Dynamics and Control of Mechanical Systems

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Lecture-20
Control of Mechanical Systems

Welcome to this NPTEL lectures on dynamics and control of mechanical systems. My name is Ashitava Ghosal and I am a professor in the department of mechanical engineering, centre for product design and manufacturing and also in the Robert-Bosch Center for Cyber Physical Systems, Indian Institute of Science, Bangalore. Till now we have looked at kinematics and dynamics of mechanical systems, from now onwards we will look at the control of mechanical systems.

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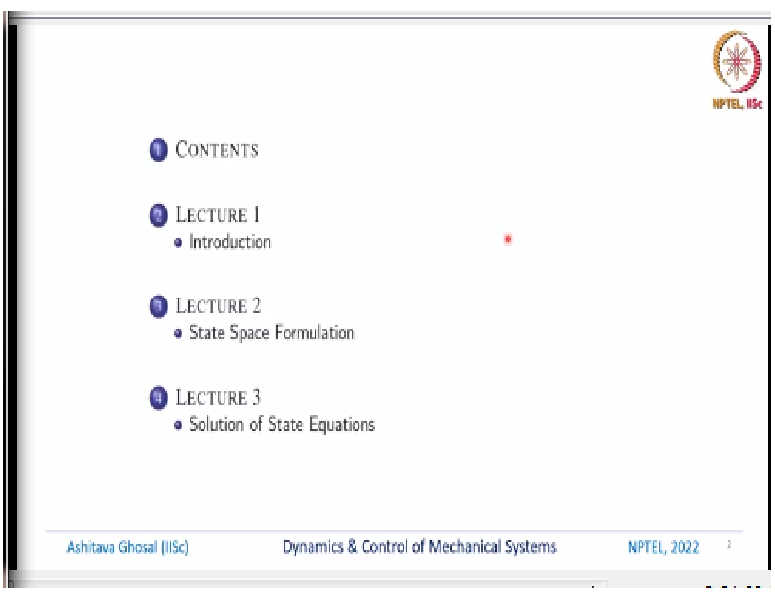

In these lectures we will first introduce the topic of control, then we will look at state space formulation for control of mechanical systems and in the last lecture we look at the solution of state equations.

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So, first lecture is an introduction.

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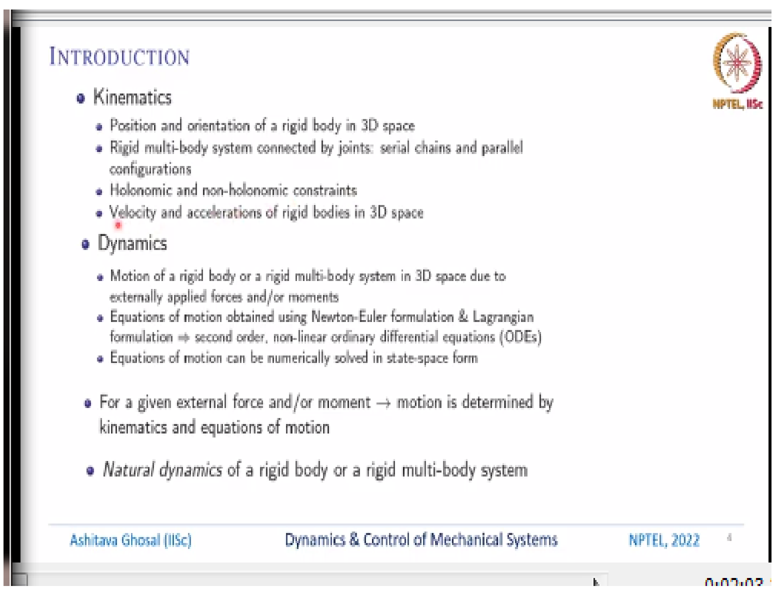

So, till now in kinematics we have looked at position and orientation of a rigid body in 3D space. We have also looked at rigid multi-body systems connected by joints and we saw that it could be in the form of a serial chain or in a parallel configuration with loops. For parallel configurations we found that there were holonomic constraints or loop closure equations. And they were also in some examples we looked at non-holonomic constraints which are basically constraints related to the velocities of the rigid body.

We also looked at velocity and acceleration of rigid bodies in 3D space. In dynamics, we looked at the motion of a rigid body or a rigid multi-body system in 3D space due to externally applied forces and or moments. We obtained the equation of motion of rigid multi-body systems using Newton-Euler formulation and Lagrangian formulation. The equations of motion for rigid multi-body systems were basically second order non-linear ordinary differential equations or ODEs.

These equations of motion could be solved numerically in state space form, I also showed you some computer tools which can be used to derive the equations of motion and also model and simulate rigid multi-body systems. So, for a given external force and or moment the motion is determined by the kinematics and equations of motion. In a sense this could be called the natural dynamics of a rigid body or a rigid multi-body system.

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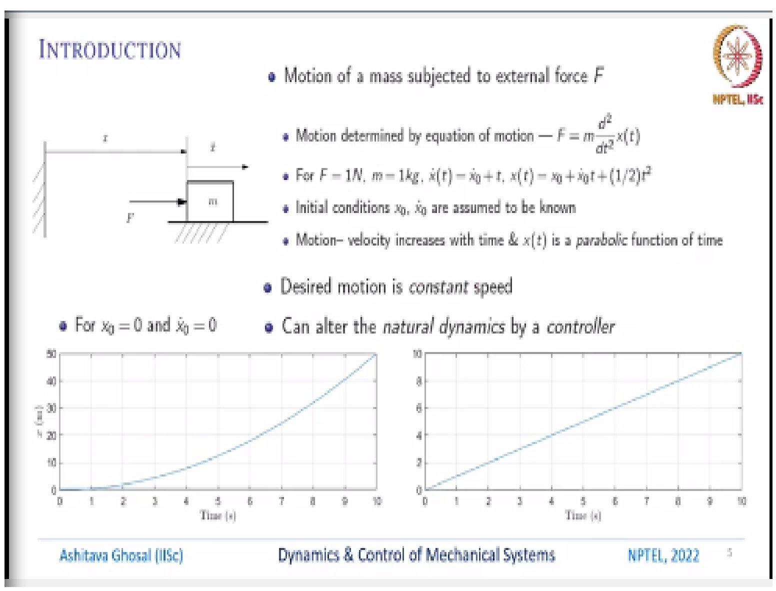

So, let us look at an example, this is an example of very simple example of a mass m in which there is an external force F which is acting from left to right. The mass is at a distance x from some reference and it has an acceleration $\ddot{x}$. So, the motion of this mass subjected to an external force is given by this equation of motion which is $F = m\, a$, everybody knows this, it is very simple straightforward, so this is $m\frac{d^2X}{dt^2}$.

So, for an assumed force of 1 Newton, mass of 1 kg you can integrate this equation once to obtain $\dot{x} = \dot{x}_0 + t$ and then we can integrate once more to obtain $x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}t^2$. So, for some initial conditions x_0 and $\dot{x}_0$ which are assumed to be known, we can obtain the motion of this mass which is basically $x(t)$. So, as you can see the velocity of this mass which is $\dot{x}$ increases with time.

And the position of this mass which is $x(t)$ is a parabolic function of time. So, in this example we have assumed $\dot{x}_0 = 0$ and $x_0 = 0$. So, we can plot this, so this looks like this parabola which is $x(t)$ is $\frac{1}{2}t^2$, so it looks like this. But suppose we want this mass to have a constant speed, so right now the speed is increasing with time. So, this is the position which is parabolic, so if I were to plot $\dot{x}$ it will be a linear.

But what I want is that the velocity should be constant speed, so what I want is $x(t)$ to be linear. So, I want $x(t)$ to be linear like this. So, is it possible to ensure that this mass has a constant speed? Not, a speed which is increasing with time. And this natural dynamics can be altered by a controller; this is what I will show you in the next slide.

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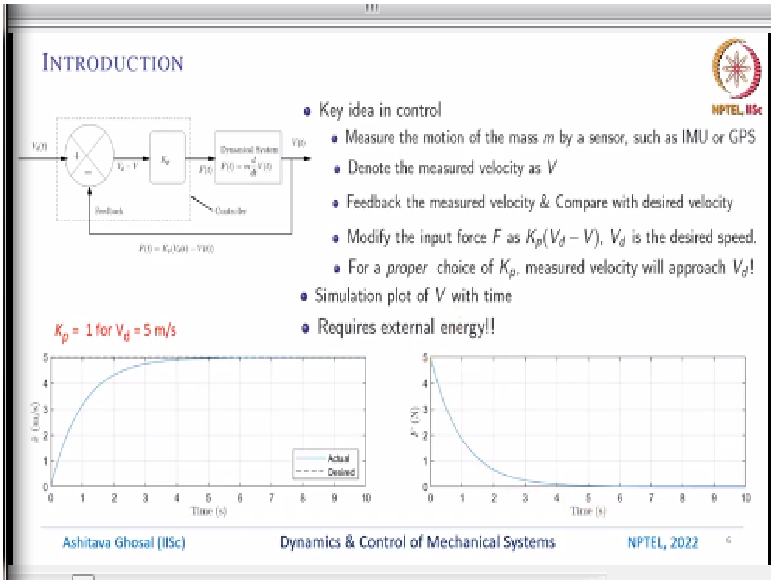

So, the key idea is the following, we have a dynamical system which is given by $F(t) = m\frac{dV(t)}{dt}$. So, the input to this dynamical system is this force which could be a function of time and the output is the velocity. So, what we are going to do is that we will measure this velocity of this mass and feed it back. So, what do we mean by feed it back? We will have some desired velocity; remember we want the velocity to be constant.

So, we measured the velocity and we feed it back and what this feedback is doing is we will subtract the desired minus the measured so $V_d - V(t)$ and then we will multiply this difference by some number, by some constant K_p and the output of this block is the force. So, this is a closed loop control system, where a force is acting on the dynamical system, due to the action of the force the mass will start to move and we are going to measure the velocity.

And then we will find the difference between the velocity which we measure and what we want and then we are going to multiply by some constant and the output of this controller will be the force. So, the key idea in control is we measure the motion of the mass m by a sensor and this could be lots of different types of sensors, it could be an IMU or a GPS. We denote the measured velocity as V, we feedback the measured velocity and compare with the desired velocity which is $V_d(t)$.

And we modify the input force to this mass as $K_p(V_d - V(t))$ where V_d is the desired speed. So, for a proper choice of K_p, I will show you that this measured velocity will approach V_d. So, remember the natural dynamics was that if I apply a force to this mass the velocity will continue to increase. The position vector was parabolic, so the derivative of the position vector was a something like C into T or some constant into time, so it would increase with time.

But what we want is that the velocity which we want this mass to have should be constant, so V_d will be a constant number. So, for this $K_p = 1$, remember there is a block here which is K_p, we will go over this much later in more detail, this is sometimes called as the proportional constant. So, K_p as 1 and we want V_d to be 5 m/s constant, so these numbers have been picked to show that we can design something or we can have a controller which will ensure that the mass has a constant speed.

And then we can plot this V as a function of time, so again we will come to later how we can plot the output as a function of time for a closed loop system like this or a feedback system like this. But let us for the moment assume that we can simulate this system and we can find what is $V(t)$. So, we can plot this, so this is $\dot{x}$ or V and what you can see is that actual velocity will slowly increase and approach this desired velocity which is 5 m/s.

And as the time is increasing, so somewhere after 5 seconds or you can say 6 seconds the velocity of this mass is now 5 m/s approximately. So, we can plot what is the desired velocity which is this dotted line and we can simulate this system and find out what is the actual output of this system which is $V(t)$. So, this is very good, it is amazing because the natural dynamics was not a constant velocity but now we have ensured that we can get a constant velocity.

So, there must be some payback it cannot be free, so it requires external energy to achieve this desired velocity and that we can see from the plot of this force versus time. So, remember in the initial case we had force of 1 Newton and that is what was giving this steadily increasing velocity and the position was a quadratic function $\frac{1}{2}t^2$. But now what you can see is in order to achieve this constant velocity we have to apply a different force.

So, we need to apply 5 Newtons and then eventually it will go to 0. Once this velocity has been achieved, once it is going at constant speed of 5 m/s we do not need to apply any more force. So, it is nothing is free, so to achieve this desired characteristic or the desired dynamics of this simple mass we need to apply external energy which is varying with time.

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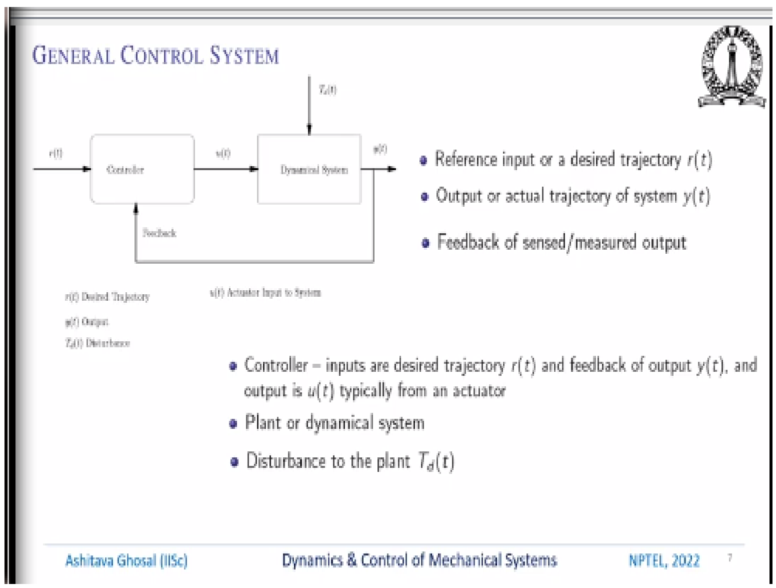

So, the general control system looks like this. So, we have a controller then we have a desired trajectory which is $r(t)$. In the previous example the desired trajectory was 5 m/s. Then the output of the controller is $u(t)$, this is sometimes also called as the actuator input to the system. So, we have to apply some force or later on we will see we have to apply some voltage to this dynamical system.

And then the dynamical system will respond and we will get an output which is y. In any control system we can also have something called $T_d(t)$, so this is the disturbance which is coming from the environment on to the dynamical system. And as shown in the previous slide we can measure this output and we can feed it back to the controller. So, this $r(t)$ is sometimes also called as the reference input or it is the desired trajectory.

The output or the actual trajectory of the system is denoted by $y(t)$ and we need to measure this output using a sensor. So, we have to sense this output, we have to measure it and then we have to feed it back into the controller. So, the controller inputs are the desired trajectory or the reference input, the feedback which is $y(t)$ and the output of the controller is $u(t)$ which is typically like some kind of a force or voltage which will go into the actuator of the dynamical system.

So, this dynamical system is also sometimes called as a plant. So, we have a controller, we have measurements, we have disturbance and this is called the plant. So, as I said $T_d(t)$ is denotes a disturbance to the plant.

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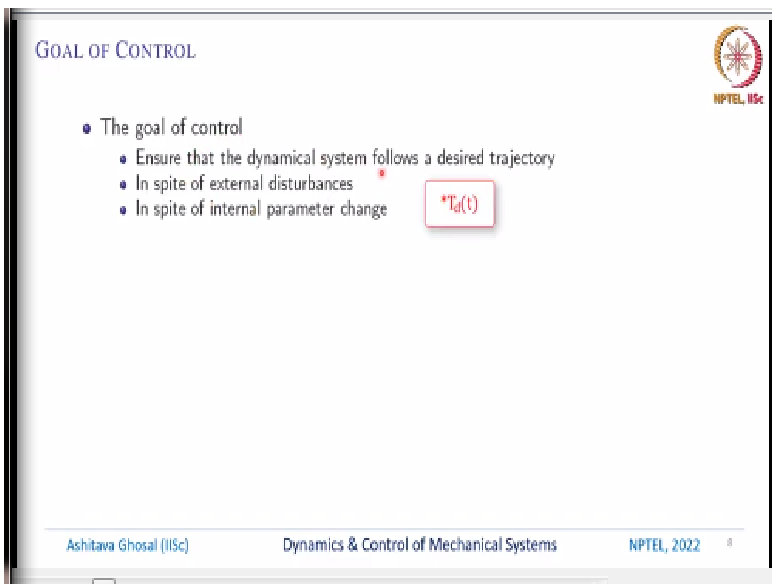

So, the goal of control is to ensure that the dynamical system follows a desired trajectory. So, as an example I showed you a mass in which a force is being applied will follow some trajectory which is the parabolic, $x(t)$ is parabolic. But we want to ensure that the dynamical system follows a desired trajectory, more so in spite of this external disturbance $T_d(t)$.

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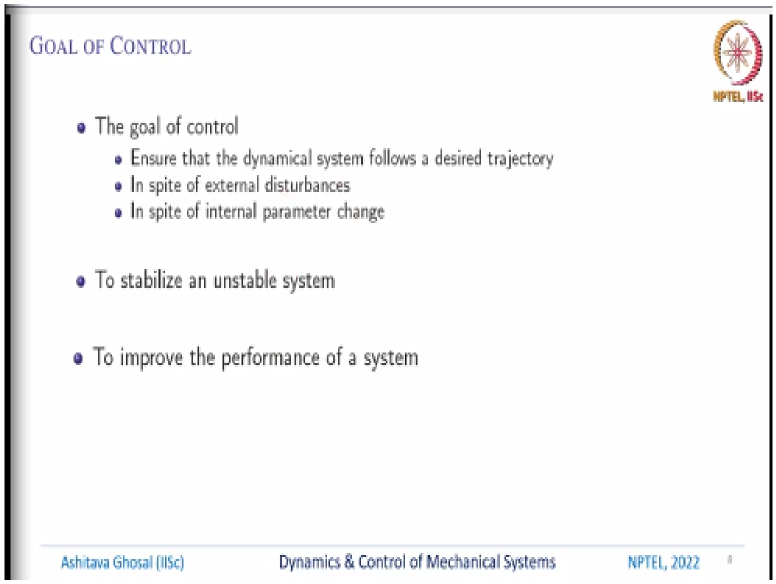

So, even if there is an external disturbance I would still like the output $y(t)$to be close to the desired $r(t)$. And secondly we would also like this system to follow a desired trajectory in spite of internal parameter change. We will come and see later what do we mean by internal parameter changes. Sometimes a control system is also used to stabilize an unstable system, so we will formally define what is a stable and unstable system but for the moment intuitively we know what is an unstable system.

So, if you give a small perturbation to the system the position or the configuration of the system changes too much, that is called as an unstable system. However if you give a small perturbation and it comes back to the place where you give the perturbation that is called as a stable system intuitively, but we will formally define what is stable and unstable and so on systems later. Sometimes the goal of control is to also improve the performance of a system.

So, I could have a system where I want this constant speed of 5 m/s but I am only getting 4.5 m/s all the time, I am not able to achieve exactly 5 m/s or closer to 5 m/s. So, I can design a controller and we will see again all these things later which will improve the performance of the system. So, basically the error between what you measure and what you want can be decreased by again by control.

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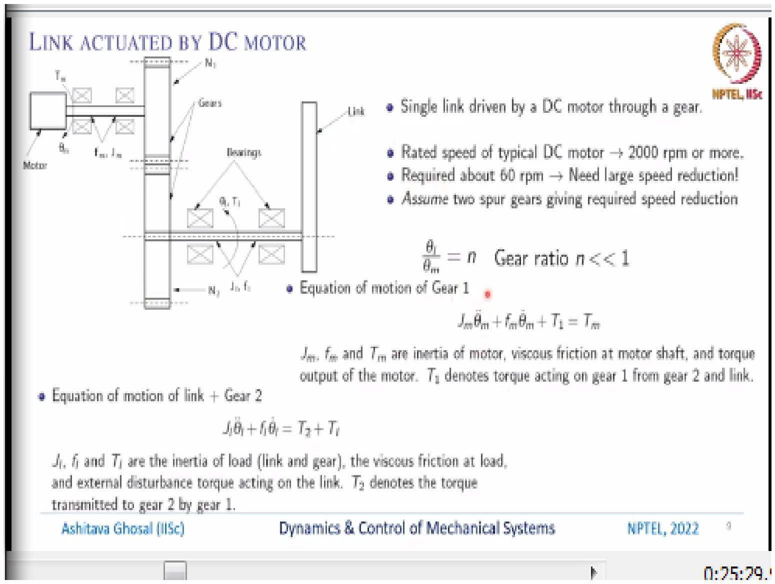

So, let us look at an example in much more detail about all these various concepts of and the goals of control. So, we will take this example of a single link which is actuated by a DC motor, so this is a very, very standard example. So, what we have is a DC motor and there is a link, so you can think of this as one of the joints driving a link in a robot or some other mechanism or it could be even simpler as something like a motor which is driving the blades of a fan.

So, we have this motor, it is applying some torque Tm and then the output shaft of the motor starts rotating and we will denote the rotation of the motor shaft by θ m, m stands for motor. And then this motor is typically connected to the link by means of gearboxes, so we normally never ever connect a motor directly to the output shaft or the output shaft to the link. So if you have a gearbox we will assume that there is a pair of gears, it need not be in a single stage.

But we will assume that it is in a single stage as far as this modeling and analysis is concerned. So, there is some N_1 teeth at the input and there is an N_2 teeth at the output and the output also shaft rotates by θ_l. We will also assume that in the input shaft there is some friction f_m and the rotational inertia of this input shaft is J_m. Similarly the output shaft has a inertia of J_l and there are some friction at these bearings and various other places or maybe even in the gears of f_l.

And one more final symbol which is T_l, so we could have some torque which is acting on this link which is denoted by T_l. So, with these symbols now we are ready to model this system. So, the purpose is I will like to model this system then analyze this system and then eventually show how we can control the motion of this link. And we can show all the nice features of control, how the control can achieve our desired trajectory in spite of internal parameter changes in spite of external disturbances and various other things.

So, as I said this is a single link driven by a DC motor through a gearbox and this is a very nice reason why we need a gearbox. So, typical rated speed of a DC motor is maybe let us say 2000 rpm, we need about only 60 rpm. Why do we need 60 rpm? So, 60 rpm is something like 1 rotation per second, whereas 2000 rpm is much, much higher. So, if you connect a link of a motor which is let us say 1 meter, so then if you have 2000 rpm you can compute the speed at the tip of this link it will be very, very high.

But typically in a robot or something like that we want this tip speed to be let us say 1 or a few m/s. So, we cannot connect a 2000 rpm motor to a link, so 2000 rpm is say 30 rotations per second and 30 rotation means 30 into 2 π , so that is something like 30 into 6 let us assume π is 3, so that is 180 radians per second. And if it is 1 meter long, so then it is 180 radian into 1 which is 180 m/s, the tip speed will be that.

Whereas we need much, much smaller, we need maybe 1 or 5 m/s, so we need to reduce the speed, so we must have a gearbox. And this is typically seen in many, many robots and many other systems where we have some kind of a gearbox to reduce the rated speed of a DC motor. So, most of the times this speed reduction cannot be achieved in 1 stage but for the purpose of this analysis we will assume that it is done by a single stage with 2 spur gears having teeth N_1 and N_2.

So, this $\theta(t)$he output or the $\theta(t)$he link divided by the $\theta(t)$he motor is n and n is much smaller than 1. So, as I showed you if you have 180 m/s and we need let us say 1 meter per second, this n will be 1 by 180, so it is much smaller than 1. So, once we have this arrangement and some simple calculations, we can now derive the equation of motion of this gear 1. So, what is the equation of motion of this gear?

We have done lots of dynamics throughout this course till now and it should be familiar to everybody; it will be like J_m into $\ddot{\theta}_m$. So, this is like $I\alpha$ + some f_m into $\dot{\theta}_m$, so this is like friction and then T_1 is the torque which is acting between these 2 gear teeth, so this is like a reaction torque. And this $T_1$is the torque which is acting on this part from this part at the where the teeth are meshing and this will be equal to the torque which is supplied by the motor.

So, J_m, f_m , T_m are the inertia of the motor, viscous friction at the motor shaft and torque output of the motor. So, T_1 denotes the torque acting on gear 1 from gear 2, so gear 1 and gear 2 and the link, so this is like the reaction torque. We can also find the equation of motion of this lin$K_p$lus gear 2. So, it is again very simple, it is some $J_l\,\ddot{\theta}_1$, so this is like J_l is the inertia of this output shaft, $\ddot{\theta}_1$ is the angular acceleration of the output shaft plus $f_l\,\dot{\theta}_l$ this will be equal to($T_2 + T_l$).

And what is T_2? T_2 denotes the reaction torque which is acting on this part from this first gear. So, it is T_2 denotes a torque transmitted to gear 2 by gear 1 and T_l as I had mentioned it could be some external disturbance torque which is acting on this link. So, we have these 2 equations, we have one which is relating the gearbox, so this entire system has 1 degree of freedom. So, whatever you do here you can easily relate this output link to the motor shaft, whatever is happening at the motor shaft.

So, we have 2 differential equations and we have one relationship between θ_l and θ m. We also need one more relationship which is how is $T_1$and T_2 related? Once we have these things I can derive the equations of motion of this entire system.

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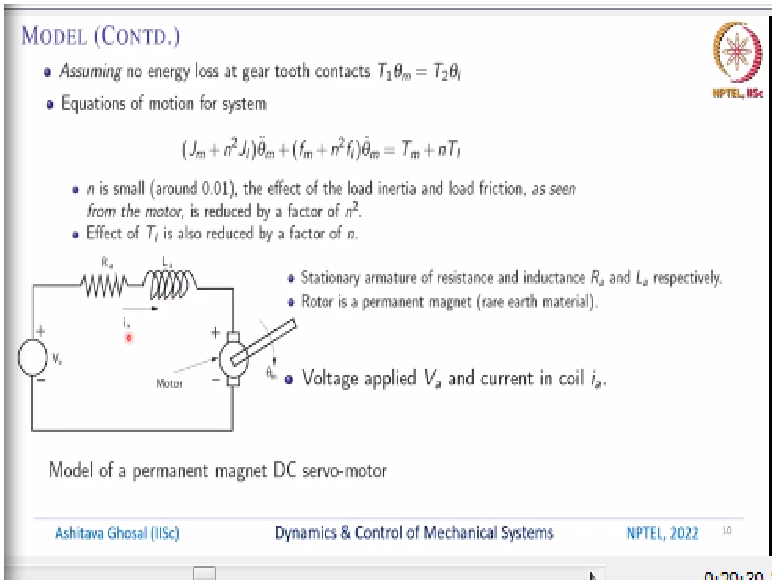

So, in order to relate what is T_1 and T_2 we will make this assumption, that there is no energy loss at the gear tooth contact. So, which is what? That $T_1\ \theta_m$ is $T_2\ \theta_l$, so $T_1\ \theta_m$ is the work done by T_1 by rotating the motor shaft by θ_m. And that should be equal to $T_2\ \theta_l$, so there is no loss at the gear tooth contact. So, once you have this equation and also the relationship between θ_l and θ_m we can get rid of θ_l, we can also get rid of T_1 and T_2 and then derive the equation of motion.

So, this you can do it, it is very easy, all you need to do is follow a few steps and do some simplification and then you can derive the equation of motion of the system. It is a 1 degree of freedom system, so it must have only one equation of motion and this is the equation of motion. What it tells you is we have some J_m which is the inertia of the motor + $n^2\ J_l$ which is the inertia of the link into $\ddot{\theta}_m$ + f_m which is the friction at the motor shaft may be at the bearings of the motor, f_l is the friction at the link and this is multiplied by n^2.

So, this into $\dot{\theta}_m$ is $T_m + nT_l$, so again you should be able to do it, it is very simple, it requires a few steps. So, what are some of the observations? n is small, remember n is like 1 by 100 roughly or even smaller. So, if n is small what is happening is you are multiplying J_l by n^2, so what it means is the inertia of the load is not seen by the motor. So, it is reduced by a factor of n square and if n is 1 by 100 then n^2 will be 1×10^4, so very little of the inertia is seen by the motor.

Likewise the effect of the external disturbance T_l is also reduced by a factor of n. So, the motor does not see what is the external disturbance which is coming on the link as it is rotating. We can also find out a model of the motor, so this is a very standard well-known model of a DC servo motor. Basically what we have is a coil which is typically the stator and then we have a rotor which is normally a permanent magnet.

So, the stator can be modeled by a resistance R_a, we can also have some kind of an inductor or inductance L_a, there is a voltage which is being applied. And then once you apply the voltage the motor shaft starts rotating with $\dot{\theta}_m$, so that is the speed of the motor shaft. So, most of the time nowadays the rotor is a permanent magnet, a rear earth material which has very high magnetic properties.

And then this stationary armature has coils which have resistance R_a and L_a. So, this is a typical model of a permanent magnet DC servo-motor. When you apply a voltage to this stator, there is a current which starts flowing in this circuit and we can denote this current by I_a, so basically this is the armature current.

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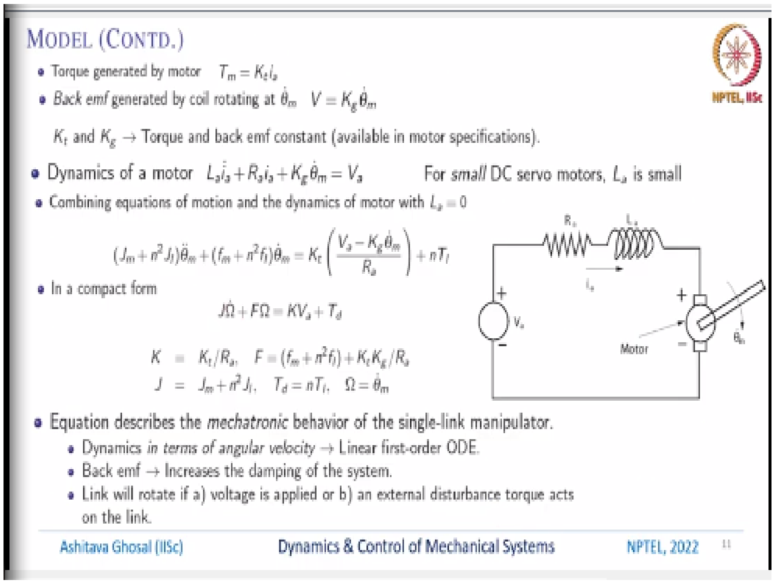

The torque generated by this motor when a current is flowing in the stator is proportional to the current so $T_m = K_t i_a$, so K_t is constant. There is also a back emf which is generated when the rotor starts rotating at some speed $\dot{\theta}_m$. And the voltage which is the back emf is $K_g \dot{\theta}_m$ and this K_t and K_g these are called the torque and back emf constants and these are typically available if when you buy a motor, it comes with the motor specifications.

The dynamics of the motor the electrical part can be written as the voltage drop across the resistance plus the voltage drop across the inductance plus the voltage drop due to the back emf will be equal to the applied voltage which is what is written here. So, it is $L_a \dot{i}_a$, the voltage drop across the inductor is proportional to the rate of change of the current with time and this is the voltage drop across the resistance very well known$R_a i_a$ and this is the voltage drop due to the back emf and this is equal to voltage applied. So, for small DC servo-motors L a is small and we can drop this term. And hence we can write, remember the right hand side was torque, so K_t times i_a but i_a can be written as $(V_a - K_g \dot{\theta}_m) / R_a$, so this is $K_t i + n T_l$, the left hand side remains as it is. So, we have something into $\ddot{\theta}_m$ plus something into $\dot{\theta}_m$.

But there is also one more term which is coming which is $\dot{\theta}_m$, that is the back emf. And finally we can simplify and rewrite this equation as a first order differential equation. So, we are going to make bunch of substitutions one is that $\dot{\theta}_m$ will be written as this Ω. So, what is this? This is J into $\Omega_d$ot. And what is J? $J = J_m + n^2 J_l$. Likewise this is $f_m + n^2 f_l$ is written as F.

But when you pull this $\dot{\theta}_m$ this side, so you will have $(f_m + n^2 f_l) + K_t K_g / R_a$. So, this is in some sense like a dissipative term, remember in a motor when you apply a current there is heating at the coils and there is some dissipation. So, this friction and this dissipation they all sort of come together. So, this is f into Ω because $\dot{\theta}_m$ is Ω and then we have K_t, so $\frac{\left(V_a - K_g \dot{\theta}_m\right)}{R_a}$, so K_t by R_a we will call it as K and this $n\,T_l$ will call T_d.

So, once we make all these substitutions we have a first order differential equation which is $J\dot{\Omega} + F\Omega = KV_a + T_d$, note that this J, F, K they are constants. So, this equation here with all these substitutions is describes the mechatronic behaviour of a single link robot or a single link being driven by a DC servo-motor. So, the dynamics in terms of angular velocity, so this is the differential equation, it is an ODE which is written in terms of the angular velocity of the motor.

So, remember Ω is $\dot{\theta}_m$ and it also includes the effect of the back emf, it also includes the effect of friction and inertia. So, what you can see from this equation is that this link will rotate if I apply a voltage. The link will also rotate if there is no voltage but there is a disturbance torque. So, you can think of this link as one of the blades of a fan, so one way this fan blade will rotate if you switch on the switch, if you apply some current, if you apply some voltage.

The other way is there is no voltage being applied but I can hit the blade with some other stick or something and then again it will start rotating. So, in this expression it tells you that there are 2 possible inputs, one is the voltage and one is the external disturbance torque and the output is Ω.

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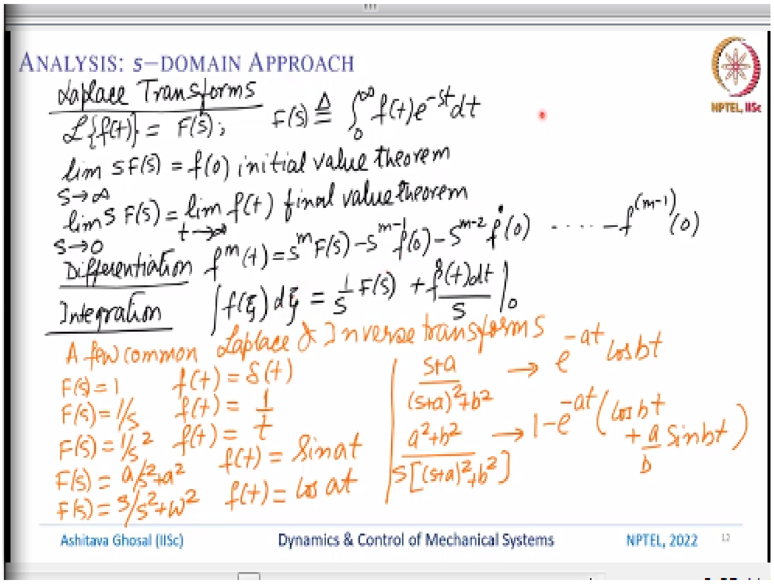

So, how do we analyze this ODE? So, in control theory one very well known or standard way is using Laplace transform. So, what is a Laplace transform? It basically converts an ordinary differential equation into an algebraic equation in s, so where s is known as the Laplace variables. So, what is the Laplace transform of $f(t)$ some function which is in time domain t? It is called $F(s)$ and it is defined as $F(s) = \int_0^\infty f(t)e^{-st}dt$.

So, I am assuming that you are familiar with this Laplace transform, how to go from time domain or $f(t)$ to $F(s)$. Nevertheless here are some very simple results which we will use for Laplace transform, please go and review and revise Laplace transform from some basic mathematics course which you must have done. So, one of the important results is that if you have $\lim_{s\to\infty} sF(s) = f(0)$ this is also called the initial value theorem.

Likewise if you have $\lim_{s\to 0} sF(s) = f(t)_{t\to\infty}$. So, we have $f(t)_{t\to\infty}$ this is also called the final value theorem. Another very, very useful result in Laplace transforms is that if you differentiate a function m times, so $f^m(t)$ means you are going to differentiate this $f(t)$ m times. So, if you differentiate it m times after doing this Laplace transform you will get this thing which is $s^m F(s) - s^{m-1}f(0) - s^{m-2}\dot{f}(0) - f^{m-1}(0)$ and so on $f^{m-1}(0)$.

So, $f^{m-1}(0)$ (m – 1) is the (m -1)th derivative of this function and 0 means the initial value of that (m – 1)th is derivative. So, this is the way to differentiate as many times as you want a function of time and you will get this a form. The Laplace transform is $s^{m}F(s)$ plus all these terms which depends on what is the initial conditions at t = 0, what is the first derivative? What is the second derivative and so on?

We can also $\int f(\zeta)\, d(\zeta)$ and that is given by $\frac{1}{s}F(s)\frac{\dot{f}(t)}{s}\, dt$ evaluated at t = 0. So, these 2 functions are very useful, one is that suppose I want to take the derivative of $f(t)$ it is s $F(s)$. If I want to integrate a function once I will get $1/\, s\, F(s)$. A few common Laplace and inverse Laplace transforms are following.

One is if $F(s)$= 1 then $f(t)$ is the derived δ function $\delta(t)$, if $F(s)$ = $\frac{1}{s}$, $f(t)$ = 1, if $F(s) = \frac{1}{s^2}$, $f(t) = t$ and if $F(s) = \frac{a}{s^2} + a^2$, $f(t) =$ sin $sin\, at$ and if $F(s) = \frac{s}{s^2} + w^2$, $f(t) =$ cos $cos\, at$, so this should be a, so this is cos $cos\, at$. So, we can also obtain the inverse Laplace transform or something like $\frac{s+a}{(s+a)^2+b^2} \rightarrow e^{-at}$ cos $cos\, bt$

And if you have $\frac{a^2+b^2}{s[(s+a)^2+b^2]} \rightarrow 1 - e^{-at}($cos $cos\, bt + \frac{a}{b}$sin $sin\, bt)$ So, these are some very standard forms of Laplace transform and what is the inverse Laplace transform. So, please go back and revise if you want to, you are not going to go very deeply into Laplace transform but whatever we need basically is given here and maybe you can go back and revise.

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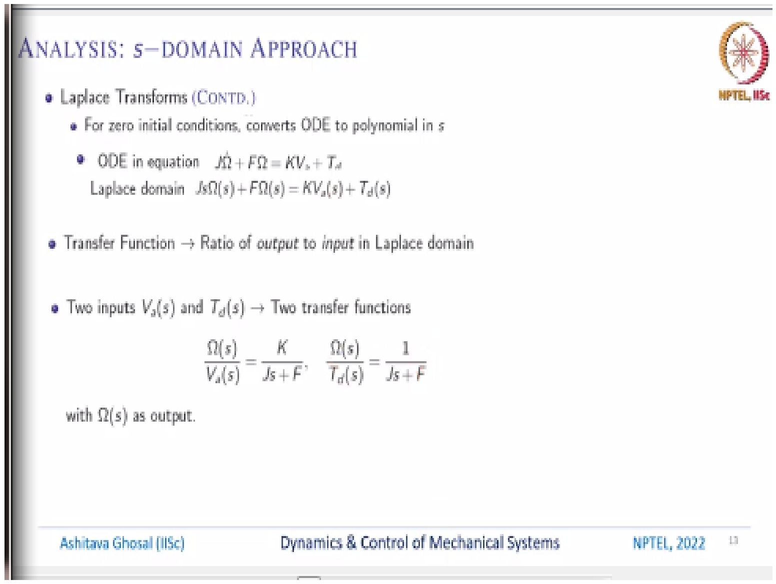

So, what do we do with Laplace transforms? So, basically we will assume 0 initial conditions and it will convert an ODE as a function of time into a polynomial in s. So, for example the ODE in this equation $J\dot{\Omega} + F\Omega = KV_a + T_d$. In Laplace domain it will be written as $J s\Omega(s) + F\Omega(s) = KV_a(s) + T_d(s)$, so $\dot{\Omega}$ became $s\Omega(s)$ but with 0 initial condition. So, most of the time in control theory or when we are doing s domain analysis of control systems we will assume 0 initial conditions.

Once we have these transforms we can define something called as a transfer function. So, the transfer function is nothing but the ratio of the output to input in Laplace domain. So, we could have 2 inputs in that example which we were discussing of a link being rotated by a DC motor. So, the 2 inputs are the voltage and the disturbance, so $V_a(s)$ and $T_d(s)$, so we can have 2 transfer functions one of them is $\frac{\Omega(s)}{V_a(s)} = \frac{K}{Js+F}$.

The other is $\frac{\Omega(s)}{T_d(s)} = \frac{1}{Js+F}$. So, how did I obtain this? In order to obtain this first transfer function we assume that the second input is 0. So, basically what we have is $J s\Omega(s) + F\Omega(s) = KV_a(s) + T_d(s)$, so we can write $\frac{\Omega(s)}{V_a(s)} = \frac{K}{Js+F}$. Likewise if you have

$\frac{\Omega(s)}{V_a(s)}$, so when we are trying to obtain this transfer function between $\frac{\Omega(s)}{V_a(s)}$, Ω is the output, T_d as the input, V_a is assumed to be 0. So, hence we can get this, so $\frac{1}{Js+F}$, very straight forward.

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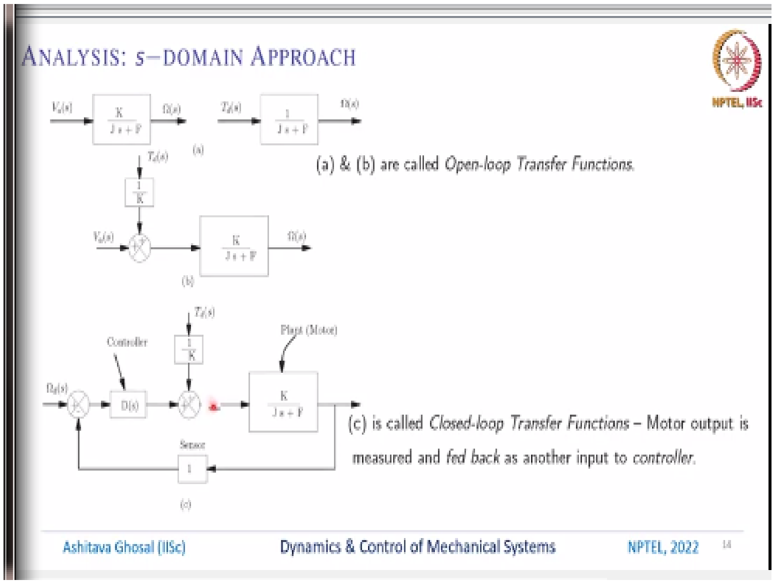

We can also describe all these transfer functions in what is called as a block diagram form. So, what is the block diagram form? We have output $\Omega(s)$, input V a s voltage applied. So, what is the block diagram which is $\frac{K}{Js+F}$. So, again it is nothing but $\Omega(s)$ divided by $V_a(s)$= $\frac{K}{Js+F}$. This is a pictorial representation that I can have an input which is voltage, output which is speed and then this block represents the transfer function which we have determined.

Likewise if the input is T_d and the output is $\Omega(s)$, then the transfer function is put in this block, it is $\frac{1}{Js+F}$. These are linear systems because remember $K,\ J,\ F$ these are all constants. So, if you have a linear system you can add the 2 inputs, so I can have T_d going into some block which is 1 by K, we can also have voltage which is coming in then we sum these 2 and then we send it together as inputs to this block which is the transfer function which is $\frac{K}{Js+F}$ and this is $\Omega(s)$.

So, this is nothing but the equation in Laplace domain. So, remember Ω into $J\,s\Omega(s)\ +\ F\,\Omega(s)\ =\ KV_a(s)\ +\ T_d(s)$, so that is exactly what you will get here. So, if you have transfer functions and block diagrams in this form that there is an input and there is an output and there is a transfer function sitting in between, a block sitting in between, these are called open loop transfer functions.

As opposed to what is called as a closed loop transfer function which is shown here. In this what you have is this part here which is T_d, $1/K$, V_a and then $\frac{K}{Js+F}$ and output is $\Omega(s)$. But let us assume that this output can be measured by a sensor whose transfer function is 1, so the output of the sensor is again Ω. So, now we have $\Omega_d - \Omega$, so at this point we have $\Omega_d - \Omega$, so this circular block is nothing but a summation block.

And summation here means that you sum the 2 inputs one of the input is with a minus sign, so it is $\Omega_d - \Omega$. And you send the output of the summation block to a controller, so at the moment we will say that the controller has a transfer function of $D(s)$and then the output of the controller is this voltage $V_a(s)$. So, what you can see is, I have taken this one, I have measured it, I have subtracted it here and sent to another block which I am calling it the controller and the output of the controller is the voltage.

So, I hope this is clear, that we could have a block diagram which is sometimes also called open loop transfer functions which denotes $\frac{\Omega(s)}{V_a(s)}$, so output by the input voltage is $\frac{K}{Js+F}$. Whereas, in some other form in basically in feedback control we also measure the output and then do some subtraction and then send it to a controller, in this case it is called closed loop.

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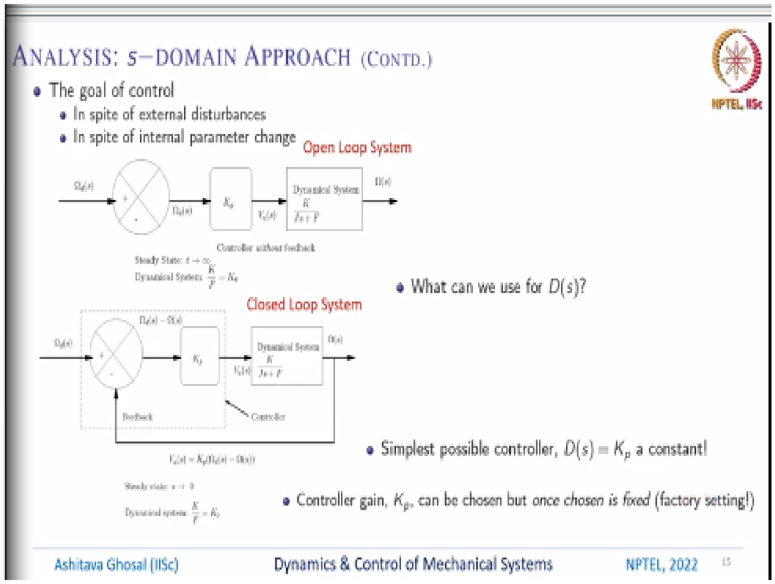

So, let us continue. So, the goal of control as I had mentioned earlier is that we want to follow a desired trajectory in spite of external disturbances and in spite of parameter change. So, let us see what happens when you have a open loop system? That means there is no feedback, there is no sensing and there is no sending back this signal and then we are doing subtracting.

So, basically the signal which was coming from $\Omega(s)$ is now broken, so we have Ω_d coming in, output of this summation block is again still Ω_d and we are sending into the controller and the output of the controller is $V_a(s)$. So, the question is this was remembered in the previous slide $D(s)$, so what can we choose as a controller transfer function? So, the simplest thing that we can choose is a constant.

So, we can use this $D(s)$as a constant because nothing can be simpler than that. So, in the case of the closed loop system again we have measurement of Ω and then we subtract this voltage which the output of this is $V_a(s)$ is nothing but $K_p(\Omega_d - \Omega(s))$. So, Ω_d is the desired and this is the measurement. So, notice in both of these I have removed the disturbance, so for the moment we will assume that the disturbance is 0.

So, we will do some simple analysis is to show that we can the control system will function right now in spite of internal parameter change, that is what we will look at first. We will also look at whether we can follow a desired trajectory with even if you have external disturbances. So, as I said in the controller transfer function block it will start with a very simple controller, simplest possible controller which is that the transfer function of that is a constant.

There are 2 more things which I wanted to mention here, that in steady state this dynamical system, basically steady state means what $t\rightarrow\infty$ or $s\rightarrow 0$. So, what is the transfer function of this dynamical system as $t\rightarrow\infty$? It is nothing but $\frac{K}{F}$. So, and we are going to denote this $\frac{K}{F}$ as K_0. So, this is an example of a controller without feedback, this is the controller with feedback, this is the closed loop system, this is the open loop system.

And we will use this symbol K_0 to find what is the transfer function of this plant as $t\rightarrow\infty$ or $s\rightarrow 0$. So, one more assumption or one more way of looking at this that I can choose this controller gain whatever I want. So, this is like an experiment, a thought experiment, however there is a condition that you can choose K_p as whatever you want but once you have chosen it you cannot change it.

So, K_p in some factory you do whatever tests you want, you do all trials and you say ok, I am going to choose K_p as 5 or 3.261. But once you have fixed that K_p, this constant, you cannot change it anymore then you will ship this controller to various places and see how it works, so that is the experiment which you want to do.

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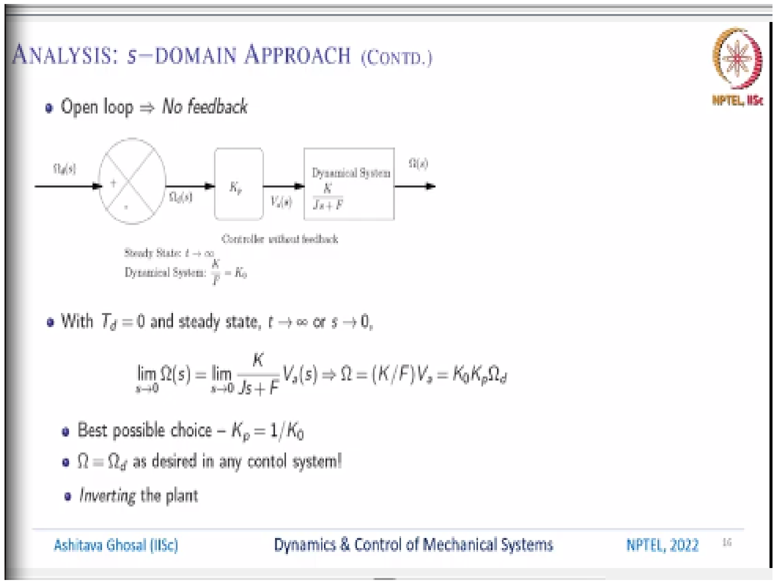

So, let us see what happens when you have open loop and no feedback? So, when you have open loop and no feedback you have Ω_d which is coming in, there is no feedback, so the output of the summation block is still Ω_d, this is K_p and there is output of this controller is Voltage and we get this. So, again with $T_d = 0$ and in steady state which means $t \to \infty$ or $s \to 0$, so what is $\Omega(s)$?

As $s \to 0$ it is $\frac{K}{Js + F}$ into the voltage which is coming in. So, what does this mean? As $s \to 0$ this Ω will be $\frac{K}{F}$ into voltage and what is this voltage applied? That is K_p into Ω_d. So, what we have is $\frac{K}{F}$ into $V_a(s)$which is same as, remember $\frac{K}{F}$ we are going to denote by K_0. So, then we have $K_0 K_p \Omega_d$. So, the output Ω in steady state that means as $t \to \infty$ is given by $K_0 K_p \Omega_d$.

So, what do you think is the best possible choice of K_p? So, clearly it is $1/K_0$, remember we are allowed to do as many experiments as you want, we are allowed to find out what is K, F, J everything and then we find out that this K_0 which is $\frac{K}{F}$ is some number. So, I can go and tune this controller or I can set the value of K_p in the controller to $1/K_0$. So, if I were to do this K_p as $1/K_0$ then I will get Ω as Ω_d, very nice, that we can have a control system where the output is exactly the same as Ω_d, whatever I want.

So, this is the goal of the controller that I would like the output to be whatever is the desired and I can do this by setting this K_p as $1/K_0$. So, at least in steady state after some time may not be initially not in the transient portion, in steady state we will have the output which is same as the desired quantity, whatever is the desired speed I will get the output as the desired speed? So, this strategy is also sometimes called as inverting the plant.

Because what is the plant? In steady state $\frac{K}{F}$ and what am I going to do with K_p? It is $1/K_0$, so I am going to invert the plant and put it in the controller. We will see later why this is not a good idea or it is not a very, very useful idea?

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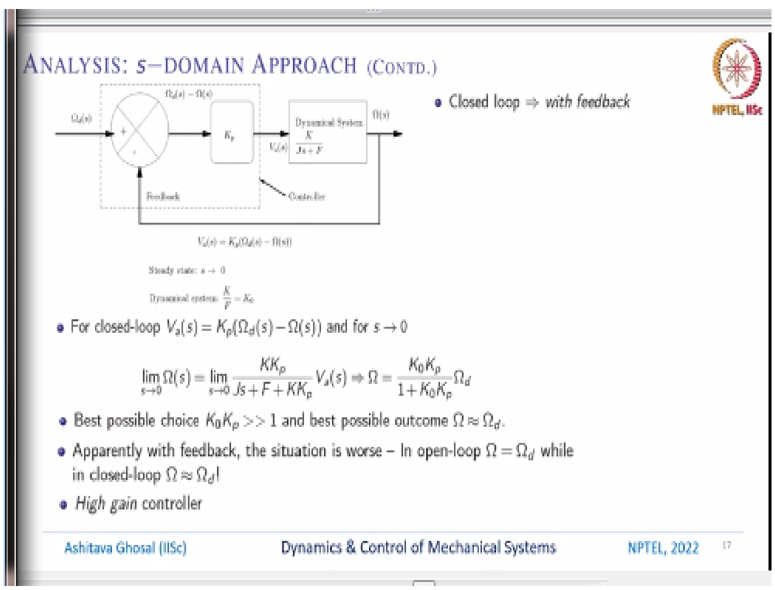

So, if you have closed loop with feedback then what do we have that this voltage which you are applying to this plant is $K_p(\Omega_d - \Omega)$. So, again in steady state we are going to assume that as $s \to 0$, the plant is $\frac{K}{F}$ which is K_0. So, now for closed loop this voltage is $K_p(\Omega_d - \Omega)$. So, we can see it is written here Voltage applied is which is this is K_p into whatever is coming into the block which is $\Omega_d - \Omega$.

So, hence in the $\lim_{s \to 0}$ the output $\Omega(s)$ is $\frac{K}{Js + F} V_a(s)$, so $K K_p / (Js + F + KK_p)$. So, you can work this out, you can see that voltage is given by $K_p(\Omega_d - \Omega)$, you put that in the left hand side and then you simplify and then you obtain $\Omega(s)$. So, basically some part of will come here and then eventually what you can see is you will get $\Omega = \frac{K_0 K_p}{1 + K_0 K_p} \Omega_d$.

So, when $s \to 0$ this part will go away, then you divide numerator and denominator by F. So, you will have $1 + K_0 K_p$ and on the top you have $K_0 K_p$. So, the best possible choice now is that $K_0 K_p$ is much greater than 1. So, suppose $K_0 K_p$ is 100, so I will choose $K_0 K_p$, so if I have numerator as 100 then 100 divided by 101 is very close to 1. So, I would choose K_p such that $K_0 K_p$ is much, much greater than 1, right, so K_0 is fixed.

So, K_0 is let us say 5, so then I will choose K_p as let us say 100, so then what do I have here? So, this is 500 divided by 501 which is very close to 1, so Ω which is the output here will be approximately same as Ω_d. So, it looks like that we are not doing so well, why because in the open loop Ω was equal to Ω_d, exactly, because I chose K_p as $1/K_0$. Whereas in the closed loop whatever you do this will be something divided by 1 plus something.

So, it will be very close if this $K_0 K_p$ is very large much larger than 1 but it will never be exactly equal to 1. So, in closed loop we have Ω which is approximately equal to Ω_d and this is the best that we can do. So, enough feedback control system with sensing and then this controller which has K_p, so I can at best choose $K_0 K_p$ much, much greater than 1 and hence the output Ω will be approximately equal to Ω_d.

So, this kind of strategy in feedback this is sometimes called as a high gain controller. So, I am going to choose K_p of large number such that $K_0 K_p$ is much, much greater than 1.

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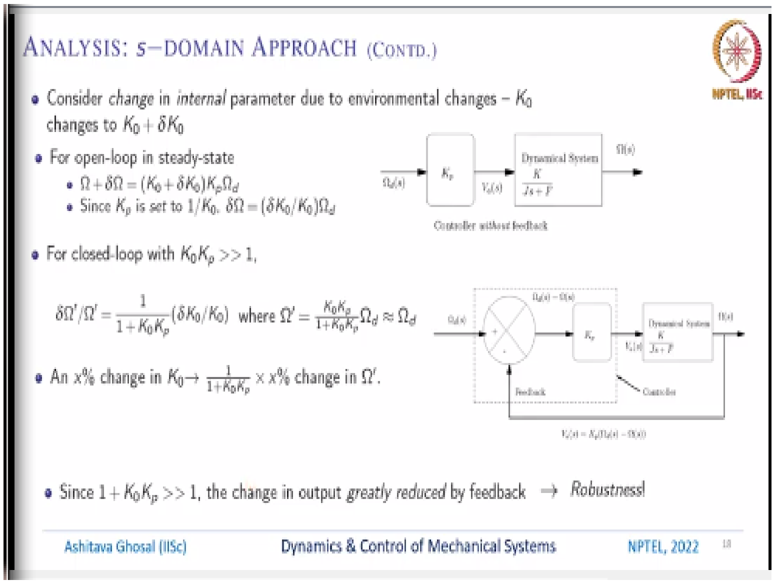

So, let us continue, let us see what we can achieve by doing this feedback. So, first thing is let us consider the open loop system and remember we have chosen K_p as $1/K_0$. So, in the open loop system controller without feedback, I have a desired Ω_d desired speed this K_p is $1/K_0$ and then the output is the voltage which I am sending to the dynamical system and we have the speed which is the output of the link.

So, let us assume that this K_0, what was K_0? $\frac{K}{F}$, so let us assume that this K_0 somehow changes to $K_0 + \delta K_0$ is that feasible? Yes, because remember this friction has in a R_a which is the resistance in the coil, it has the friction in the bearings. So, you could do all your simulations and studies and other things in one place and you send it to the deserts which are very high temperature in that case the resistance will change, the friction will change.

So, this $\frac{K}{F}$ can change and let us assume that this K_0 changes to $K_0 + \delta K_0$. So, if you have an open loop system in the steady state where $s \rightarrow 0$ then the output will change to $\Omega + \delta\Omega$ and that will be given by$(K_0 + \delta K_0)K_p\Omega_d$. So, your $\frac{K}{F}$ has changed from K_0 to δK_0 and remember Ω was $K_0K_p\Omega_d$, so now there will be an $\Omega + \delta\Omega$ which is $(K_0 + \delta K_0)K_p\Omega_d$

So, since K_p is set to $1/K_0$, this $\delta\ \Omega$ will be $\left(\frac{\delta K_0}{K_0}\right)\Omega_d$. Remember, we were doing this thought experiment that I can choose K_0 whatever I want but once I choose it I cannot change it, so that is like a factory setting. So, I have done lots of experiments and I had chosen K_p as $1 / K_0$ but then I send it to somewhere else this system this rotating link where the resistance of the coil or the friction has changed and K_0 is changed.

So, then the output change which is $\delta\ \Omega$ is same as $\delta\ K_0 / K_0$. So, if my system changes by 5% my output will change by 5%. Let us now consider what happens when you have feedback. When you have feedback then the voltage which you are applying is $K_p(\ \Omega_d - \ \Omega)$. So, for closed loop we have $K_0\ K_p$ much, much greater than 1 that is the way we chose it.

So, now when you have an X% change in K_0 you can show that you will get 1 divided by 1 + K_0 K_p into X% change in Ω '. Where Ω ' is $\delta\ \Omega'/\Omega' = \frac{1}{1+K_0K_p}\left(\frac{\delta K_0}{K_0}\right)$, this is what you will get. So, what is the moral of the story? That if I have a feedback and I have chosen $K_0\ K_p$ as let us say 100 much, much greater than 1.

So, then an X% change in $\delta\ K_0 / K_0$ will be X divided by 101% change in the output speed. So, what is the net result? The net result is that due to a change in the internal parameters of the system the output is not affected much, the changes are in outpost is greatly reduced by feedback. And this means there is a term in control theory; it means it is giving you robustness. In the sense that small changes in the internal parameters, small changes in K_0 does not change the output too much.

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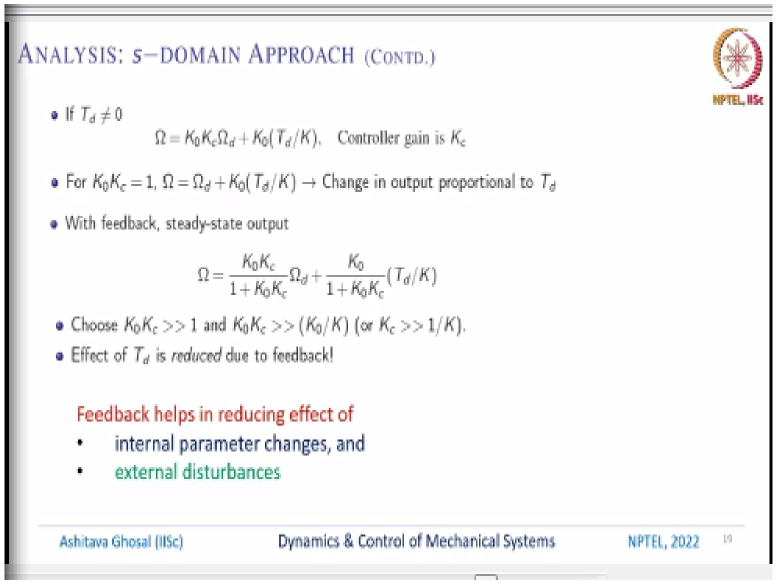

Let us continue, if T d were not to be 0 suppose we had some external disturbance then again in the steady state you can show that this Ω will be K_0 into some control again, I am intentionally using K_c here not K_p into Ω_d + K_0 into $\frac{T_d}{K}$, so this you can derive. So, if you have K_0 into K_c as 1 which is the open loop system then Ω will be Ω_d but then you will have this additional term which is K_0 into $\frac{T_d}{K}$.

So, whatever change or whatever is the disturbance T_d will be reflected directly in the output. So, if T_d were 0 then your Ω is same as Ω_d but if T_d is some small changes in the speed, some disturbance, it will show up in the Ω because it is Ω_d plus something into T_d. Whereas, if you have feedback then again the steady state output which is $s \rightarrow 0$, you can derive that $\Omega = \frac{K_0 K_c}{1 + K_0 K_c} \Omega_d$.

So, it is nothing but the same expression instead of K_p I am using K_c here. But the term which is coming from the disturbance is $\frac{K_0}{1 + K_0 K_c} \frac{T_d}{K}$. So, K again was coming from the motor, so K *and* F, they were coming from the model of the link which is driven by a DC motor. So, if you choose K_0 K_c to be much greater than 1 and K_0 K_c to be much greater than K_0 / K.

The another way of saying is I am going to choose K_c much, much greater than $1/$ K, then the effect of this disturbance is also reduced. In open loop what was I doing? I was choosing K_c or K_p as $1/$ K but now I need to choose K_c which is the controller gain or K_p whichever what I was using earlier as much, much greater than $1/$ K. So, then not only we can reduce the effect of the change in the internal parameter which is K_0 but also the effect of T_d.

So, I hope you realize that I have shown you that with feedback I can reduce the effect of the internal parameter changes and I can also reduce the effect of the external disturbances. I can reduce what is happening with $K\ and\ F$ which is K_0 if that changes then the output is not changing as much as K_0 but K_0 something divided by $1 + K_0\ K_c$. Likewise if you have a disturbance T_d which is acting on the system again by choosing the controller gain K_c much, much greater than $1/K$, I can make sure that the disturbance is not seen in the output.

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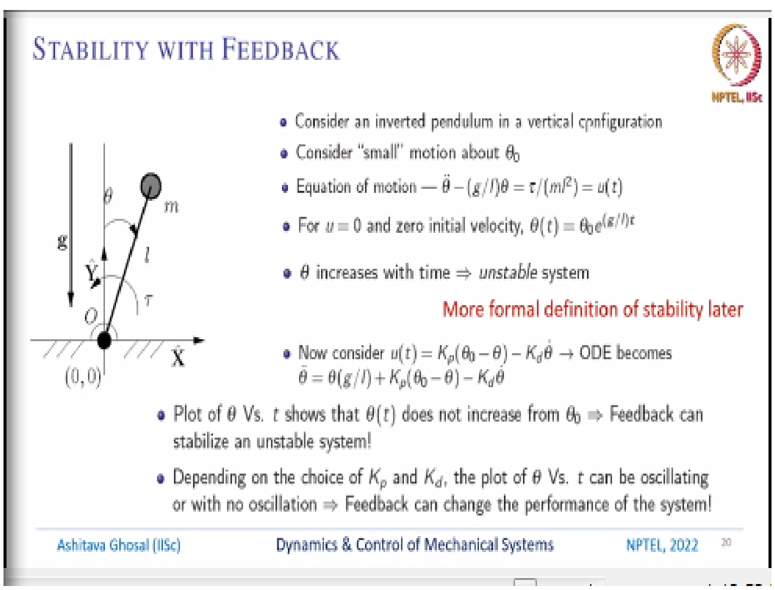

Let us continue, as I said one more purpose of controller is to stabilize an unstable system. So, again we will do these things in much more detail and rigorously later on but right now I am taking a very simple example. So, this is what is called as an inverted pendulum, so I have a joint here, a rotary joint and then there is a rod and there is a mass and gravity is acting downwards.

So, for u = 0 that means there is no input and the 0 initial condition, so we can show that $\theta(t)\ is\ \theta_0\ e^{\left(\frac{g}{l}\right)t}$, where am I getting this from.

So, we take this inverted pendulum and we considered a very small motion about this θ = 0. The equation of motion for a small θ is, what do we mean by small θ? We are going to use sine θ = θ is given by $\ddot{\theta} = -\left(\frac{g}{l}\right)\theta = \tau/ml^2$ that is equal to $u(t)$. As for u = 0, so it will be $\theta_0\ e^{\left(\frac{g}{l}\right)t}$. So, hence as long as θ_0 is not 0 even if it is a small number, so e to the power some positive number into time.

So, as time goes on this $\theta(t)$ will increase, so any small perturbation from the vertical which is $\theta_0 = 0$, $\theta(t)$ will increased with time and this is what is called as an unstable system. We will discuss this stability and unstability business later and we will give you a very formal definition of stability later. So, now let us consider this $u(t)$ in this differential equation of motion as something like K_p which is a constant into $\theta_0 - \theta$. So, what is $\theta_0 - \theta$?

So, we are going to measure this θ and then we are going to subtract from θ_0 and then multiplied by K_p, so one part of $u(t)$ which is this torque, remember $u(t)$ is tau divided by ml square, so one part is that. And I also want to add one $-K_d$ into $\dot{\theta}$, so what does this assume? That I am going to measure θ, I am also going to measure $\dot{\theta}$ and this is the way I am going to compute $u(t)$ and I am going to send to the plant to my system.

So, remember $u(t)$ is the output of the controller. So, once you have this then this ODE which is this inverted pendulum, we have derived the equation of a pendulum in the dynamics portion of this course, so this is known earlier. So, then this ODE becomes $\ddot{\theta} = \theta\,\frac{g}{l} + K_p\,(\theta_0 - \theta) - K_d\,\dot{\theta}$. And I will now solve this differential equation, this is not very hard to solve.

In fact this is a linear equation we can solve it but nevertheless I want to try different values of K_p and K_d and I am going to numerically solve this differential equation and plot θ versus time. And I am going to show you that for some values of K_p and K_d this θ will not increase with time, without K_p and K_d it is like this θ_0 to the power $\frac{g}{l}$, so it will increase with time.

But with K_p and K_d it will not increase with time and if you accept this definition of stability that $\theta(t)$ does not increase from θ_0 then it is considered to be stable. Then I can claim that this way of computing $u(t)$ is now stabilizing a system. So, the output of the controller which is $u(t)$ which is K_p into this $-K_d\dot{\theta}$ will stabilize the unstable system. And again I will show you plots that I can choose this K_p and K_d with different values.

And I can show you that the nature of the output in this case $\theta(t)$ will be different for different K_p and K_d. And which another way of saying is I can change the performance of the system, so I will show you. In one case it is oscillating about what I want and in another case there is no oscillation.

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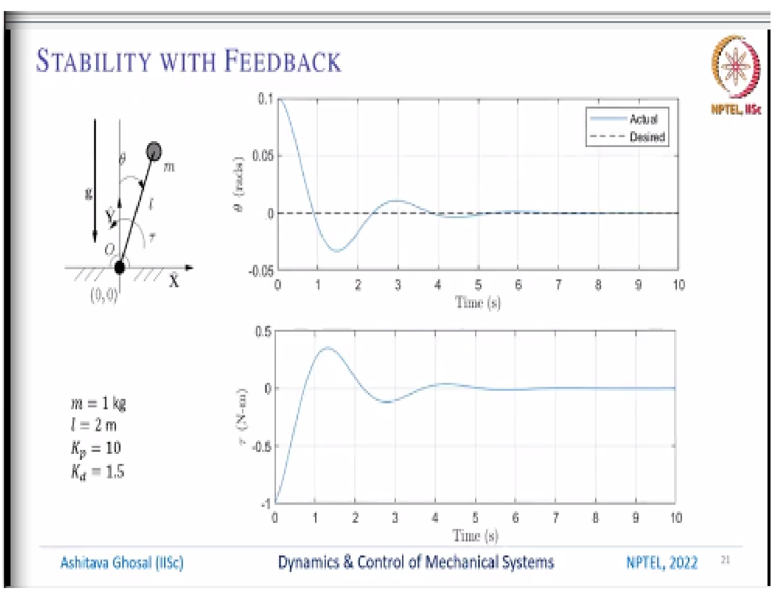

So, here is a plot of $\theta(t)$, for again in this case we have chosen mass as 1 kg, l as 2 meters, K_p as 10 and K_d as 1.5. So, I want this pendulum to remain vertical, so θ is at t = 0 is vertical, then I give some small perturbation but I want to make it sure that it stays vertical. So, θ_0 the desired θ should be 0, so in this case for this kind of K_p and K_d what you can see is that plot is oscillating but eventually it is settling at θ = 0.

So, the desired θ is this dotted line, dashed line and the actual θ is this blue curve. And we can also plot the torque which looks like this, so the motor needs to supply a torque which is changing with time. But nevertheless initially it was θ as θ_0 into e to the power some constant into time, it would constantly increase with time, here it is not increasing with time, it is oscillating about 0 but more or less staying around 0.

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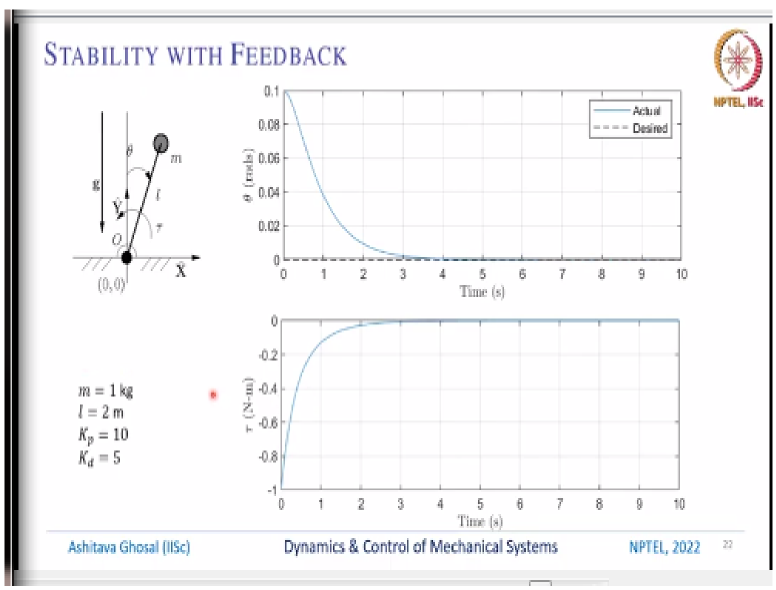

As I said I can choose a different value of K_p and K_d, everything else remains same, m is 1 kg, l is 2 meters but now I increase K_d to 5. And in that case I wanted θ_0 to be 0 and here as you can see there is no oscillation within a very short time of about maybe 4 or 5 seconds it will settle at θ = 0 and the torque also profile looks different. So, what have I done? In the controller I have changed this K_p and K_d, initially it was 10 and 1.5, now it is 10 and 5.

And by changing this controller gains we will see later this is called as the proportional gain and this is called as the derivative gain that is what p and d sort of stands for. By changing these gains I can make the system behave very differently which is no longer the natural dynamics of the system. The natural dynamics of an inverted pendulum is that I give a small perturbation it will fall until starts oscillating here and it will go oscillate all over the place.

Whereas here by using this controller which is basically I am giving some torque which is related to the measurement of θ and $\dot{\theta}$, I can ensure that the pendulum is staying vertically all the time. After some short time, initially there might be some transients or initially if the K_d is smaller it will oscillate but it should stay at $\theta = 0$.

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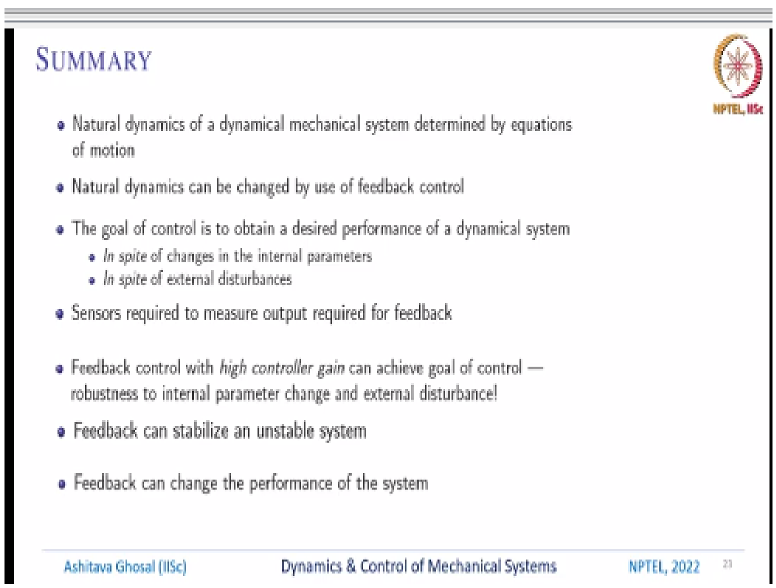

So, in summary the natural dynamics of a dynamical system is determined by the equations of motion, $f = ma$. So, if I tell you what is f, I can at least conceptually find out what is a and then I can integrate a to find velocity and I can integrate again to find the position, I can plot position as a function of time, so same thing if it is rotating. If you have $\tau = I\alpha + \omega \times I\omega$ if I give you the torque I can.

And if I know the inertia I can find ω and from that ω I can find the θ or some rotation angle. The natural dynamics can be changed by use of feedback control. I have shown you 2 examples, one which is a mass which is being pushed and one which is this inverted pendulum. In both cases the first I can change the natural dynamics, in the first case the position would increase quadratically like in the form of a parabola.

But I wanted the velocity to be constant not increasing with time and I could do that, similarly with the case of this inverted pendulum, I can make it stand vertically. The goal of control is to obtain a desired performance of a dynamical system in spite of changes in internal parameters, in spite of external disturbances. And this I have shown you that it is possible to do so. And feedback control helps in both of these that the effect of feedback is to reduce the effect of the changes in the internal parameters and to reduce the effect of the external disturbances.

In controls we need to have sensors, we need to measure the output and then feed it back. The feedback control with high controller gain can achieve the goal of control. So, it can be robust to internal parameter change and external disturbance, I showed you that. And feedback can also be used to stabilize an unstable system and I can use feedback to change the performance of a system.