Dynamics and Control of Mechanical Systems
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Lecture-16
Equation of Motion Using Computer Tools

In the next lecture we will look at how to derive these equations of motion using computer tools. So, this is a little bit of descriptive, if you want to learn how to use computer tools then you have to sit in front of a computer, just listening to these slides or these lectures are not enough. But nevertheless we will go through step by step as to how to start or use these computer tools. So, that we can finally derive the equations of motion.

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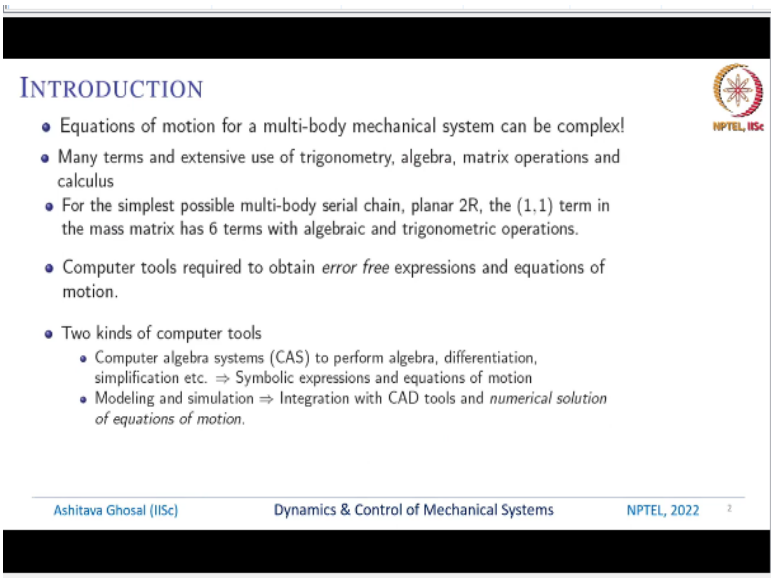

The equations of motion for a multi-body mechanical systems can be very complex, we have seen very, very simple systems of 2R and maybe a disk which is rolling but any mechanical system which is reasonably big can have many, many parts and hence the equations of motion can be very complex. Even in the examples which I have showed you there are many terms and the extensive use of trigonometry algebra, matrix operation and calculus.

We have done derivatives, partial and time derivatives. Even for the simplest possible multi-body serial chain the planar 2R, if you go back and see the mass matrix the 1, 1 term. So, the first term in the mass matrix has 6 terms with algebraic and trigonometric operations. So, think of it if you have a robot with 6 degrees of freedom with many parts you will have many, many more terms.

And each of these terms will have many algebraic and trigonometric operations. So, we will have sine and cosine of angles and so on. So, it is imperative that we find some computer tools which can give you this error free expressions and equations of motion. It is physically almost impossible to derive by hand or manually the equations of motion of any reasonable mechanical system.

There are 2 kinds of computer tools which are available; one is what is called as a computer algebra system or CAS. These systems perform algebra differentiation simplification etcetera on the symbolic expressions. So, we will not give numbers, we will give symbolic expressions as some kind of a text and then these computer algebra systems can do addition, multiplication then differentiation of the symbolic expressions.

So, if you do all these symbolic expressions, for example in the Lagrange formulation where we do partial derivatives with respect to q_i and $\dot{q}_i$ and so on. We can arrive at the equations of motion by doing all those expressions symbolically. There is also another kind of computer tools which are for modeling and simulation of mechanical systems and other systems also.

So, most of these tools come integrated with a CAD tool and basically they will do numerical solutions of the equations of motion.

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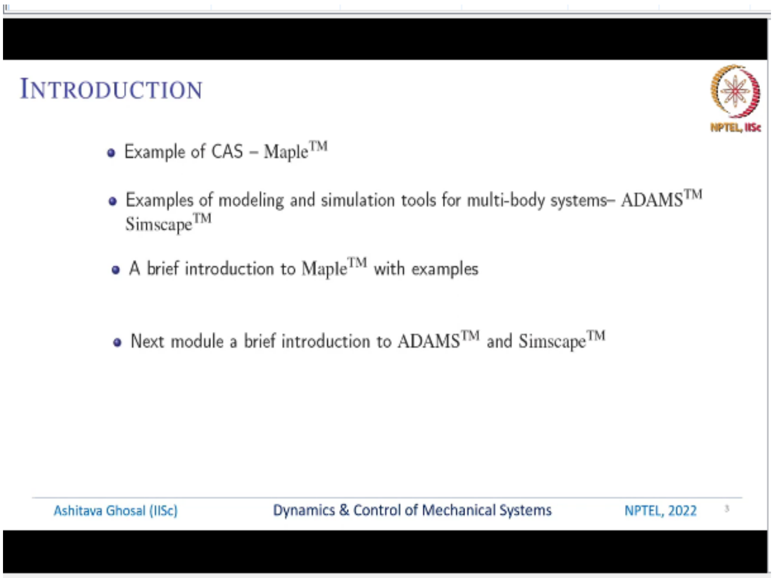

So, we will see some of these expressions or tools later on. So, let us continue. So, an example of a computer algebra system is Maple. So, we have used Maple extensively in this course and for various applications to obtain equations of motion and even for rotation matrices. So, I thought we can just go over very quickly how does Maple work, what are some of the important elements of Maple.

We can also have examples of modeling and simulation tools for multi-body systems; two of the well-known ones are ADAMS and Simscape. So, we will start with a brief introduction to Maple with examples. So, this requires a lot of practice with Maple. So, the purpose of this talk or this lecture is to tell you how to go about starting Maple and do some very basic simple operations.

Once you start doing this then you can see it is a very powerful tool which can be used for many applications. In the next module next week we will look at a brief introduction to ADAMS and Simscape. The ADAMS and Simscape are basically modeling and simulation tools where we can model a multi-body mechanical system and then we can ask it to simulate. Basically we can solve equations of motion for given initial conditions.

And for given parameters of the mechanical system and find how the mechanical system moves.

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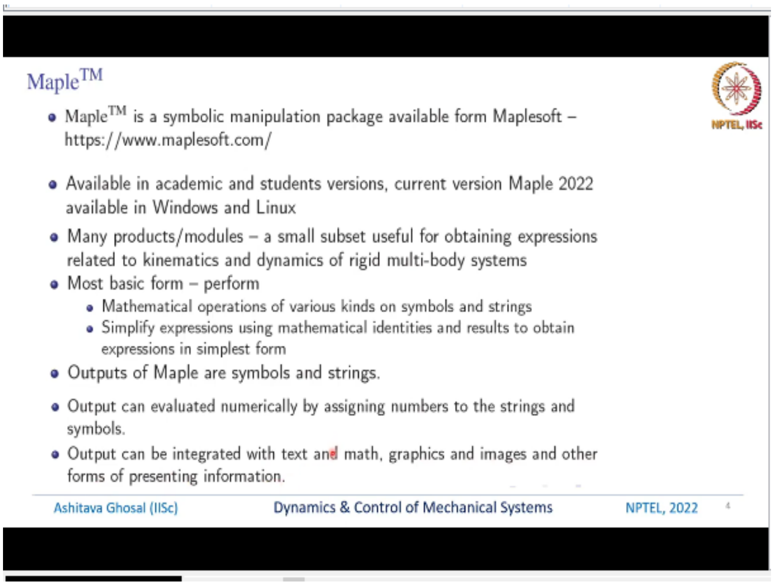

So, let us continue we have Maple. Maple is a symbolic manipulation package available from Maple soft. So, this is a company, and you can go to this website www.Mapleshopsoft.com for more details about Maple. It is available in academic and student versions; the current version of Maple is 2022 and it is available in Windows and Linux. Many products and modules are given by this Maple soft.

We are only going to look at a small subset which are useful for obtaining expressions related to kinematics and dynamics of rigid multi-body systems. There are many other modules; please go to the website here and you can see what all features and what all packages and what all specialized packages are available from this company. In its most basic form Maple can perform various mathematical operations which are represented as symbols and strings.

It can simplify expressions using mathematical identities and results to obtain expression in simplest form. So, for example if you have some $\theta + \theta$ appearing somewhere in your derivation of equation of motion; it can convert that using trigonometric identities to one. The output of Maples are also symbols and strings. They are not numbers, they can be numbers but you can use Maple to obtain expressions of equations of motion and I will show you examples.

The output can also be evaluated numerically by assigning numbers to the strings and symbols. So, if you have an output which says $\cos cos\, \theta$ and if you give the value of theta as some angle, it will evaluate $\cos cos\, \theta$ and give you a number. The output can also be integrated with text and math, graphics and images and other forms of presenting information it can give you plots of some variable as a function of time and so on.

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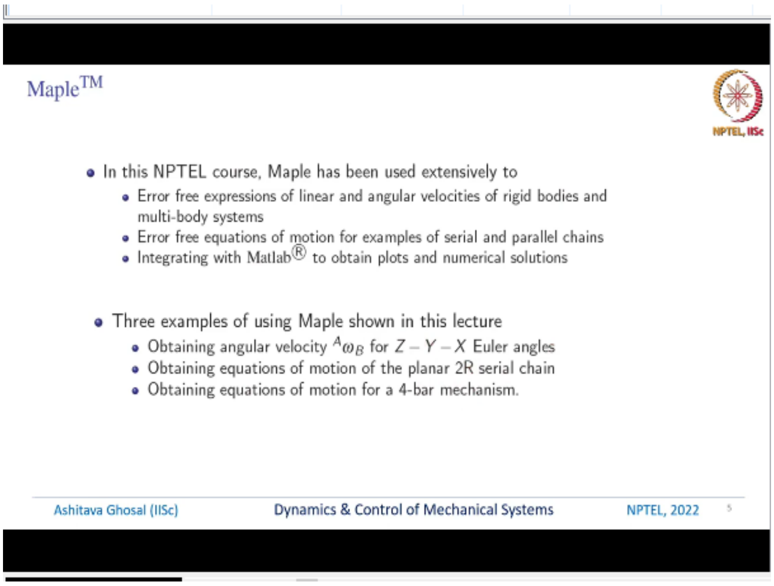

In this NPTEL course, as I have mentioned Maple has been used extensively to obtain error free expressions of linear and angular velocities of rigid bodies and multi-body systems. The error free equations of motion for examples of serial and parallel chains. So, for example the planar 2R or even the 4-bar mechanism. The Maple outputs have been integrated with Matlab to obtain plots and numerical solutions. There are three examples which I am going to show you of using Maple in this lecture.

First is we obtain the angular velocity of a rigid body B in a reference coordinate system A for Z-Y-X Euler angles. So, I will show you how we can obtain the rotation matrix; then we can obtain $[\dot{R}]\ [R]^T$ and then from the skew symmetric matrix we can extract the components of the angular velocity vector. I will also show you how to obtain the equations of motion for a planar 2R serial chain.

In fact in the examples which I have shown earlier the equations of motion were obtained using Maple and likewise I will show you how to obtain the equations of motion for a 4-bar mechanism. So, this is a parallel chain, this is a serial chain and this is some simple kinematics requirement to obtain angular velocity of a rigid body.

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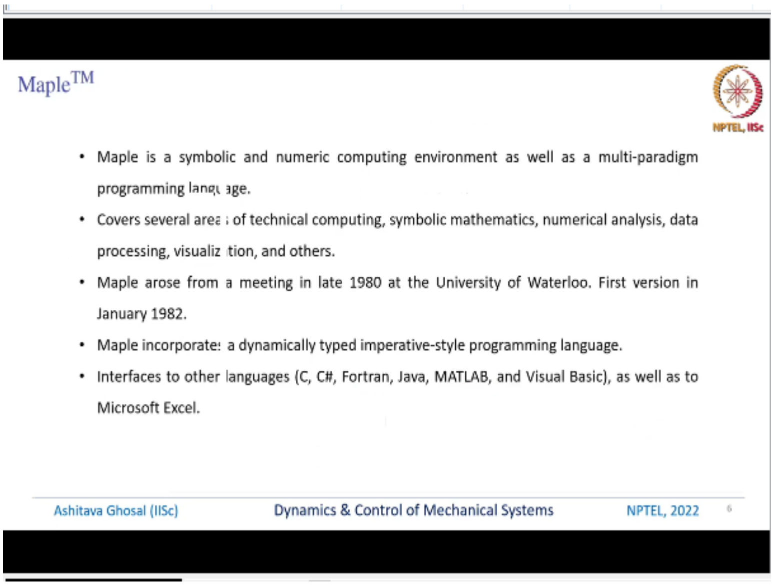

Just continue little bit of introduction of Maple more. Maple is a symbolic and numeric computing environment as well as a multi-paradigm programming language. Please go to the Maple website and you can understand what are some of the really powerful features of Maple? It covers several areas of technical computing symbolic mathematics, numerical analysis, data processing visualization and others.

Maple arose from a meeting in late 1980 at the university of Waterloo in Canada. The first version was available in January 1982. So, it is a fairly old and mature software. Maple incorporates a dynamically typed imperative style programming language; we will see what it means. So, basically you will have a command prompt and then if you type and you can ask it to find out what it means or what it evaluates.

Maple also has interfaces to other languages C, C#, Fortran, Java, Matlab and various other languages and it can also in integrate with Microsoft excel.

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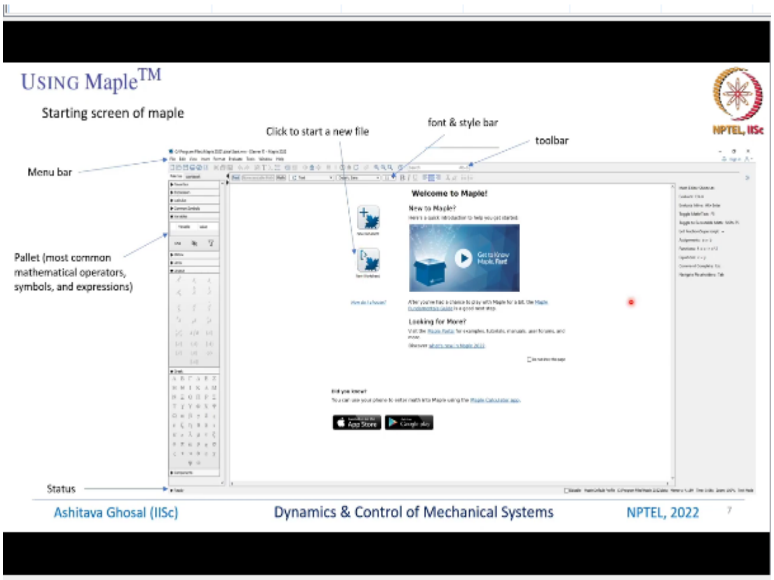

So, here is a starting screen of Maple. So, once you download Maple and then if you initiate Maple. So, then you can get this something like this screen. So, this screen has various features. First is this menu bar. So, like any other windows operating system you have file, edit, view, insert, format and various other things. Below that you have this Maple tool bar.

It tells you we will go into little bit more detail, it tells you what are all the things it can do, then there is on this left side there is a pallet which is basically it gives you all the most common mathematical operators, symbols and their expression. And we can easily use this tool bar to do in this workspace, what to do or how to initiate conversation with Maple? So, for example you can also do it click some start button to start a new file. There are many others help files which are also available.

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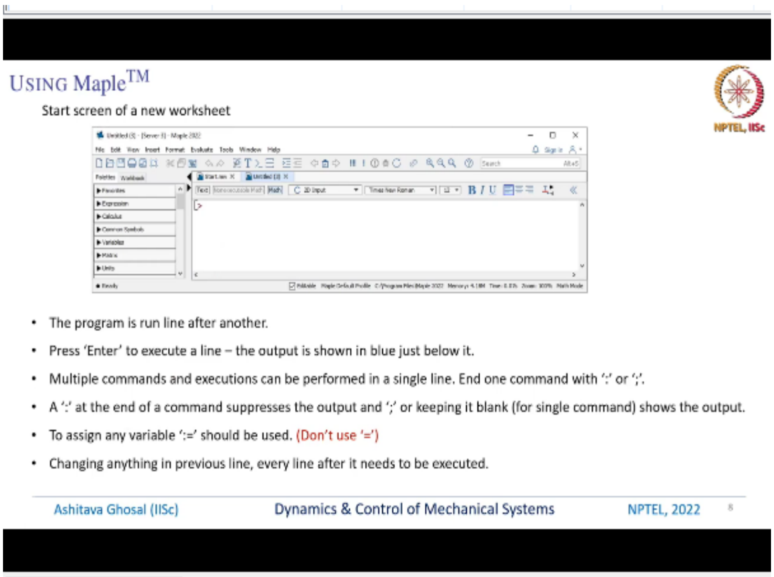

So, the start screen of a new workspace looks like this. So, you have some favorite, expressions, calculus and so on. And when you start it you will get a prompt which looks like this. So, in Maple the program is run line after another. So, if you type something here then when you enter it will execute the line and the output is shown in a blue line just below it and I will show you examples.

You can give multiple commands and executions can be performed in a single line. So, for example if you can end one command with this colon or this and you can keep on typing all the commands one after another. So, a ':' at the end of the command suppresses the output. So, even though you enter there will be no output which is shown and a ';' not keeping it blank for a single command shows the output.

To assign any variable we have to say a colon equal to something, we cannot use equal alone and if you change anything in the previous line every line after it needs to be executed because it does not know what you have changed in the previous line.

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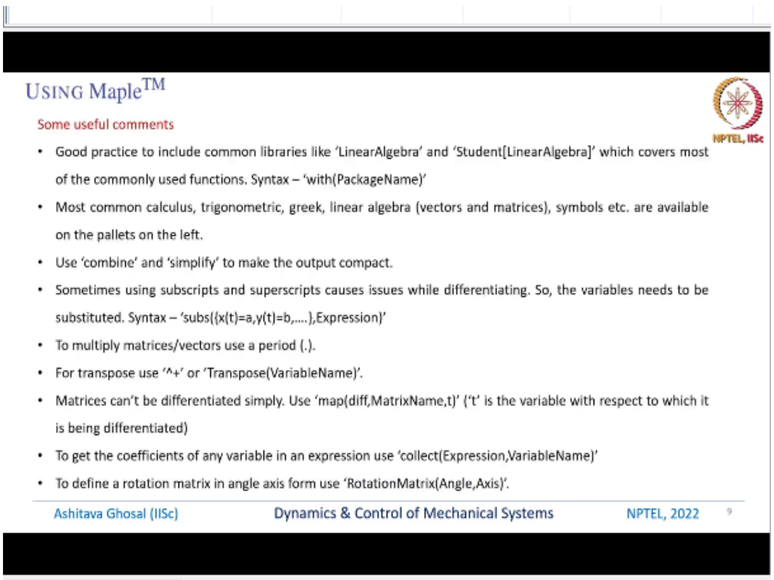

So, some useful comments about Maple, it is a good practice to include common libraries like linear algebra or student linear algebra which covers most of the commonly used functions. The syntax is you initiate this line with code with package name. So, with linear algebra. Most common calculus trigonometric. greek, linear algebra, vectors and matrices symbols etcetera are available on the palette on the left.

So, remember I showed you all palette; these are for common trigonometric and calculus and linear algebra operations. You can use combine and simplify to make the output compact. So, as I gave you an example if you have $\theta + \theta$ you can say simplify and it will give you back one, it will give some inherent trigonometric identities which are pre-programmed to simplify those expressions.

Sometimes using subscript and superscript causes issue while differentiating. So, if you want to differentiate $\cos cos\,\theta$, Maple if you say differentiate $\cos cos\,\theta$ it will give you minus $\sin sin\,\theta$. So, you need to practice once in our little bit to find out exactly what this is happening. So, sometimes you need to substitute the variables with which you want to differentiate.

So, for example syntax is substitute $x(t) = a,\ y(t) = b$. So, you cannot differentiate maybe x (t) but you can differentiate a and there are some expressions like that. To multiply matrices and vectors we have to use a dot like this and again these examples will see. For transpose we can use this '^+' , not like t which is what I have shown in the text or you can say transpose a variable name.

So, if you have a x and you want to do $x^T\ .\ y$ then we have to use this code plus this '^+'. The matrices can be differentiated by quite simply, all you say is use $map(diff,\ MatrixName,\ t)$ with which you are differentiating each element of the matrix. So, t is like time, it could be some other theta. You have to tell this element of the matrix need to be differentiated with this variable t.

To get the coefficients of any variable in an expression use collect. So, if you have a long expression and if you want to find out what are what is the coefficient of $\ddot{\theta}_1$. Remember in the equations of motion I had $\ddot{\theta}_1$ and inside the bracket I had $I_1 + I_2 + M_1 r_1^2$ and so on. So, how did I find all the coefficients of $\ddot{\theta}_1$? You can use command like this.

And we can use very simply rotation matrix in the angle axis form. So, these are some specialized commands which are useful for this course because we can use these commands to do some kinematics and find rotation matrices and so on.

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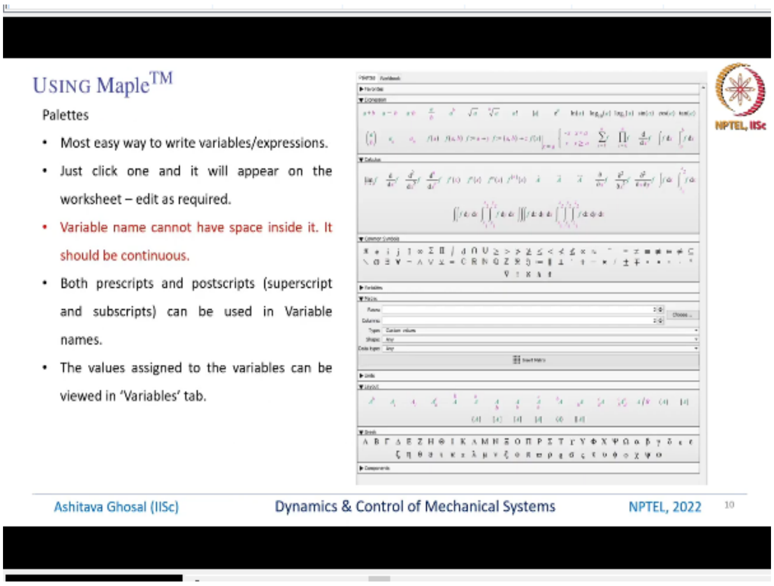

So, let us continue we want to use Maple. So, what is in the palette? So, I hope you can see it here, we have some a + b, a – b, a by b. So, all these various operations are already there, we can also find the derivatives and you can find limits and then you can also integrate. So, Maple allows you or it will evaluate certain integrals which it knows and then you have this common symbols and then you have variables matrices units and so on.

So, most easy to write variables and expression just by clicking on this. So, you click on these symbols and it will appear on the worksheet, you can also edit it if you want to. Variable name cannot have spaces in it, this is something which you know once you start using it you will find. So, it should be continuous. So, you cannot have a blank b, it will not like it, you have to write a b.

Both prescripts and postscripts, superscript and subscripts can be used in variable names. So, you can write x_1 and subscript and so on. So, you can see here that there is a to the power b so, you can write variables name like a to the power b. The values assigned to variables can be viewed in the variables tab. So, there is a variable tab here and if you say that x is some 2 or 3 or some number then that can be seen here.

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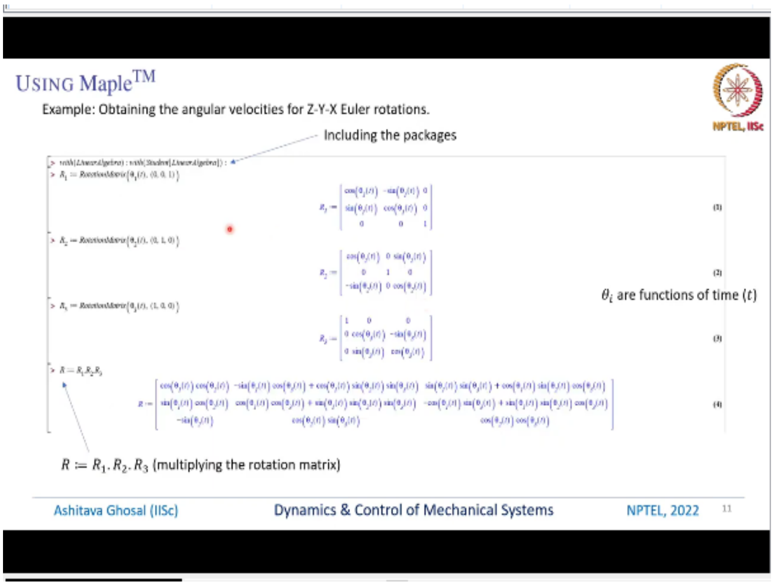

So, let us use Maple to obtain the angular velocities for Z-Y-X Euler angles. So, this is one of the example which I thought. If you go through it then you can see what exactly we need to do. So, the first thing is in this prompt we say with with student linear algebra, this is including the packages linear algebra and student linear algebra. One thing what you can see is I have written something $R_1 := RotationMatrix(\theta_1(t), (0\ 0\ 1))$.

So, what it means is that I want to rotate about a vector 0 0 1 which is nothing but the z axis by an angle θ_1. So, if you type this in that command prompt in Maple then you will get this matrix. So, you can see R_1 *is* $\cos cos\,\theta_1(t)\ -\ \sin sin\,\theta_1(t)\ 0,\ 0\ 0\ 1,\ \sin sin\,\theta_1(t)\ \cos cos\,\theta_1(t)$ and so on. This is a simple rotation about the z axis and it is very, very simply you can do it in one step and it comes out as I said written in blue.

So, this is the output of Maple when you type this command and you can see that this is nothing but a rotation about the z axis by an angle θ_1. Then we can do similarly R_2 which is the rotation about y-axis by an angle θ_2. So, you will get R_2 like this and remember we are going to later on differentiate. So, we want to ensure that θ_1 θ_2 and likewise θ_3 are functions of time.

So, $R_3 := RotationMatrix(\theta_3(t), (1\ 0\ 0))$. this is the x axis. So, I want to find out what is the resultant rotation matrix after Z-Y-X. So, we multiply it in the order which we have done these rotations. So, first Z then Y then X you multiply and then you will get this little bit long expressions. But if you just carefully see what it is telling you is for example this element which is the r_{31} element is - sin $sin\,\theta_2\ \ (t)$.

This one is $cos(t)\ \ cos\theta_3(t),\ \ this\ one\ is\ cos\ \ cos\ \ (t)\ \ -\ sin\ \ \ cos\ \ (t)\ \ +\ \ so\ on$. So, this is the first column vector which is the x b in the a coordinate system. This is the y b in the a coordinate system and this is the z b in the a coordinate system and you can go back to your notes and see that this is exactly what we had written in the text when we were looking at Euler angles.

So, as you can see with these three steps and one more which is here I can find the rotation matrix corresponding to Z-Y-X Euler rotations.

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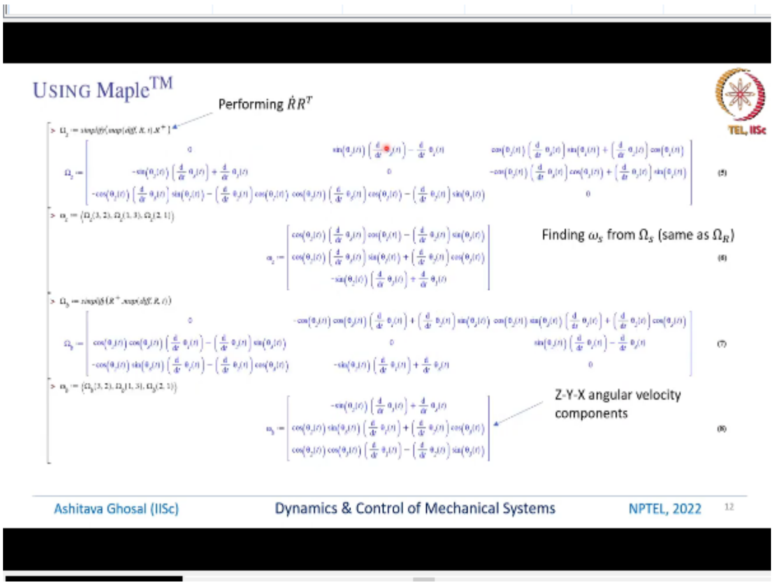

Then let us take steps to find out the angular velocity. So, I want to do actually $[\dot{R}]\ \ [R]^T$ because I want to find the space fixed angular velocity vector corresponding to Z-Y-X Euler rotations. So, what you can see is we need to type omega s which is the skew symmetric matrix is $simplify(\ map(\ diff,\ R,\ t).\ R^{+})$. So, when you say $diff,\ R,\ t$ it is trying to do $\dot{R}$.

And then when you say this '^+' So, remember transpose corresponds to a plus, so when you type $R^T\ \ R^{+}$ it will go back and see what was R earlier and then it will do transpose of that matrix and then this is $\dot{R}$ and this simplify is to write it in a compact form. So, wherever possible it will try to simplify using trigonometric identities and any other if something is 0 it will make it 0.

So, remember in a skew symmetric matrix the diagonal term should be 0 and it will automatically make it 0. So, how does it know this is skew symmetric matrix? It will basically after doing simplification it will see that these elements are 0 and hence we can find what is this skew symmetric matrix. So, you can see the diagonal terms are 0, the off diagonal terms are $a_{ij} = \; - \; a_{ji}$.

So, for example this term is sin $sin\,\theta_2$ and this is $\dot{\theta}_3$, this is the way $\dot{\theta}_3$ is represented. So, this is - $\frac{d}{dt}$ θ_1 (t). So, this quantity is $\dot{\theta}_1$, this is$-$ sin $sin\,\theta_2$ $(\dot{\theta}_3)$ $+$ $\dot{\theta}_1$and these two terms as $a_{ij} = \; - \; a_{ji}$. Likewise we can find all other terms. So, as you can see we obtain clearly little bit different than how you write with hand but all the terms are there.

So, this is $\dot{\theta}_3$, $\dot{\theta}_1$, - sin θ_2, if you look at let us say another this point, this is - cos $cos\,\theta_2$, this is $\dot{\theta}_3$ cos $cos\,\theta_1$ + this is $\dot{\theta}_2$ sin $sin\,\theta_1$. So, this matrix comes out when you do these steps in Maple. Then we can see that this is the skew symmetric matrix, this is minus and this is plus, this is minus and this is plus and so on. It has all the rules of a skew symmetric matrix.

So, we can extract from the skew symmetric matrix the space fixed angular velocity vector which is the x component is 3, 2 which is this 1 here then 1, 3 and then 2, 1. So, this is the z component. So, we can obtain all these components and then as soon as you type this it will come out and show you that these are the components of the angular velocity vector. So, in our nodes we have called this Ω_R .

So, this is right multiplication we started with $\dot{R}R^T$. is identity then we took the derivatives and so on. So, here in this slide it is being called as Ω_S and you can see ω_x is so cos $cos\,\theta_2$ $\dot{\theta}_3$ cos $cos\,\theta_1$ - $\dot{\theta}_2$ sin $sin\,\theta_1$. So, you can go back and verify because these expressions are given earlier in the text that this is indeed the x component of the angular velocity vector.

And then if you want to see whether it can be written in a more compact form you can say simplify $[R]^T\ [\dot{R}]$ which is the body fixed angular velocity vector also we want to find out and then again we can find this these are the components of $[R]^T\ [\dot{R}]$ and the body fixed angular velocity vector is again the (3, 2), (1, 3) and (2, 1) components of this skew symmetric matrix.

So, again you can see that we can get the angular velocity vector components either in the space fixed which is Ω_S or in the body fixed which is in this form. So, the summary of this is we can start with a rotation matrix which is nothing but rotations about z, y and x then we do this differentiation of this rotation matrix and then we type these commands and we get the angular velocity in either space fixed or body fixed angular velocities.

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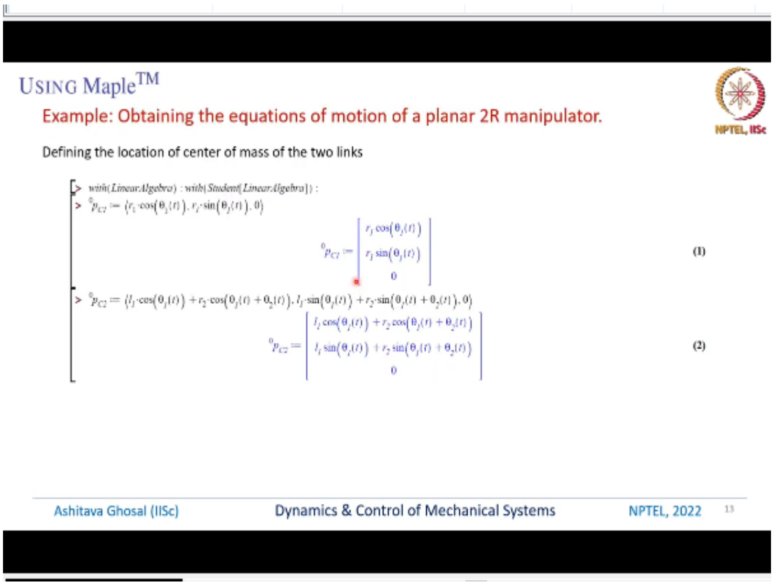

Let us continue we want to show you how to obtain the equations of motion of a planar 2R manipulator. So, this will take a little bit more time, but nevertheless it is a good way to see the power of Maple and how we can get error free equations of motion. So, again we start with some packages which is linear algebra and student linear algebra. First thing is we say what is the location of the CG of each link.

So, $0p_{C1}$ is the position vector of the first link center of mass or center of gravity and **if** you can see in your notes that this is nothing but r cos $\cos\theta_1$ and r sin $\sin\theta_1$, we have to say explicitly that θ_1 is a function of time because later on we will take the derivatives. So, as soon as you type this and since there is you are not suppressing the output you will get this vector that $0p_{C1}$ is given by r_1 cos $cos\ \theta_1\ (t)$ r_1 sin $\theta_1\ (t)$ and the z component is 0.

Likewise the location of the CG of the second link which is after the first link is $l_1 c_1 + r_2 c_{12}$, $l_1 s_1 + r_2 s_{12}$. So, again you can see it is $l_1 c_1(t) + r_2 c_{12}(t)$ and so on and as soon as you type this you will get the position vector. So, these are consistent with what is mentioned in the nodes and this is very easy. So, you have to give this input. So, you need to know that this r_1 cos $cos\ \theta_1\ (t)$ is the x component.

And r_1 sin $sin\ \theta_1\ (t)$ is the y component but that is very simple we are not doing very, very tricky things. From the figure you can see what is the position vector of the center of mass of each link.

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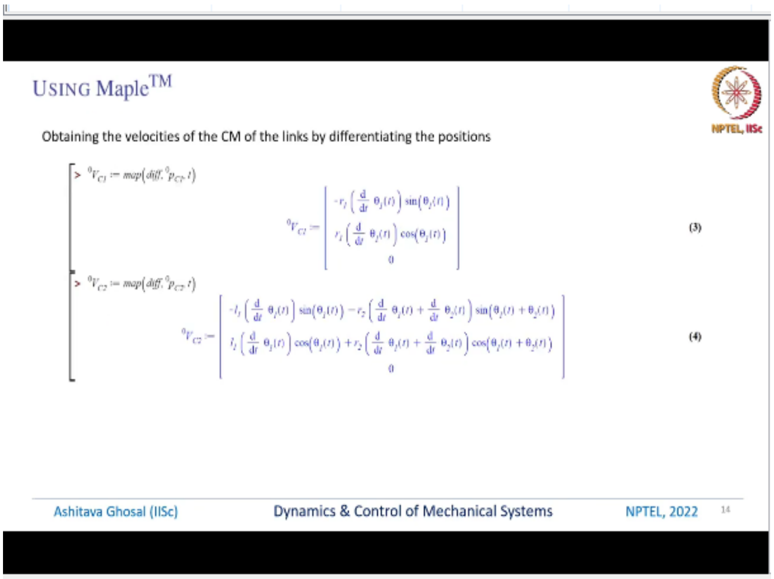

Then we can find the velocity of the center of mass and as I said we differentiate $0p_{C1}$ which is the position vector with respect to time and this map this means that you want to differentiate this position vector and when you differentiate you can see the output is $-r_1 \frac{d}{dt}\theta_1(t)$. So, this is like $\dot{\theta}_1$ into sin $\theta_1(t)$. So, x coordinate was r_1 cos $cos\ \theta_1\ (t)$, y coordinate was r_1 sin $sin\ \theta_1\ (t)$.

So, it does this differentiation of symbolically and it will give you $r_1 \frac{d\theta_1(t)}{dt} sin\, \theta_1(t)$. written in this form. Similarly, the y component was the derivative of r_1 sin $sin\, \theta_1\ (t)$. So, you will get $r_1 \frac{d}{dt}\theta_1(t)$ cos $cos\, \theta_1\ (t)$ and likewise we can find the velocity of the CG of second link again by taking the derivative of $0p_{C2}$ with respect to time, this tells you that you are taking the derivatives with respect to time.

And again you will get various terms. So, remember it was $l_1 c_1 + r_2 c_{12}$ that was the x coordinate of the center of mass of the second link. So, when you take the derivative it will be $-\ l_1 \frac{d}{dt}\theta_1(t) sin\, \theta_1(t)$ then $-\ r_2\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t)\right)$, you can see even these brackets come up nicely. So, you know exactly this is what is $\dot{\theta}_1$ and $\dot{\theta}_2$ and then we have sin $sin\, \theta_1$ ($\theta_1(t) + \theta_2(t)$).

The y component is similarly derivative of sine. So, you will get cos and then $\dot{\theta}_1$ and so on. So, you can find the velocity of the CG of the first link and the second link using Maple. Again these are error free, the way it is written this computer tool, it is not going to make mistakes. So, it will not going to put some different sign here by mistake.

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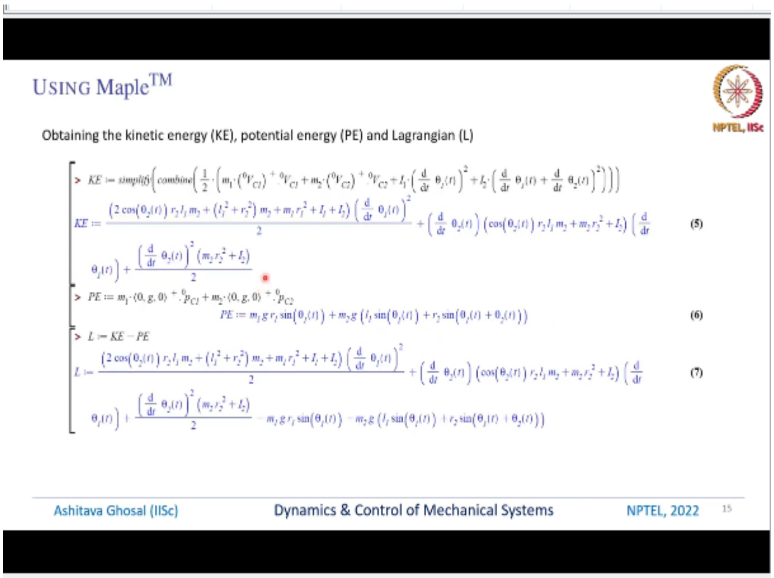

The kinetic energy now can be obtained because we know the linear velocity of the center of mass of each link. So, what is the kinetic energy? It is nothing but $m_1 \ 0V_{C1}^T \ 0V_{C1}$. So, this transpose means basically it is like $0\dot{V}_{C1} \ 0V_{C1}$. So, dot is like $0V_{C1}^T \ 0V_{C1}$. So, if you have vector $\dot{A}$ vector B. I can write it as $A^T B$. So, that is what is exactly being done here.

So, this is $\frac{1}{2}(m_1 \ 0V_{C1}^T \ 0V_{C1} + m_2 \ 0V_{C2}^T \ 0V_{C2} + I_1\left(\frac{d}{dt}\theta_1(t)\right)^2 + I_2\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t)\right)^2)$. So, this is I_1 which is the moment of inertia of the first link and we are only interested in the z component that we need to make sure that we are doing a planar example. So, only the z component will appear. So, you type this and the Maple will give you this output.

So, this output is a little bit different than how you write it by hand, but basically you see that this whole thing is divided by 2. So, you have basically something like $I_1 + I_2$ and then $\dot{\theta}_1$ square half. So, this half is coming here like this and then you have this $\dot{\theta}_2$ into some term which is dependent on cos $cos\ \theta_2$. Remember in the kinetic energy there was one term which was function of θ_2 and the rest were constant.

So, that is what this term will come. We can also find out the other terms. So, this is for the second link, this is $\dot{\theta}_2^2 (m_2 r_2^2 + I_2)$ and then this half will appear because it is like half m v square. So, $r_2\dot{\theta}_2$ is like V and $\frac{1}{2} mV^2$. So, this half will come here. This is the output of Maple. So, it looks a little different than how you would write by hand or do it manually, but nevertheless this is correct if you just see little carefully.

The potential energy can also be obtained by $m_1 g$ dot this position vector, remember that is the way we defined the position vector, it is like mgh. So, instead of h we have a dot product with g vector which is along the y direction into this. So, we will get

$m_1 g r_1 \sin sin\,\theta + m_2 g (l_1 \sin sin\,\theta_1 + r_2 \theta_1 + \theta_2))$, because the second CG is after end of the first link which is $l_1 c_1$ and $l_2 s_1$, $l_1 s_1$ and $l_1 c_1$ and then you have this r_2.

You go along the second link by a distance r_2. So, we can find the potential energy and then we find the Lagrangian which is KE - PE and again as soon as you type this it will give you this long expression. So, is this correct? Yes because you need to go through a little bit carefully and see that this taken the kinetic energy here which is this long term and then it has taken this potential energy and it is subtracted.

So, it looks. So, we can find the Lagrangian and importantly it is in a symbolic form. So, we should be able to take the partial derivatives and time derivatives of various terms whenever required because it knows that θ_2 is a function of time, $I_1\, r_2\, m_2$ these are not functions of time, they are constants.

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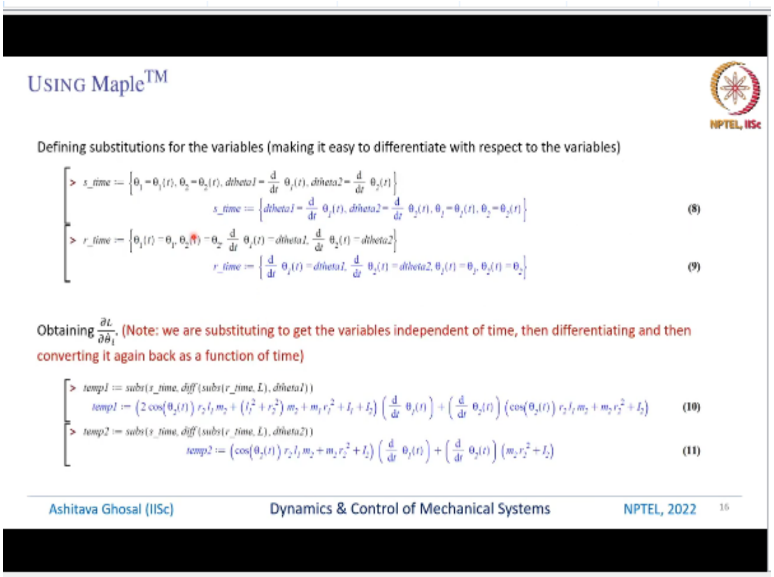

So, we define a substitution for the variables making it easy to differentiate with respect to the variables because we say that we want to differentiate with respect to θ_1, θ_2 and also with $\dot{\theta}_1$ and $\dot{\theta}_2$. So, we define these new variables $d\,\theta_1$, $d\,\theta_2$ and θ_1 and θ_2. Then we can say that these are like you can switch the order and say that $\frac{d}{dt}\theta_1$ is basically this new variable $d\,\theta_1$.

And $d\,\theta_2$ is this variable $\frac{d}{dt}\theta_2$ and we obtain $\frac{\partial L}{\partial \dot{\theta}_i}$, we are substituting to get the variables independent of time. So, when you do partial derivatives you are taking derivative with respect to $\dot{\theta}$. $\dot{\theta}$ itself is a function of time. So, we do not want to confuse Maple into thinking that you should do it like a chain rule.

First with respect to $\dot{\theta}_i$ and then again d theta by dt and things like that. So, we will take this partial derivatives with respect to the theta I dot and then later on if when required we will differentiate with time and then we will convert it back to actual functions of time. So, what do we do? We assign two variables temp 1 and temp 2. So, temp 1 is this long expression. Remember this was part of the kinetic energy, temp 2 is this also reasonably long expression and then we will take these partial and time derivatives.

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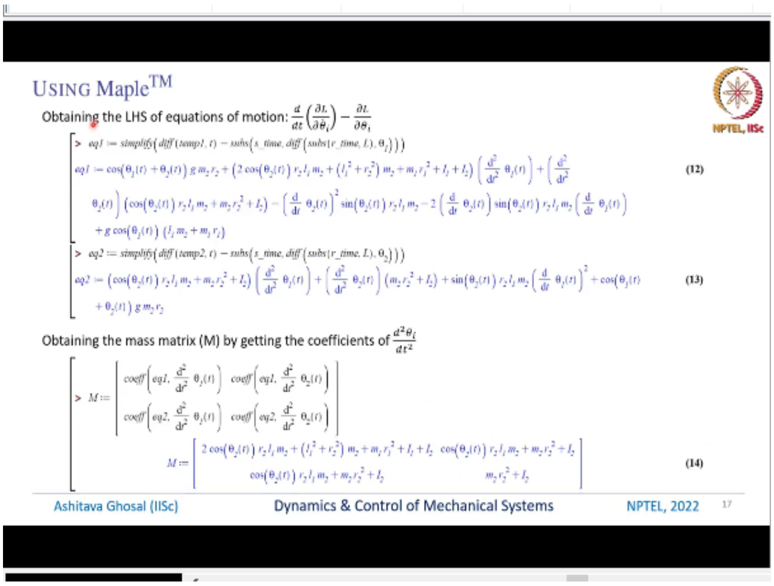

So, the first equation is implementation of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0$. So, this is the partial of the Lagrangian with theta I, this is the partial of the $\frac{\partial L}{\partial \dot{\theta}_i}$ and then the time derivative of this. So, what you can see is we are taking this r time which was defined earlier and L and we are going to differentiate with respect to θ_1 here because this is the first equation.

Similarly here we are going to differentiate temp 1 which was there with time and then if you simplify this whole expression you will get terms like this. So, the first equation is

$\cos cos\,(\theta_1 + \theta_2)\ m_2 r_2 g$. So, this is like $m_2 r_2 g\, c_{12}$ then this is like $2\, c_2$ m_2 $l_1 r_2$. So, this is also a term in the equations of motion if you go back and see then you will have $m_2(l_1^2 + r_2^2)$.

And this is $m_1 r_1^2$ and this is $I_1 + I_2$ whole everything is multiplied by $\frac{d^2}{dt^2}\theta_1$. Likewise we will get terms which are other terms. So, for example you will get something like $(l_1 m_2 + m_1 r_1)g \cos cos\,\theta$. So, once you start using it you will see that this is parts of the equation but not in the most simplified form. Similarly the equation 2 we take the derivative with respect to θ_2 partial then we take the derivative of this with respect to time and then we again simplify.

So, the equation 2 contains $(m_2 l_1 r_2 \cos cos\,\theta_2 + I_2 + m_2 r_2^2)\ \ddot{\theta}_1$ and that is correct, this one is very easy to see. The m22 element the second element in the second equation, so m22 that is $(m_2 r_2^2 + I_2)\ \ddot{\theta}_1$. So, these equations you can obtain using Maple by simply basically taking the required derivatives, $\frac{\partial L}{\partial \theta_i}, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right)$.

This part and then of course you have to simplify. So, you need to get used to it try it a few times and then you will see that it does make sense, it is not the way we write it by hand because we would write it like $m_2 l_1 r_2 c_2$, but here Maple is giving the output the string or the text output is in some sense in the opposite way. So, this is more or less.

This is maybe the way you would write it by hand also, maybe you would write it as $I_2 + m_2 r_2^2$ because this is the use of the parallel axis theorem from CG you are going to some other point. Next once we have these two equations we want to write it in the standard form which is M mass matrix into $\ddot{\theta}_i$ coriolis centripetal term plus gravity term and then we will say this is equal to $\tau_1\ and\ \tau_2$.

So, how do I find the element of the mass matrix? So, what you can see is if you type coefficient 1 of equation 1 which is this long expression and then you see what are the coefficients of $\frac{d^2r}{dt^2}\theta_1(t)$. So, basically what is the coefficients of $\ddot{\theta}_i$ in equation 1? So, if you type this it will search and collect all the coefficients which are multiplying $\ddot{\theta}_1$.

Similarly this is the coefficient of $\ddot{\theta}_2$ in equation 1, this is equation 1 and the mass matrix element this one is the coefficient of $\ddot{\theta}_1$ in equation 2 which is this and this is m_{22} element which is the coefficient of $\ddot{\theta}_2$ in equation 2 straight forward. So, then you type this and then Maple will give you this and here you can see that it is looks.

So, the first element m_{11} is this long term which $is\, I_1 + I_2 + m_1 r_1^2 + m_2 r_2^2 + m_2 l_1^2$. So, Maple automatically will combine these two and then you have 2 cos $cos\, \theta_2\ m_2 l_1 r_2$, the 1 2 element is $I_2 + m_2 r_2^2 + m_2 l_1 r_2$ cos $cos\, \theta_2$ and you can see this is a symmetric matrix. So, 1 2 is same as 2 1 and the 2 2 element is this.

So, if you go back and see the notes or if you go back and see the lectures where we derive the equations of motion that this is indeed exactly the terms which you will get written slightly differently but nevertheless these are exactly the terms that you will get for the mass matrix.

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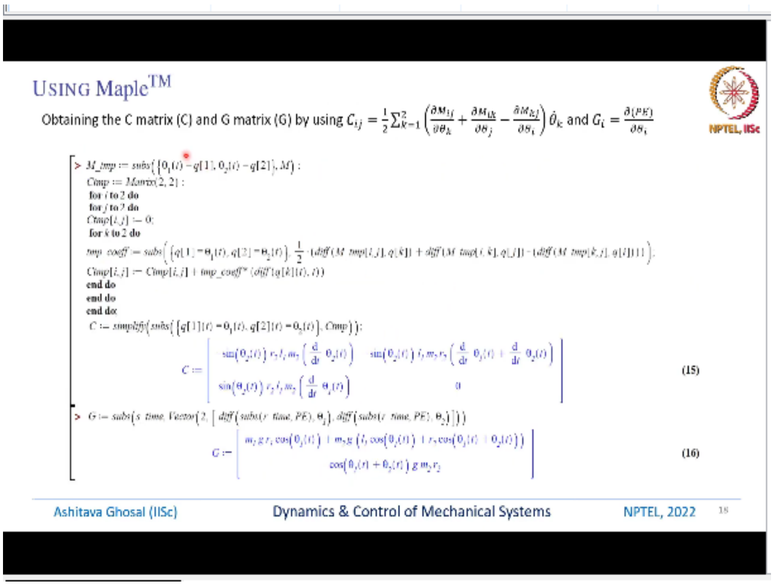

How do we obtain the coriolis term and the gravity term? We will use this nice interesting formula for the coriolis term, this was shown in the lecture that you take the $\frac{1}{2}\sum_{k=1}^{2}\left(\frac{\partial M_{ij}}{\partial \theta_k}+\frac{\partial M_{ik}}{\partial \theta_j}-\frac{\partial M_{kj}}{\partial \theta_i}\right)\dot{\theta}_k$.

In this case there are only two of them. So, ij will be the variables which are not summed over. okay So, indices which are not summed over and you will get C_{11}, C_{12} and so on. So, this is the matrix corresponding to the coriolis and centripetal term. How about gravity? We know what is the potential energy, we derived what is the expression for the potential energy. So, the first ith term in the gravity will $\frac{\partial PE}{\partial \theta_i}$.

And once you do all these things and then you type this command which is M. So, we are going to do some temporary things which is $\theta_1 \, is \, q_1$, $\theta_2 \, is \, q_2$ and this is the mass matrix because we want to take the derivatives of the elements of the mass matrix we type a few terms like this and you can find that the coefficients of the elements of the coriolis C_{ij} will come out.

So, C_{11} is this, C_{12} is this, C_{21} is this and C_{22} is 0 and the gravity terms are nothing but the derivatives of the potential energy and with respect to θ_1 and θ_2 and again we will get these terms. So, you can see yourself, open the other notes and then you can see that these are indeed the terms corresponding to the gravity and these are indeed the term corresponding to the coriolis and centripetal term.

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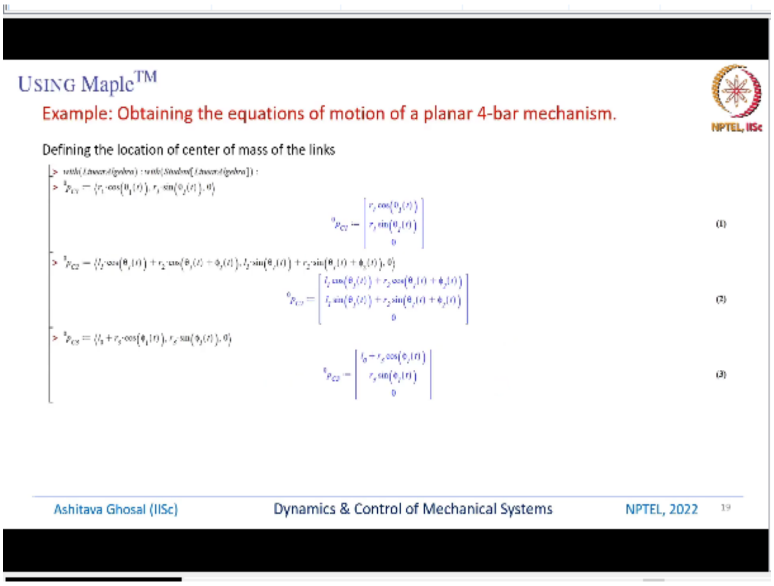

So, let us continue, I also want to show you how we can obtain the equations of motion of a planar 4-bar mechanism. So, again basically we start with some packages linear algebra and student linear algebra. Then we write down the position of the center of mass of each one of these links. So, 1, 2 and 3 and again you can see that the third link is you go along the x axis by l_0 and this is $r_3 \cos cos\ \phi_1$ *and* $r_3 \sin sin\ \phi_1$. So, these are the position vectors for the 3 position of the CGs of three of these links.

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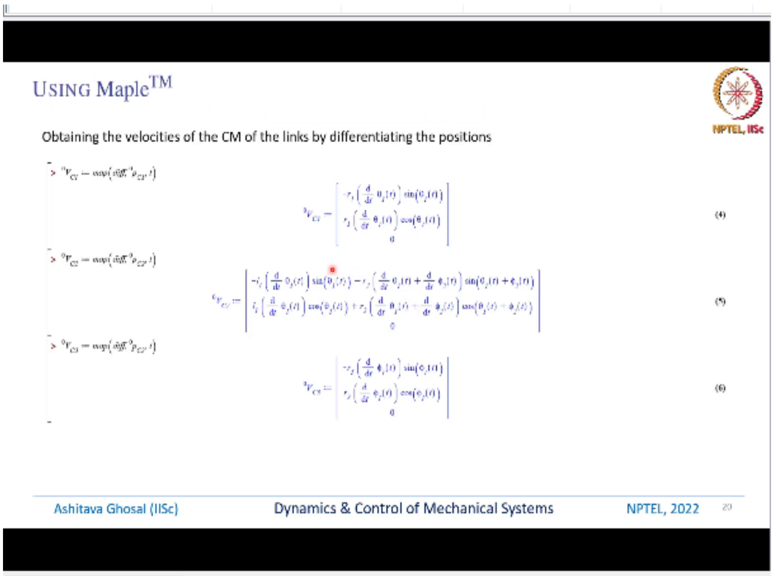

The velocity of the center of mass of the center of gravity can be obtained by taking the derivatives of this position vectors with time and again you will get $-r_1\dot{\theta}_1 \sin sin\,\theta_1$ and so on. Similarly the velocity of CG of the second link can be obtained by taking the derivative of $0p_{C_2}$ with time and again we will get all these terms. So, we will have $\dot{\theta}_1$.

But you will also get $\dot{\phi}_2$, because remember the position vector of the end of the CG of the second link has θ_1 and ϕ_2, whereas the position vector of the last link the output link is contains only ϕ_1 and when you take the derivative you will have $-\ r_3\dot{\phi}_1 \sin sin\,\phi_1\ ,\ \ r_3\dot{\phi}_1 \cos cos\,\phi_1$.

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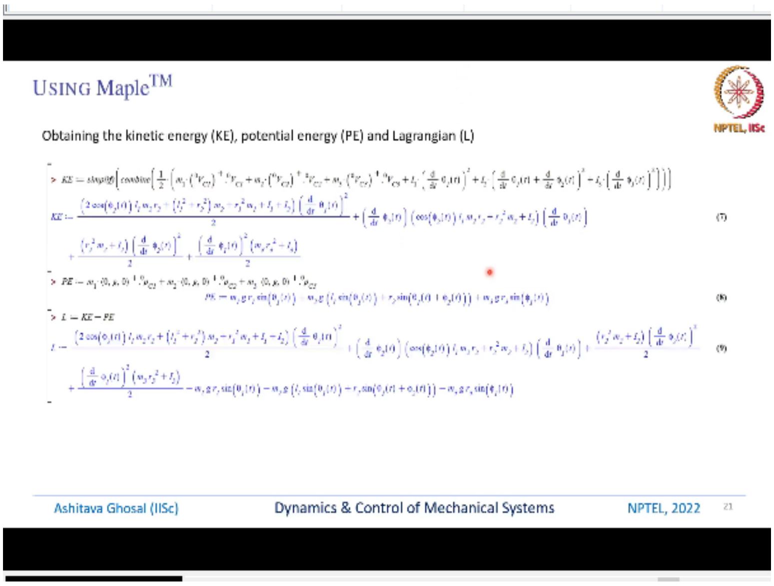

Once we have the linear velocity of the center of mass of link 1, link 2 and length 3 we can find the kinetic energy of link 1 which is $\frac{1}{2}(m_1\ \ 0V_{C1}^T\ \ 0V_{C1} + m_2\ \ 0V_{C2}^T\ \ 0V_{C2} + m_3\ \ 0V_{C3}^T\ \ 0V_{C3} + I_1\left(\frac{d}{dt}\theta_1(t)\right)^2 + I_2\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\phi_2(t)\right)^2 -$. So, once you type this in Maple you will get the expression for the kinetic energy which looks like this.

The potential energy of each of this link can also be obtained by taking the dot product of the gravity vector with the position of the CG the vector and remember this one is plus means here transpose and dot means multiplying these two vectors. So, likewise this is

$m_2gh + m_3gh$ roughly speaking. Ofcourse we will have sine and cosine of the angles θ_1 and then ϕ_3 and ϕ_1 and also, we have sin θ_1 and then we have r_2 sin sin $(\theta_1 + \phi_2)$ and so on.

The sine component is the height from the zero potential energy surfaces. Once we have the kinetic and the potential energy we can obtain the Lagrangian which is now a function of θ_1, ϕ_2 and ϕ_1 and also the derivatives of this. So, the kinetic energy contains the derivatives of θ_1, ϕ_2 and ϕ_1. And the potential energy is a function of only θ_1, ϕ_2 and ϕ_1.

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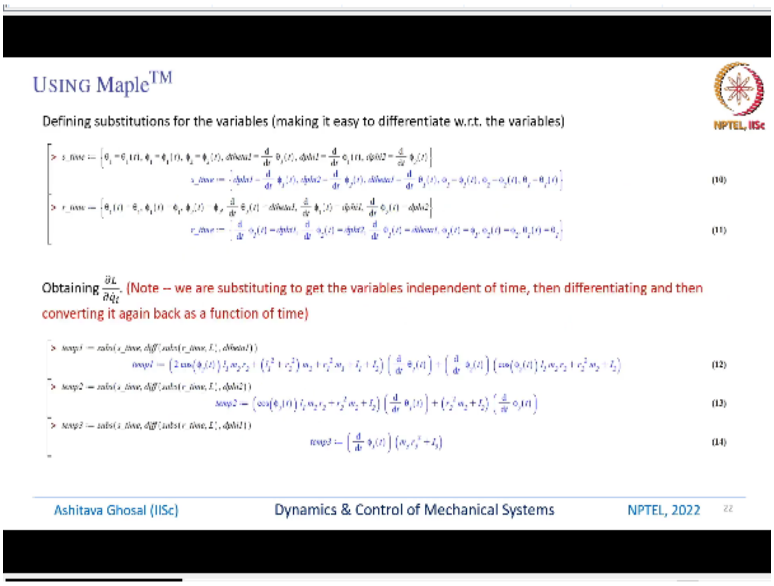

Similar to this case when we did for the 2R we define intermediate variables s time and r time which are basically θ_1 to say that θ_1 is a function of time, ϕ_1 is a function of time, $d\theta_1$ is basically $\dot{\theta}_1$but it is assigned a new variable $d\theta_1$ because when we take the partial derivative of this Lagrangian with respect to $\dot{\theta}_1$ this is the easy way to do it or this is the simplest way to do it.

And likewise we have another variable r_time which is again as used in the case of the planar 2R we find it is useful to introduce these two variables. Then once we obtain the Lagrangian we can take the derivative with respect to $\dot{q}_i$ not again we are substituting to get the variables independent of time. We want to $\frac{\partial L}{\partial \dot{q}_i}$, $\dot{q}_i$ itself is a function of time.

Inherently a function of time but we do not want to differentiate with respect to time, we want to do $\frac{\partial L}{\partial \dot{q}_i}$. Then later on we can always go back and substitute and find it as an explicit function of time. So, we have this two variables temp 1 and temp 2. Temp 1 is basically $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1}$ and temp 2 is a $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) - \frac{\partial L}{\partial \phi_2}$.

So, hence if you do these derivatives then you can show that temp 1 will give these terms, these are some parts of the equation of motion, this is only the partial derivatives with respect to $\dot{q}_i$, this is the partial derivative with respect to $\dot{\phi}_2$ and then similarly this is the partial derivative with respect to $\dot{\phi}_1$. So, we will get these three terms.

And we have assigned this temp 1, temp 2, temp 3 because later on we have to take the time derivative of each of these terms. Remember $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

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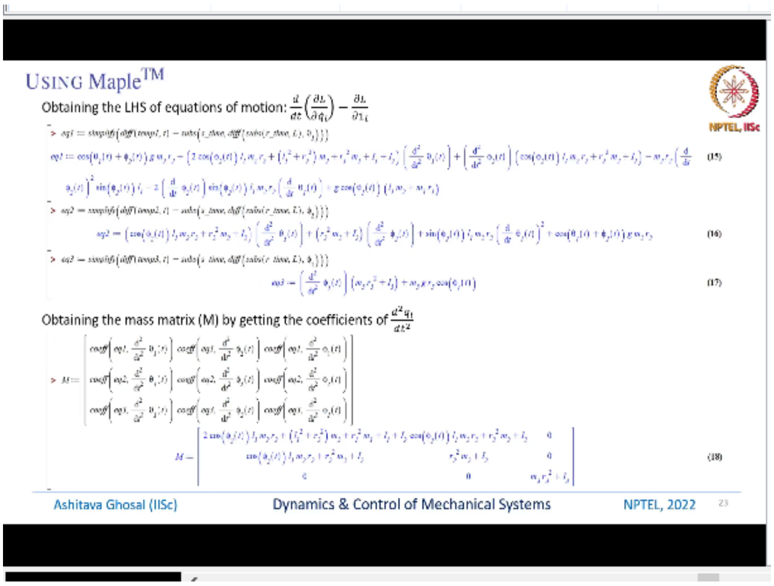

So, we obtain the $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$ and hence we get three equations, one is equation 1 then we have equation 2 and then we have equation 3. So, these are the terms of the left hand side of the equations of motion. So, as you can see we have some $(m_3 r_3^2 + I_3)\ddot{\phi}_1$ and then we have this gravity term which is $m_3 g r_3 \cos cos\, \phi_1$.

So, this is for the output link. The output link has no connection with θ_1 and ϕ_2 yet, we will connect θ_1, ϕ_2 and ϕ_1 using the constraint equations. So, we now have equation 1 which contains all the terms which we are interested in which is $\ddot{\theta}_1$, $\ddot{\phi}_2$ and also the coriolis and centripetal terms. So, you can see here there is a $\dot{\phi}_2^2$.

And likewise for equation 2 we have which is $\ddot{\theta}_1 + \ddot{\phi}_2$ plus we have this coriolis term which is $\dot{\theta}_1^2$ and of course the gravity terms. Once we have these three equations equation 1, equation 2 equations 3. So, this is nothing but the left hand side. So, we will later on equate it to the external torques but before we do that we can collect the terms in the mass matrix.

In this case the mass matrix is 3 by 3 because we have 3 variables, we have θ_1, ϕ_2 and ϕ_1 and just like before we collect the coefficients of equation 1 which are $\ddot{\theta}_1$ into what is multiplying the $\ddot{\theta}_1$ and likewise we find the coefficient of $\ddot{\phi}_2$ from equation 1 and coefficient of $\ddot{\phi}_1$ from equation 1. So, this will be the first row of the mass matrix.

The second row will be given by this and the third row will contain the coefficient of $\ddot{\theta}_1$ in equation 3, $\ddot{\phi}_2$ in equation 3 and $\ddot{\phi}_1$ in equation 3. So, once we do these steps we can see what the mass matrix looks like. It is very similar to the planar 2R. So, the top 2 by 2 so m_{11}, m_{12}, m_{21}, m_{22} are exactly the same as the terms in the mass matrix in the plane 2R

Why because we have broken the 4-bar mechanism into a 2R mechanism, planar 2R chain and a single 1R chain. So, the 2R part looks exactly like this except we do not have θ_2, in the planar 2R we have θ_1 and θ_2. Now we have θ_1 and ϕ_2. In the planar 2R the mass matrix was a function of θ_2, we had cos $cos\ \theta_2$ and so on. So, here we have 2 cos ϕ_2.

The interesting part is here. The third column is 0 0 $m_3 r_3^2$ and the third row is 0 0 with this 3 3 element being $m_3 r_3^2 + I_3$. So, what you can see is this third column and the third row except this term everything is 0 and there is no relationship with θ_1 and ϕ_2. This will multiply $\ddot{\phi}_1$ plus whatever is the centripetal and coriolis term. There is actually no coriolis term plus gravity term for the third equation.

So, in the third equation you can see $(m_3 r_3^2 + I_3)\ \ddot{\phi}_1 + \ m_3\, g\, r_3 cos\ \phi_1$. So, this is nothing but a pendulum with r_1 r link, a single link which is being rotated at the angle with angle ϕ_1. So, the mass matrix does not have any relationship between the top 2 by 2 and the last 3 by 3, 3 comma 3 elements.

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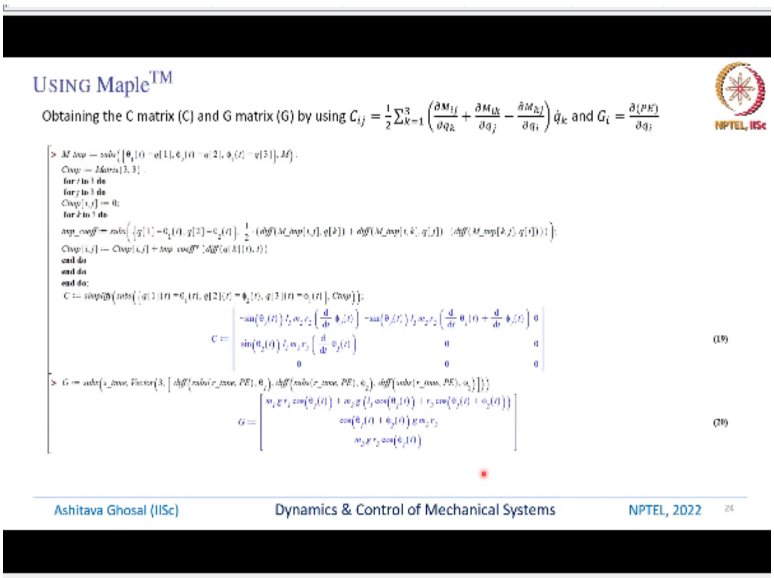

This will come once we obtain the constraint equations, but before we go to constraint equation let me again show you how we can compute the coriolis and centripetal term and the gravity term. So, the gravity term is again the partial derivatives of the potential energy with respect to q_i, here q_i is θ_1, ϕ_2 and ϕ_1 and similarly we have all these partial derivatives of the elements of the mass matrix $\sum_{k=1}^{3} (\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i})$.

Because now there are three of these elements, q's are three dimensional. So, once we have this we can obtain what are the elements of the coriolis and centripetal matrix or terms which is given by C and that we obtain by substituting $q[1]\ (t)\ \theta_1 q[2]\ (t)\ as\ \phi_2\ q[3]\ (t)\ as\ \phi_1$ and then we look at this stamp which is some way to capture what is the derivatives of this mass matrices.

So, once we do all that we can see that the coriolis matrix there is one $C_{11}\ term,\ C_{12}\ term,\ C_{13} = 0$. So, this part is 0, because you can see this 0 because the coriolis term for the out output link $m_{33}$is constant. So, m_{33} is constant. So, derivative of m_{33} with everything will go to 0. The gravity term likewise is a partial derivative of the potential energy with respect to q_i.

Again you will get some mg cos $cos\ \theta$ and then for the third or the output link we wiil have $m_3\ g\ r_3 cos\ \phi_1$. So, this will be θ_1 then you will have some $\theta_1 + \phi_2$ and the y component will be $m_2\ r_2\ g\ cos(\theta_1\ +\ \phi_2)$. So, we can find the mass matrix, we can find the coriolis matrix C_{ij} and we can find the gravity terms. So, this is a vector, this is a 3 by 1 vector. Coriolis is a 3 by 3 matrix and mass matrix is of course 3 by 3.

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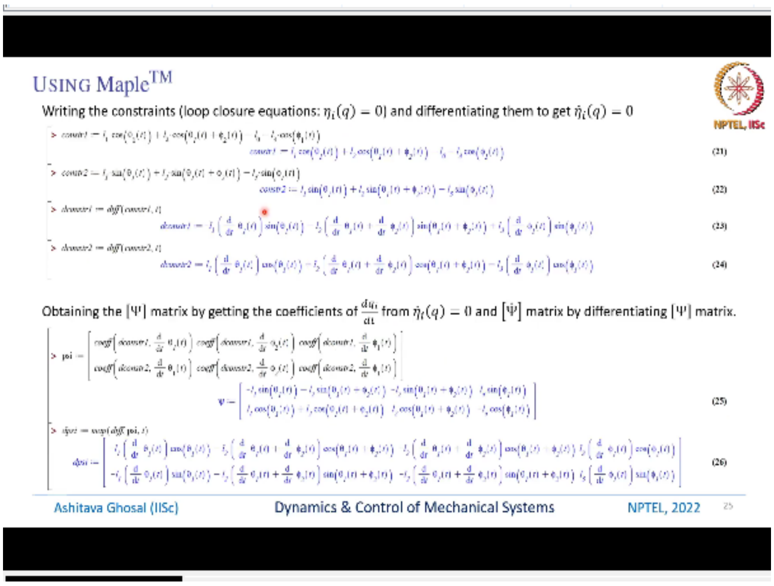

Now let us look at the constraint equations. So, what do we have? We have two constraint equations; remember for a 4-bar mechanism when we break at the joint 3. So, we can find the position vector of the joint 3 which is l_1 cos $cos\ \theta_1$ + l_2 cos $cos\ \theta_1$ + ϕ_2 and this will be equal to the vector going from the other direction which is $l_0 + l_3$ cos ϕ_1 t. So, one of the constraint equation is l_1 cos $cos\ \theta_1$ + $l_2\ \theta_1$ + ϕ_2) - l_0 - l_3 cos ϕ_1 = 0.

This is what is given here. So, we define one constraint. Then the second constraint is the y component which is l_1 sin θ_1 + l_2 sin $sin\ (\theta_1 + \phi_2)$ this will be equal to l_3 sin ϕ_1 and so you can take this on the one side and this is equal to 0. So, constraint 1 = 0, constraint 2 = 0. These are the two constraint equations. In the earlier part of this week I had said $\eta_i(q)$. So, this is $\eta_1(q)$. and this is $\eta_2(q)$.

So, there are only two constraint equations. So, then we can differentiate this constraint equation and find the constraint matrix. So, the constraint matrix is given by ψ and how do I find out? We find the derivative of this constraint equation 1 with time, the derivative of the constraint equation 2 with time and then we say that these are $dconstr\,1$ $dconstr\,2$ and we get some terms.

This will contain $\dot{\theta}_1$, then it will contain $\dot{\phi}_2$ and so on because we are taking the derivative of this constraint equation .So, we will have to use chain rule. So, l_1 cos $cos\ \theta_1$ will become - l_1

$\sin sin\, \theta_1$ and then also one more term with $\dot{\theta}_1$. So, this is what you will get so we find the derivative of this constraint equation and label them as $dconstr\,1$ $dconstr\,2$.

So, the coefficients of the [ψ] the constraint matrix can be obtained by finding the coefficients of $\dot{\theta}_1$ from $dconstr\,1$. Similarly, the coefficient of $\dot{\theta}_1$ in $dconstr\,2$ the second derivative of the constraint equation and likewise the coefficient of $\dot{\phi}_2$ from the first equation, $\dot{\phi}_2$ from the second equation and then $\dot{\phi}_1$ from the first equation and $\dot{\phi}_1$ from the second equation.

So, we will get a [ψ] as 2 by 3. So, there are two rows and three columns and these are the terms. So, this is one term, this is the second term and this is the third term. So, we get the psi matrix which is a 2 by 3 matrix, we also need to find the derivative of the psi matrix which is denoted by $dpsi$ and how do I find the derivative of the matrix? We use this command $map(diff,\ psi, t)$.

So, basically we take the derivative of each element of this matrix with time and then we label it as $dpsi$. So, here again we will get the derivatives of each one of these terms and these are given here. So, we can take a look at one of the terms. Let us say $l_3 \sin \phi_1(t)$. So, what will be the derivative of this? It will be $l_3\dot{\phi}_1 \cos \phi_1(t)$, this simple chain rule.

So, the derivative of $-l_3 \cos \phi_1(t)$, is $l_3\dot{\phi}_1$ into $\sin \phi_1(t)$, So, we can find each of these elements the derivatives and arrange them. So, the dimension of $\dot{\psi}$ is also 2 by 3.

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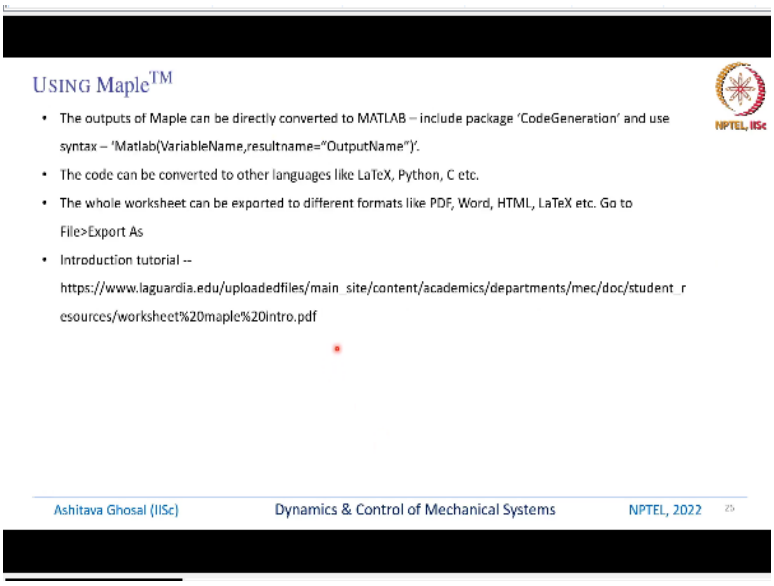

So, once we have found out all these things we can use the elements of the Jacobian of the constraint matrix, we can find the equations of motion and then we can derive the equations of motion. One thing which we can do is we can convert whatever we have obtained to Matlab because eventually we have to solve these equations of motion in Matlab. Maple also allows you to solve equations of motion.

But Matlab is more better known, it has better GUI and it you can get plots and other things very easily. So, we can convert whatever Maple is producing all the symbols and expressions into Matlab by using a package called code generation and basically we use the syntax Matlab variable name, result name and output name. So, we can give these variable names and we can get a Matlab code which we can run in Matlab.

You can also convert the output of Maple to LaTex, Python and C. LaTex is for type setting. So, it will look nicer. So, we can obtain what exactly the equation of motion looks like in the slides. The whole works sheet can also be exported to different formats like PDF, word, HTML, LaTex and so on. If you want more material on how to use Maple you can please go to this site and then there is a very nice way of instruction tutorial on Maple.

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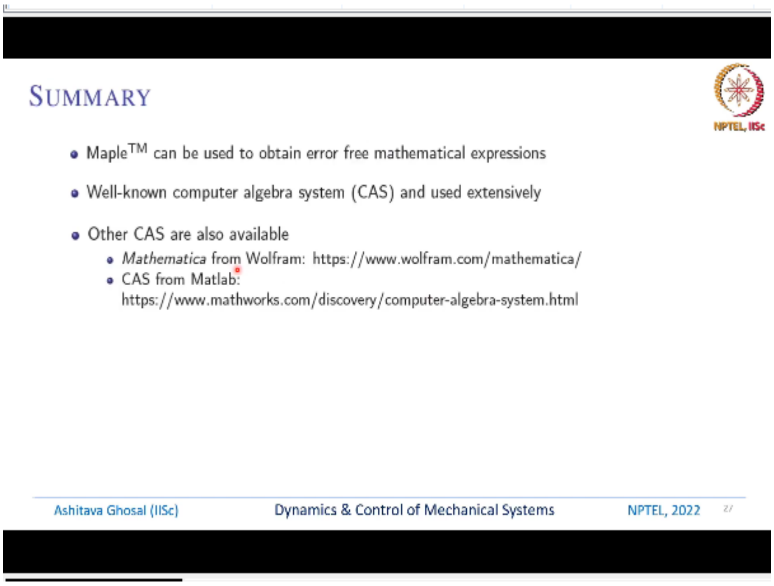

So, in summary Maple can be used to obtain error free mathematical expressions. So, I can find out the elements of the mass matrix, the elements of the coriolis term, the elements of the gravity term, the elements of the constraint matrix, the derivative of the constraint matrix and so on. Then we can form the equations of motion and solve the equations of motion in Matlab or some other software which allows you to use numerical numbers or numerical tools like ODE, solvers and so on which can solve this differential equations of motion.

So, what I have shown you is one of the computer algebra system which is well known Maple it is used reasonably extensively. There are other computer algebra systems which are also available, there is another very well known CAS system which is called mathematica, it is available from this company called Wolfram and this if you want if you are interested in mathematica please go to this website wolfram.com slash mathematica.

Matlab also gives you its own computer algebra system, this is slightly less well-known and less powerful than either Maple or mathematica but nevertheless you can use the computed algebra system from Matlab. For simple examples like the plane 2R or the angular velocity vector or even the 4-bar mechanism we can obtain everything from the computer algebra system given by Matlab.

So, in this NPTEL course you have access to Matlab and all its tool boxes. So, it is a good idea if you have time to try out the computer algebra system from Matlab, more details about the computer algebra system from Matlab is available in this website.