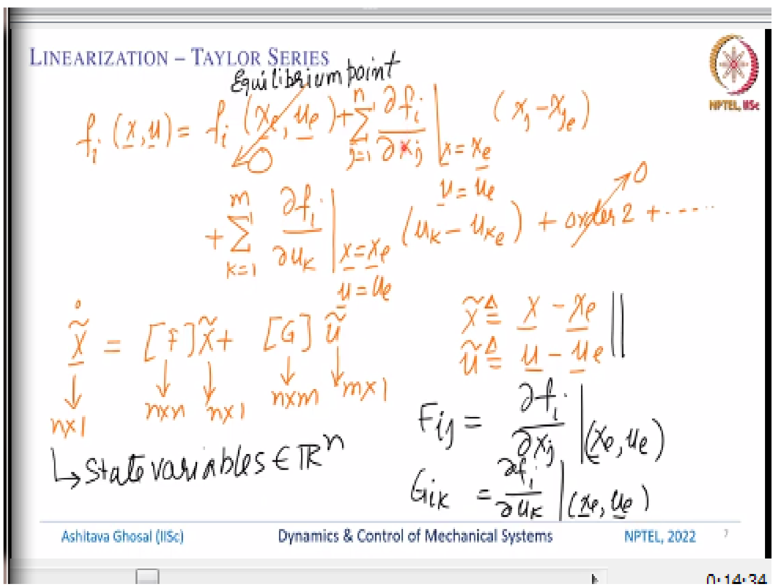

So, what is the expression for Taylor series? So, any one of these functions, so let us say $f_i\,(x, u)$ can be written as $f_i\ (\underline{X_e},\ \underline{u_e})\ +\ \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}|_{\underline{X}=\underline{X_e},\ \underline{u}=u_e}(X_j\ -\ X_{je})$. So, this is the first term in the expansion of the Taylor series and what is $\delta\ X$? It is $\left(X_j\ -\ X_{je}\right)$equilibrium point. We can also look at the $\sum_{k=1}^{m} \frac{\partial f_i}{\partial u_k}|_{\underline{X}=\underline{X_e},\ \underline{u}=u_e}(u_k\ -\ u_{ke})$ because remember there is $(x,\ u)$.

So, again we have $\sum_{k=1}^{m} \frac{\partial f_i}{\partial u_k}|_{\underline{X}=\underline{X_e},\ \underline{u}=u_e}(u_k\ -\ u_{ke})$. And $\delta\ u$ is again $\left(u_k\ -\ u_{ke}\right)$ and then plus we have order 2 and order 3 and all the higher order terms. So, we are going to drop all the higher order terms and this term is also equal to 0, the first term because why this is the definition of an equilibrium point. So, $f_i\left(\underline{X_e}, \underline{u_e}\right) = 0$.

So, what you are left with is one part which $\sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}|_{\underline{X} = \underline{X_e},\ \underline{u}=u_e} (X_j - X_{je})$ and then $\sum_{k=1}^{m} \frac{\partial f_i}{\partial u_k}|_{\underline{X} = \underline{X_e},\ \underline{u}=u_e} \left(u_k - u_{ke}\right)$. So, both terms can be written as some $\dot{\underline{\tilde{X}}} = [F]\tilde{X} + [G]\tilde{u}$, so what is the dimension of $\dot{\underline{\tilde{X}}}$? It is $n\times 2$ because there are n of these state variables. So, what is the dimension of $[F]$? It is $n\times n$.

And what is $\tilde{X}$? So, $\tilde{X}$ and $\tilde{u}$ are nothing but $\underline{X} - \underline{X_e}$, $\underline{u} - \underline{u_e}$. So, it is this small deviation about the equilibrium point. So, what is the dimension of $[G]$? It is $n\times m$, remember u is an element of R^m, so there are m inputs and n state variables $\underline{X}$ is n dimensional, $\underline{u}$ is m dimensional. So, now we have $[G]$ into $\tilde{u}$, so this is $n\times m$ and $m\times 1$. So, these are the state variables element of R^n and what are these matrices?

They are nothing but the $\frac{\partial f_i}{\partial X_j}|_{\underline{X_e},\ u_e}$. And what are the elements of $[G]$? This is $\frac{\partial f_i}{\partial u_k}|_{\underline{X_e},\ u_e}$.

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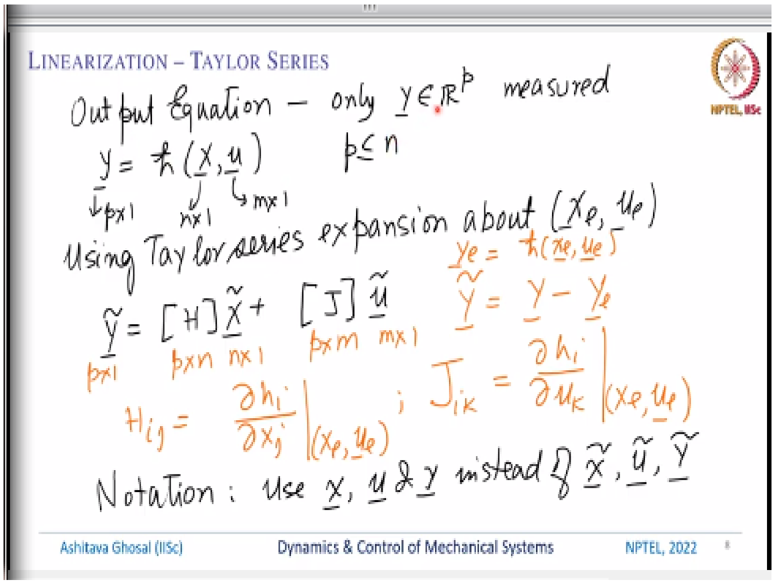

Similar to the state equation we can also look at the output equation. And here we are going to assume that $\underline{y}$ denotes the output, it is also a vector but there is only p of them. So, the number of state variables are n, the number of $\underline{u}$ is m and the number of measurements or output variables are p and p will be less than or equal to n. So, again we can use Taylor series and expand this non-linear equation about $\underline{X}_e$, $\underline{u}_e$.

So, similar to the state equations we will get some $\dot{\tilde{\underline{Y}}} = [H]\,\tilde{\underline{X}} + [J]\,\tilde{\underline{u}}$ and what is $\underline{Y}_e$? $\underline{Y}_e$ is the equilibrium point which is the $h(\underline{X}_e, \underline{u}_e)$ and $\tilde{\underline{Y}} = \underline{Y} - \underline{Y}_e$. So, what is this element of the H matrix? So, $H_{ij} = \frac{\partial h_i}{\partial X_j}\Big|_{\underline{X}_e, \underline{u}_e}$. The elements of $J_{ij} = \frac{\partial h_i}{\partial u_k}\Big|_{\underline{X}_e, \underline{u}_e}$. So, what are the dimensions of $\tilde{\underline{Y}}$?

Remember, it is p of them, so this is clearly $p \times 1$. How about the dimension of $[H]$? It is $p \times n$, $\tilde{\underline{X}}$ is $n \times 1$, the dimension of $[J]$ is $p \times m$ and $\tilde{\underline{u}}$ is $m \times 1$. So, in the rest of the course or in the rest of this lecture we will drop this tilde unnecessarily we are carrying around one more symbol. So, we will use $\underline{X}$, $\underline{u}$ and $\underline{Y}$ instead of $\tilde{\underline{X}}$, $\tilde{\underline{u}}$ and $\tilde{\underline{Y}}$.

Remember $\underline{X}$ is a vector of dimension $n\times 1$, $\underline{u}$ is a vector of dimension $m\times 1$ and $\underline{Y}$ is a vector of dimension $p\times 1$. And they are nothing but $\underline{\tilde{X}}$, $\underline{\tilde{u}}$ and $\underline{\tilde{Y}}$, we are just going to drop this tilde from the rest of the treatment.

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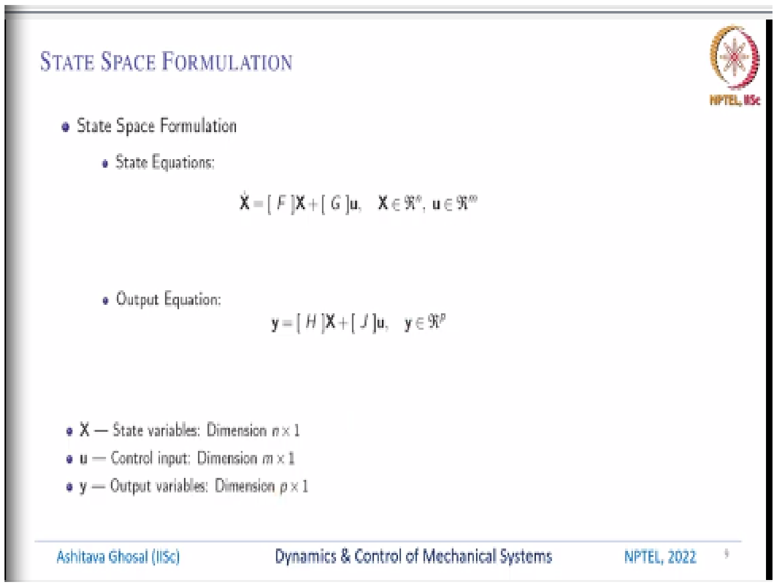

So, in summary we can obtain the state space formulation of a linear time invariant system in these following 2 sets of equations. One is called as the state equation which is $\dot{X} = [F]X + [G]u$, X is an element of R^n , u is an element of R^m. And then we have an output equation which is $y = [H]X + [J]u$ and y is an element of R^p . So, this is X is again to repeat at the state variables, it has dimension of $n\times 1$, u are the control inputs which has dimension of $m\times 1$, y are the output of the measured outputs and this has dimension of $p\times 1$.

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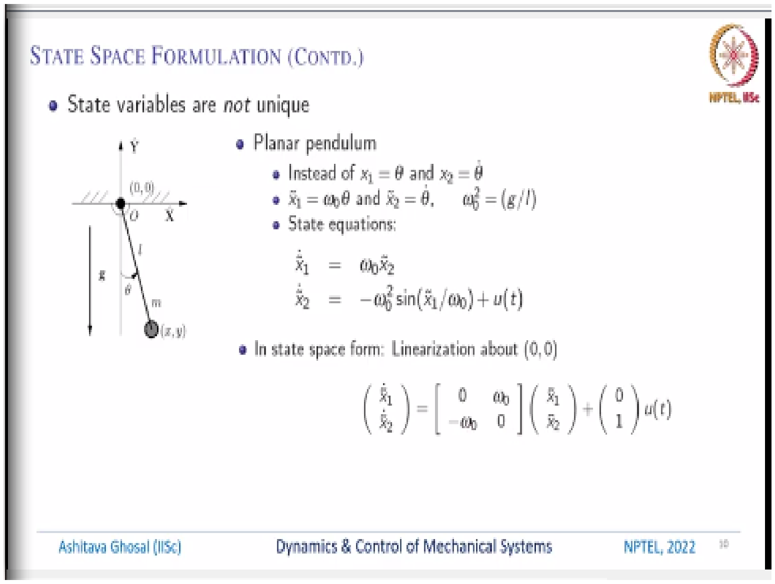

So, let us continue one of the most basic issues in state space formulation is that state variables are not unique and I will show you an example. Again we will use this planar pendulum, so now we have this pendulum with the mass which is oscillating here with an angle θ, gravity is this way and we had chosen the 2 state variables x_1 and x_2 as θ and $\dot{\theta}$. We could also use another set of state variables which are $\tilde{x}_1$ and $\tilde{x}_2$.

$\tilde{x}_1$ is defined as ω_0 θ and $\tilde{x}_2$ is same as $\dot{\theta}$, so $\tilde{x}_2$ is same as earlier x_2. However $\tilde{x}_1$ is a scaled version of the previous x_1 and what is ω_0 ? $\omega_0^2 = g/l$. So, now if you obtain the state equations you will get these following state equations which is $\dot{\tilde{x}}_1 = \omega_0 \tilde{x}_2$ and $\dot{\tilde{x}}_2$ dot is $-\omega_0^2$ sin sin $(x_1 /$ $\omega_0)$ +u(t).

Again if you linearize about $(0, 0)$ we will get $\tilde{x}_1$ and $\tilde{x}_2$, so this is like $\dot{x}$ $[F]0$ ω_0, $-$ ω_0 $0 \left(\tilde{x}_1 \tilde{x}_2\right)$ and $(0\ 1)\ u(t)$. So, you can see that the $[F]$ $matrix$ has changed, in the initial [F] matrix it was 0 1 and then something like $\frac{g}{l}$ and 0 .

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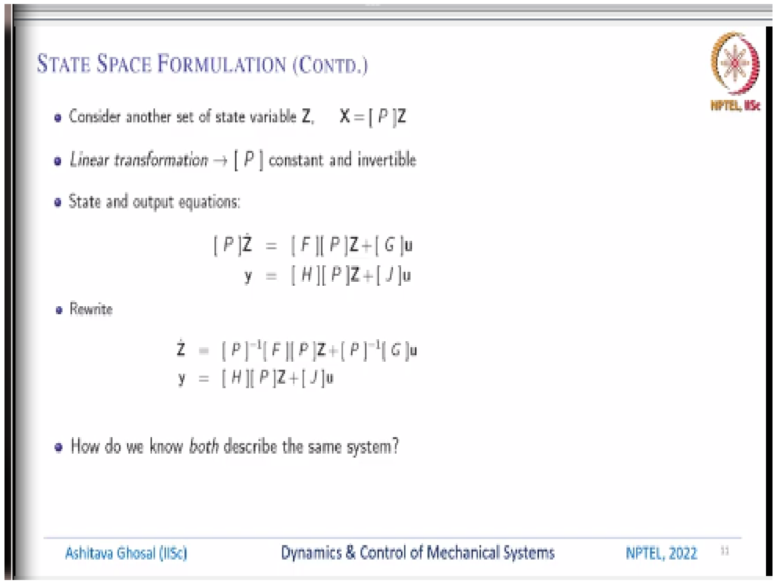

In general we can have another set of state variable Z and let us assume that Z and X are related by this transformation $X = [P]Z$ and it is a linear transformation. Meaning, that this $[P]$ is constant and this $[P]$ is full rank, so $[P]$ is invertible, so X is $n\times1$, so Z is also $n\times1$and this $[P]$ is $n\times n$ matrix and it is constant elements, and it is invertible. So, now in terms of Z we can write the state and output equations in this form.

So, basically we had $\dot{X} = [F]X + [G]u$, now we have X is $[P]Z$, so we have $[P]\dot{Z} = [F][P]Z + [G]u$. Remember $[\dot{P}]$ is 0, so if you were to worry about what happened to when you did $\dot{X}$ should we get a $[\dot{P}]$ Z, no, that term is 0. So, $\dot{X} = [P]\dot{Z}$ which is what is here and likewise we have $y = [H][P]Z + [J]u$. So, we could rewrite these state an output equations as $\dot{Z} = [P]^{-1}[F][P]Z + [P]^{-1}[G]u$ and $y = [H][P]Z + [J]u$.

Remember, $[P]$ is invertible, so $[P]^{-1}$ exist. So, the first basic question is how do we know that both these just systems are same? So, if you are a analyst somewhere which you are using X as your state variables you would have got $\dot{X} = [F]X + [G]u$. If you are in another analyst somewhere else and you are using Z as your state variables, then you will get $\dot{Z} = [P]^{-1}[F][P]Z + [P]^{-1}[G]u$ and the output equations are now $y = [H][P]Z + [J]u$.

So, it is not clear and remembers these are all numbers, it is not clear that we are describing the same system. So, what is common between this description using Z and the previous description of the state equations and the system using X, so let us continue?

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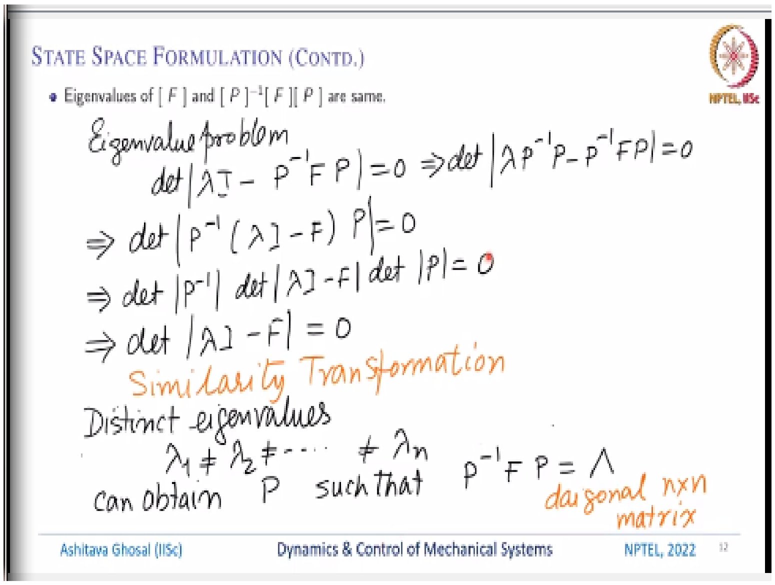

So, what is common between $[F]$ and $[P]^{-1}[F][P]$ let us start with that. So, one important thing that if you have studied linear algebra you can look at the Eigen values of this $[F]$ and $[P]^{-1}[F][P]$. So, what is the Eigen value problem? We have to solve a characteristic polynomial which comes from setting determinant of $\lambda[I]$ - $[P]^{-1}[F][P] = 0$, so $[I]$ is the identity matrix. So, this we can simplify as determinant of $\lambda\ [P]^{-1}[P]$, $[I]$ could be written as $[P]^{-1}[P] - [P]^{-1}[F][P] = 0$.

And again we can simplify again and we can write this expression as determinant of $[P]^{-1}(\lambda[I]$ - $[F])$, so basically we will take $[P]$ outside. And then we will have $[P]^{-1}(\lambda[I] - [F])[P] = 0$. So, the determinant of a product of 3 matrices are the determinant of each one of these 3. So, we can write as determinant of $[P]^{-1}$ then determinant of $\lambda[I] - [F]$ and then determinant of $[P] = 0$.

So, this itself can be simplified because now these are scalars, so we can take determinant of $[P]^{-1}$ and then we can go back and put this on this side, so you will get determinant of $[P]$. So,

$[P]^{-1}$ P is identity matrix, so you are left with determinant of $\lambda[I] - [F] = 0$. So, what have we showed you that the Eigen values of $[P]^{-1}[F][P]$ is the same as the Eigen values of $[F]$, so this is called as similarity transformation.

So, if these Eigen values are distinct meaning there are n Eigen values of $[F]$ and let us assume that λ_1, λ_2 all the way to λ_n are the n Eigen values and if they are not equal to each other then you can obtain $[P]$ such that $[P]^{-1}[F][P]$ is a diagonal matrix. So, this is a diagonal $n \times n$ matrix and each of these diagonals are λ_1, λ_2 all the way to λ_n.

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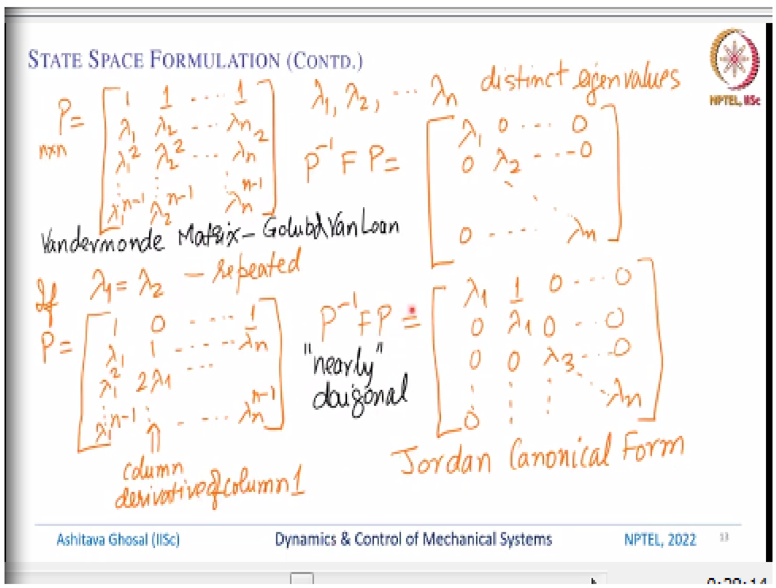

So, how do I find out what is $[P]$? Firstly $[P]$ is an $n \times n$ matrix because we are doing $[P]^{-1}$ F $[P]$, so all of them are $n \times n$ matrices. So, there is a very nice formula to obtain $[P]$, this is called as the Vandermonde matrix. You can look at this very good book on linear algebra by Golub and Van Loan. So, what it says is to obtain this $[P]$ we put in the first column 1, λ_1, λ_1^2 all the way till λ_1^{n-1}.

Second column is 1 λ_2, λ_2^2 all the way to λ_2^{n-1} and the last nth column is 1 λ_n, then $\lambda_n^2 \lambda_n^{n-1}$. And remember we are discussing the case of λ_1 λ_2 all the way to λ_n are distinct Eigen values. So, if you were to use this matrix $[P]$ then $[P]^{-1}$ $[F][P]$ will be a diagonal matrix with λ_1 λ_2 all the way till λ_n on the diagonal and all other elements are 0.

If $\lambda_1 = \lambda_2$, so let us say 2 of them are repeated then the $[P]$ can also be determined. In the first column of the $[P]$ is 1 λ_1, λ_1^2, λ_1^{n-1} . Then the second column is which is corresponding to λ_2 is 0, 1, $2\lambda_1$ all the way till λ_1^{n-1} into λ_1^{n-2}. So, basically how did I get this second column? It is the derivative of this first column.

And since we are only considering 2 of them repeated, the rest of the matrix remains same, the nth column is 1 λ_n all the way till λ_n^{n-1}. Now if you compute this $[P]^{-1}$ $[F][P]$ what you will get is λ_1, 1 comma λ_1 in the second column and again 3, 4, 5 all the way till n are distinct. So, it is not exactly a diagonal form but this is the best that you can do, this is called as the Jordan canonical form.

In the Jordan canonical form and in this example there are only 2 repeated Eigen values. You will have first column is λ_1, 0, 0, 0 all the way, the second column is 1, λ_1, 0, 0, 0 all the way, the third column is 0, 0 λ_3 zeros and so on.

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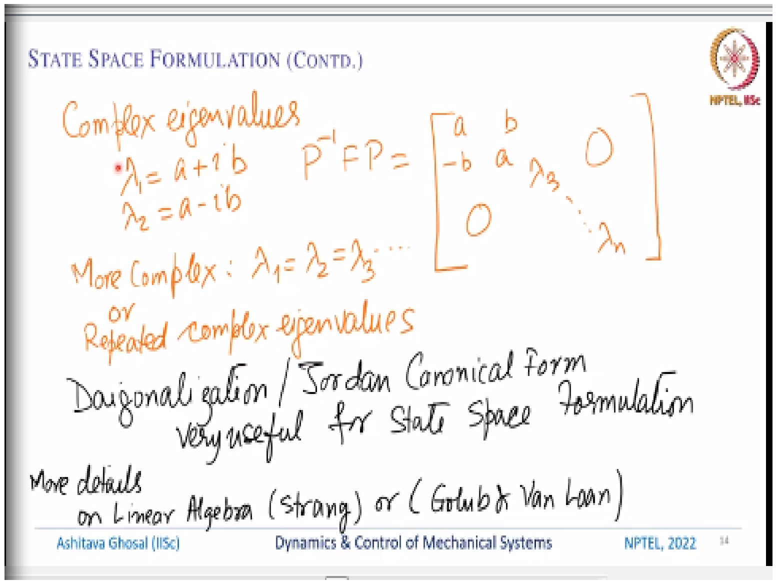

So, while you are solving these eigenvalues of a matrix you can also get complex and Eigen values. So, we can have a $\lambda_1 = a + ib$ so the second Eigen value will be $a - ib$ this is a property of complex numbers. So, one of them is an Eigen value which is $a + ib$ the other one must be $a - ib$. So, in that case if you do $[P]^{-1}[F][P]$, then what you can see is you will get $a\ b\ -b\ a$ and again since we are taking only 2 of them are equal then we will have λ_3 all the way till λ_n and all these off diagonal terms to be 0.

So, this is the best we can do if you have complex conjugate Eigen values. Of course there could be more complications; you can have many of these Eigen values equal. So, if there are 3 of them equal then you will have some things which is happening in the first column of the P, the second column of the P, the third column of the P, the third column is the derivative of the second column, the second column is the derivative of the first column and so on.

So, you will have some more complications, so it will not be exactly diagonal but it is still the Jordan form. You can also have repeated complex Eigen values, so you can find out what happens when you have a repeated complex eigenvalue, what happens to $[P]^{-1}[F][P]$. So, this diagonalization or this Jordan canonical form these are very useful for state space formulation, so we will see that in the next few slides.

For more details on this diagonalization or Jordan form or complex Eigen values you should look at a modern or a good textbook on linear algebra. So, one such recommended textbook is by Strang or you can also go back and see Golub and Van Loan.

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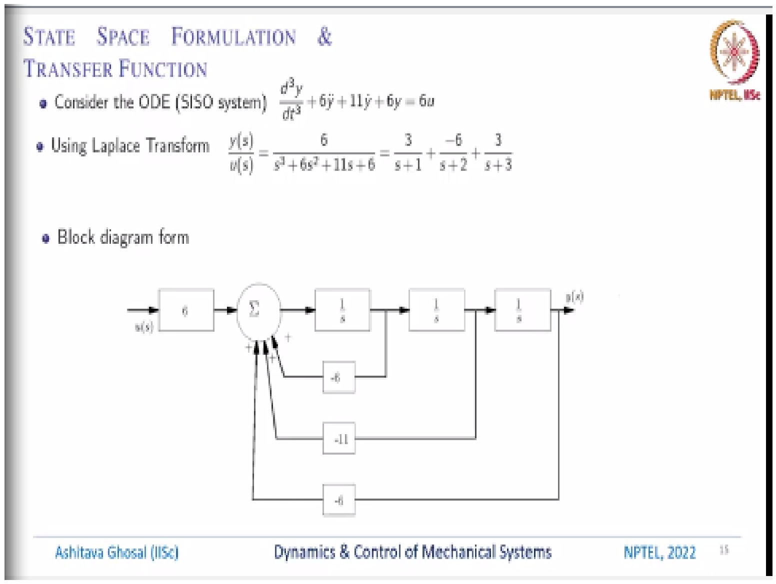

So, let us go back to state space formulation and transfer function. So, let us consider an ODE or and in particular a single input single output system. The ODE is given in this form, so $\frac{d^3y}{dt^3} + 6\ddot{y} + 11\dot{y} + 6y = 6u$. So, this is my ordinary differential equation which describes a single input single output system. So, if you were to use Laplace transform on this, so what would you get?

So, the output is $y(s)$ input is $u(s)$, so this will be some s^3y, this will be $6s^2y$, this will be $11sy$ and this is $6y$, all the initial conditions are 0. So, whenever we take Laplace transform we assume that the initial conditions are 0. So, what is the transfer function between y and u? It is $\frac{6}{s^3 + 6s^2 + 11s + 6}$. So, I have chosen this example because we can now very simply do a partial fraction of this.

So, if you go back and remember what partial fraction is, we can write this as some A divided by s + something + B divided by s + something and C divided by s + something, so it turns out it can be written as $\frac{3}{s+1} - \frac{6}{(s+2)} + \frac{3}{s+3}$. So, this transfer function $y(s)/u(s)$ can be written as sum of these 3 fractions. So, in the block diagram form I can represent this in the following. So, what do I have?

I have an input which is $u(s)$ and output which is $y(s)$. So, first thing you can see is this is multiplying by $6u(s)$ and then $6u$ into $u(s)$ is given by all these terms. So, you have $6u(s)$ is $y(s)$ into all this $s^3 + 6s^2 + 11s + 6$. So, that could be written in this way. So, what do we have here? So, you have a block which is $1/s$, this is called as an integrator, remember Laplace transform of y($1/s$) is nothing but the integration of that quantity.

So, we have $1/s$ and the output of that you multiply by -6 and then you put it back here. Then you integrate again and then the output of that you multiply by -11 and put it back here and final output of this you integrate and you multiply it by -6 and you push it back here. So, basically what are we doing? We are finding that $(1/s)$ 6 + 11 s and integral and then -6 s, so we are putting back all these things here and then if you sum it up then you will get these. So, this is a block diagram form of this transfer function.

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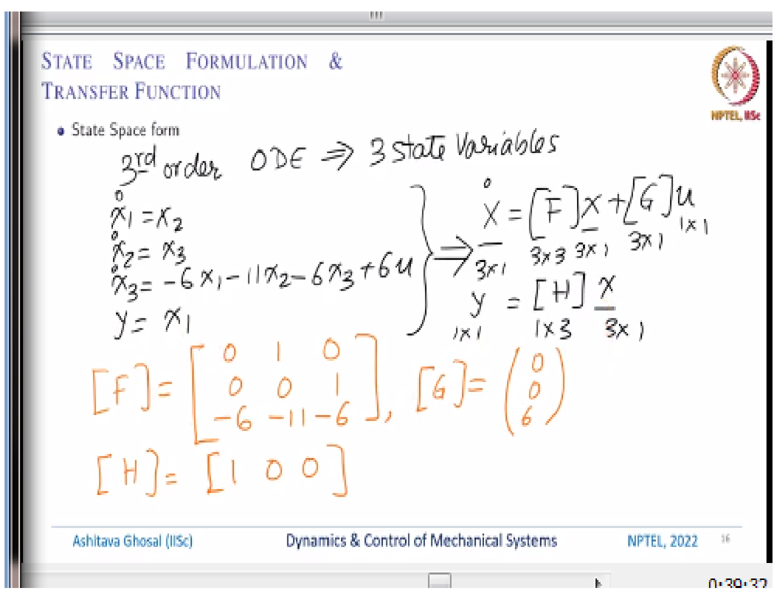

So, let us look at the state space form of that equation. So, this is the third order ODE, so there are 3 state variables, so first equation is

$\dot{X}_1 = X_2$, $\dot{X}_2 = X_3$ *and* $\dot{X}_3 = -6X_1 - 11X_2 - 6X_3 + 6u$. And what is the output variable? That is X_1. So, we can rewrite these 3 first order equations as $\dot{X} = [F]X + [G]u$, so what is X here? It is 3×1, it is a vector of X_1, X_2, X_3, so $\dot{X}$ is $\dot{X}_1$, $\dot{X}_2$, $\dot{X}_3$ that is equal to $[F]X$.

So, what is the first row of X that you can see from this equation? So, $\dot{X}_1 = X_2$ which is 0 1 0 into X_1, X_2, X_3, so this will be the 1 here. The second equation is $\dot{X}_2 = X_3$, so what will be the second row of $[F]$? It will be 0 0 1 and then the third equation is $\dot{X}_3$ is this $-6X_1 - 11X_2 + -6X_3 + 6u$. So, this is $-6X_1 - 11X_2 + -6X_3 + 6u$ and what is the rest of it, what will be $[G]$, u here is only 1, right.

So, there is only one input and one output, so this is a single input single output system. So, $[G]$ will be 0 0 6 into u because remember there is a $6u$ term. And how about the output equation y = $[H]X$, so what is the single measurement we are doing? A single output which is X_1, so it is 1 0 0 into X. So, y is 1 0 0 into X_1, X_2, X_3, so basically $y = X_1$.

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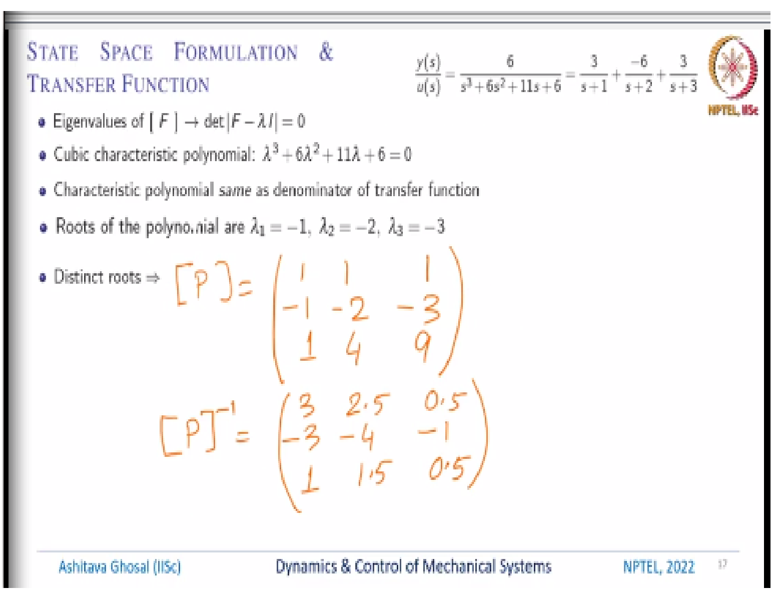

So, the Eigen values of $[F]$ are obtained by solving determinant of $[F] - \lambda[I] = 0$. So, this $[F]$ is a 3×3 matrix, so the characteristic polynomial would be cubic and it turns out when you expand when you put all the whatever is $[F]$ there, all the elements of $[F]$. You will get $\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$. So, the characteristic polynomial is same as the denominator of the transfer function; this is a very, very useful observation.

And what are the roots of the polynomial? Their $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$. So, these are distinct roots. So, hence we can find $[P]$ using that Vandermonde matrix, so remember the first column was 1, then λ_1, then λ_1^2, second column was 1, λ_2, λ_2^2, third column was 1, λ_3, λ_3^2. So, λ_1, λ_2, λ_3 are given here, we can compute λ_1^2, λ_2^2, λ_3^2 and all the terms.

And we will get $[P]$ which is this, $[P]^{-1}$ can also be calculated, this is a reasonably simple matrix and you can also go back and use MATLAB. So, if you type this matrix in MATLAB and say I want the $[P]^{-1}$ you will get back this. So, the $[P]^{-1}$ has first column $3\ -3\ 1$, second column $2.5\ -4\ 1.5$, third column $0.5\ -1\ 0.5$.

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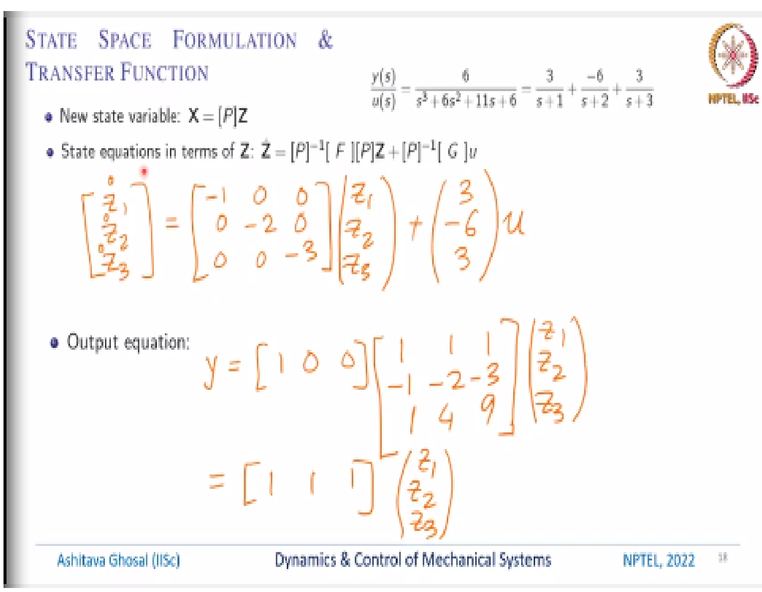

So, as I said let us choose a new state variable Z, $X = [P]\,Z$. So, the state equations in terms of Z are $\dot{Z} = [P]^{-1}[F][P]\,Z$, go back and see the formulation. And then $[P]^{-1}[G]\,u$, so what do we get? We know what is $[P]$, we know what is $[F]$, so you can substitute, and we can do some simplification. Then you will get $\dot{Z}_1$, $\dot{Z}_2$, $\dot{Z}_3$ and what do we expect? We expect it to be diagonal, remember $[P]^{-1}[F][P]$ will become a diagonal matrix that was the purpose of finding that $[P]$ matrix.

So, $\dot{Z}_1 = -Z_1 + 3u$, $\dot{Z}_2 = -2Z_2 - 6u$ and $\dot{Z}_3 = 0\ 0 - 3Z_3$ or $3 - 3Z_3 + 3u$ and how about the output equation? Which is $Y = [H][P]Z$, again $[P]$ is we know that, so what you will see is if you simplify this you will get 1 1 1, this into this is 1 1 1 into Z_1, Z_2, Z_3. So, what have we got? So, we had one form which is $\dot{X} = [F]X + [G]u$ and now we have another form which is $\dot{Z}$ = some matrix times Z + some other matrix times u and likewise we had $Y = [H]X$, now we have y is some matrix times Z.

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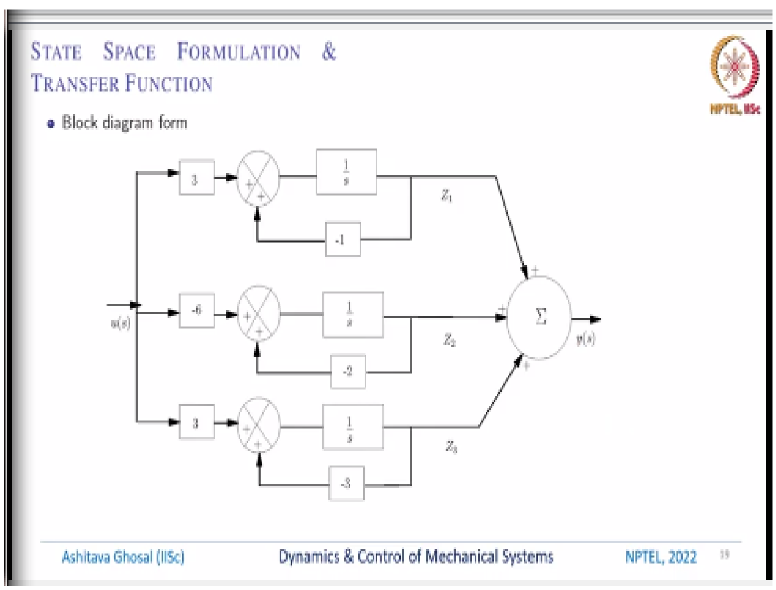

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So, let us write this transfer function in a block diagonal form. So, now what you have is u which is coming in and then you can see that you have to multiply u by $3, -6$ *and* 3. And then whatever is the output here you integrate once and you will get Z_1 and then you feedback -1. Similarly, whatever you do integrate once you will get Z_2 and then whatever you integrate here output of this input and you will get Z_3. And then when you add $Z_1 + Z_2 + Z_3$ you will get $y(s)$, remember y is 1 1 1 into Z_1, Z_2, Z_3.

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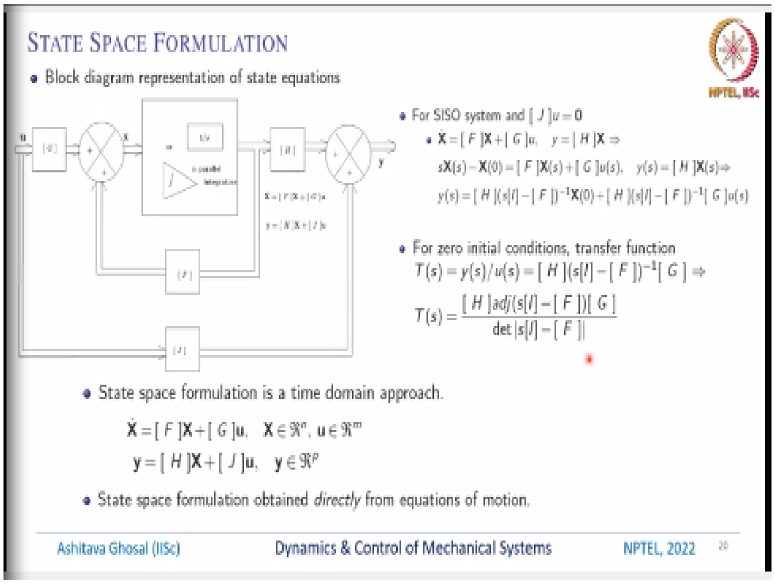

So, in general if you have a system which is $\dot{X} = [F]X$ and then you have $y = [H]X + [J]u$, so you have $\dot{X} = [F]X + [G]u$ and $y = [H]X + [J]u$. We can also write it symbolically in a block diagram form. So, this is the block diagram representation of the state equations. So, what do we have? We have u which is coming in, so this is $[G]u$ and then we also need to add $[F]X$, so we will get $\dot{X} = [F]X + [G]u$.

And how do I get x here? That is the integral of this $\dot{X}$, so that is what is shown here you will have $\frac{1}{s}$, actually there are if it is a vector you will have n parallel integrators. So, each of those $\dot{X}$ needs to be integrated and you will get X, you multiply by $[F]$ and then you add to $[G]u$, you will get X, so this is the feedback part. The output of this integrator is X, so you can multiply it by $[H]$ and you also need $[J]u$.

So, you take this u which is input directly multiplied by $[J]$ matrix and then you add these 2 and you will get y. So, this is a nice way to represent what is happening in the state equations. So, if you just have state equations like this which is $\dot{X} = [F]X + [G]u$ and $y = [H]X + [J]u$ then you do not see this nice structure that there is some integration which is happening, input to the block is $\dot{X}$ output after integration is X and then you can feedback.

So, the state space formulation which is this can be obtained directly from the equations of motion. Remember, even if you have a non-linear equation of motion you can linearize about an equilibrium point and then you will get this $\dot{X} = [F]X + [G]u$ and $y = [H]X + [J]u$. So, this term $[J]u$ is also some times called as the direct term. So, whatever is the input this is directly going to the output, it is not going through the plant, nothing is happening, no integration or some other feedback is happening to the $[J]u$ term.

So, for most a single input single output system and $[J]u = 0$, we will have $\dot{X} = [F]X + [G]u$ and $y = [H]X$. So, we can also think of this as you can take the Laplace transform of this ordinary differential equation, it is a vector equation but we can take the Laplace transform. And we can write it as $sX(s) - X(0)$, so let us keep the initial conditions for a while, $[F]\ X(s) + G\,u(s)$.

And the output equation can be written as $y(s) = [H]X(s)$. So, both of these 2 can be now written in a simple form after some manipulation as $y(s) = [H]X$ what is X? X is $\left(s[I] - [F]\right)^{-1} X(0) + [H]\left(s[I] - [F]\right)^{-1}[G]\,u(s)$. So, think about it that we want to rewrite this we have to take X this side. So you will have $s[I] - [F]$ and then you will have something which is multiplying $X(0)$ and something which is multiplying $u(s)$.

So, for 0 initial conditions which is $X(0)= 0$ then the transfer function is nothing but $y(s)/u(s)$. Remember, now we are looking at single input single output system, so y is 1 dimensional, u is also 1 dimensional. So, this same expression here $y(s)/u(s)$, you drop this term which is $X(0)$, so what do we left with is $[H]\left(s[I] - [F]\right)^{-1}[G]$. So, the transfer function of a state equations which is of this form $\dot{X} = [F]X + [G]u$ and $y = [H]X + [J]u$.

For a single input single output system with initial condition 0 is $[H]\left(s[I] - [F]\right)^{-1}[G]$ and this can be written as adjoint. So, the inverse is nothing but the adjoint divided by the determinant. So, we have $\frac{[H]adj\left(s[I]-[F]\right)[G]}{det|\left(s[I]-[F]\right)|}$.

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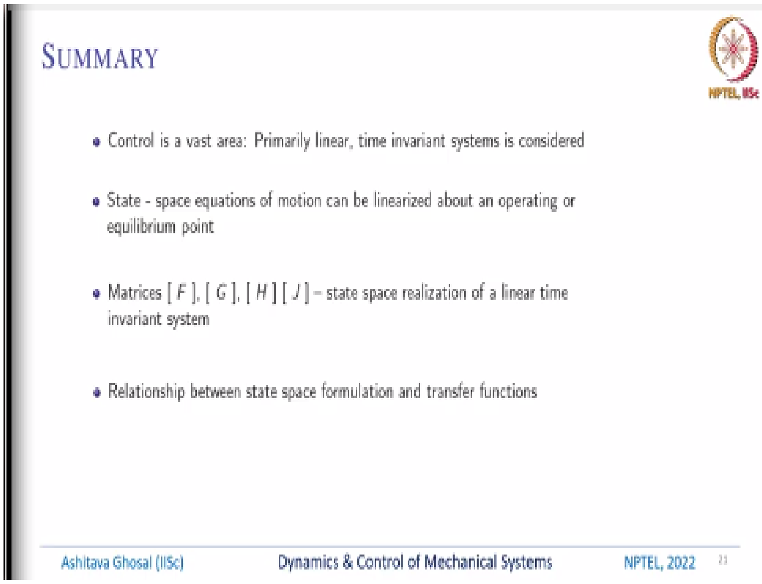

So, in summary control is a very vast area. We have linear system, non-linear systems, linear time in varying system, linear time varying system, SISO system, MIMO system all kinds of classification of dynamical systems we have. It is a very vast area; we are primarily interested in linear time invariant systems. We can obtain the state equations of motion, state space equations of motion from the dynamics of the system they can be linearized about an operating or equilibrium point.

And then we get these matrices $[F]$, $[G]$, $[H]$ and $[J]$, so these 4 matrices are sometimes called as the state space realization of a linear time invariant system. If this was a function of time, this is also a function of time, $[H]$ is a function of time then you will have linear time varying system. And I showed you there is a relationship between the state space formulation and the transfer function. So, the transfer function for a SISO system $y(s)/u(s)$ is nothing but $[H]\left(s[I] - [F]\right)^{-1}[G]$.