Dynamics and Control of Mechanical Systems
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Lecture-15
Examples of Equations of Motion

In the last 2 lectures we have looked at how to obtain the equations of motion of multi-body systems using the Newton-Euler formulation and the Lagrangian formulation. In this lecture we will see several examples of equations of motion derived using the Newton-Euler formulation or the Lagrangian formulation.

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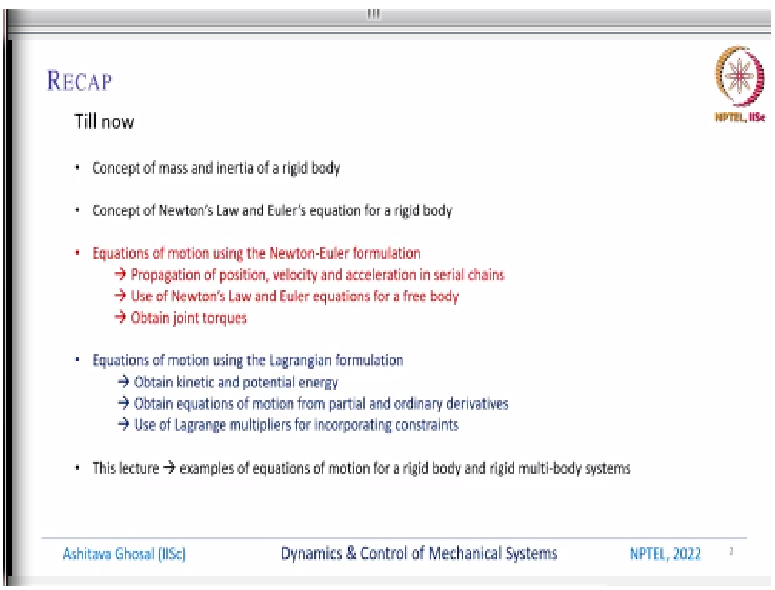

To recap, till now we have looked at the concept of mass and inertia of a rigid body. I have also showed you what is the concept of Newton's law and Euler's equation for a rigid body. We have also obtained the equations of motion or a way to obtain the equations of motion using Newton-Euler formulation. So, in the Newton-Euler formulation basically we propagate the position velocity and acceleration in serial chains.

We use Newton's law and Euler equations for a free body and we have obtain joint torques. We can also obtain the equation of motion using Lagrangian formulation. In the Lagrangian formulation basically we obtain the kinetic and potential energy then we obtain the equations of motion from the partial and ordinary derivatives of a function called the Lagrangian, a scalar function called the Lagrangian.

And then if you have a closed loop system or a system with constraints then we can use Lagrange multipliers for incorporating the constraints. In this lecture we will show examples of equations of motion for a rigid body and rigid multi-body systems.

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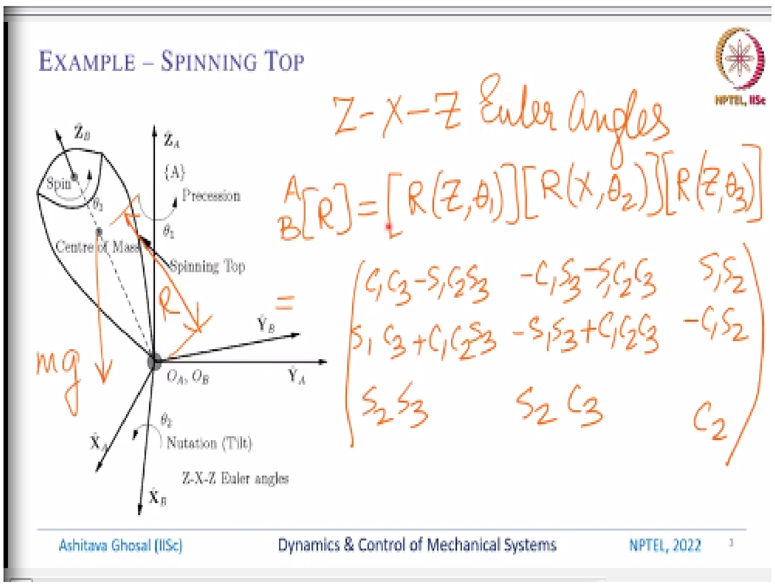

The first example is a very classical example we are going to consider the spinning top, we have already considered the scene what a spinning top does when it is not subjected to an external force earlier on. But now we will look at a spinning top when there is a mg force which is acting at the center of mass here. So, if you recollect the spinning top can be described by Z-X-Z Euler angles. So, what is Z-X-Z Euler angles?

We can obtain the $BA[R]$, so A is the reference coordinate system and B is a coordinate system attached to the spinning top as some rotation matrix Z about θ_1 about Z by θ_1, rotation matrix X, θ_2, rotation matrix Z, θ_3. So, where θ_1 is this angle which is also called the precession angle, θ_2 is the tilt or the mutation angle and θ_3 is the spin which is the rotation of the top about the Z-axis.

So, if you expand all these things we have looked at these 3 simple rotations Z, X and Z and if you multiply the matrices in that order we will get a rotation matrix which contains terms like this. So, 3-3 term is cos θ_2, this is sin θ_1 sin θ_2, -cos θ_1 sin θ_2, this is sine θ_2 sin θ_3, $s_2\, c_3$. And 1-1 term is little bit more complicated but we have seen this earlier it is not completely new.

So, we have 1-1 term as $c_1\, c_3 - s_1\, c_2\, s_3$, the 1-2 term is $-c_1\, s_3 - s_1\, c_2\, c_3$ this term is $s_1\, c_3 + c_1\, c_2\, s_3$, the 2-2 term is $-s_1\, s_3 + c_1\, c_2\, c_3$. So, this is nothing but obtained from the multiplication of 3 simple rotation matrices, once about Z then about X and third about again Z.

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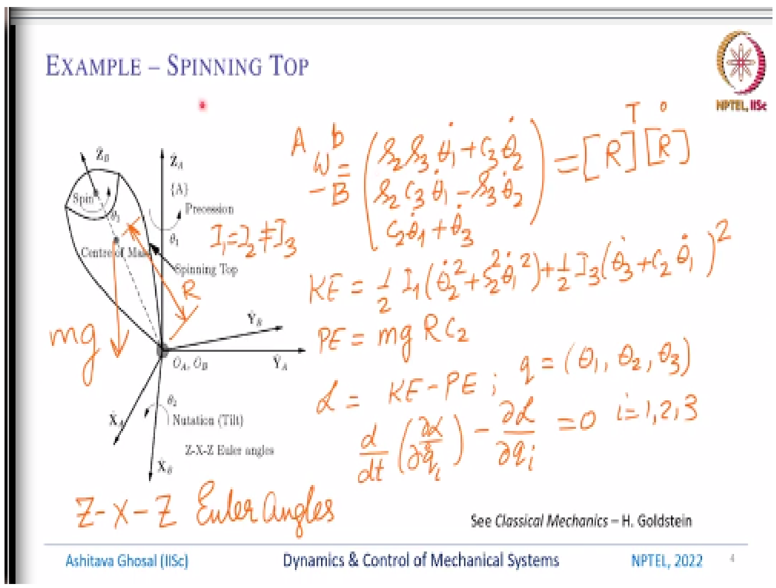

So, once you have this spinning top rotation matrix we can obtain the angular velocity, so we will find $A\omega_B^b$ and this is the body fixed angular velocity which is nothing but $[R]^T\,[\dot{R}]$. Again we have done this several times but you can again do it once more and after simplification after extracting from the skew symmetric matrix we will get ω_x as $s_2\, s_3\, \dot\theta_1 + c_3\, \dot\theta_2$. Then $s_2\, c_3\, \dot\theta_1 - s_3\, \dot\theta_2$ and then $c_2\, \dot\theta_1 + \dot\theta_3$.

Once we have this angular velocity we can obtain the kinetic energy. To obtain the kinetic energy we need to know what are the inertia I_{xx}, I_{yy}, I_{zz}, so these are denoted by I_1, I_2 and I_3. So, in this example for top I_1 is normally equal to I_2 and which is not equal to I_3, I_3 is about the Z axis. So, once we have this body fixed inertia matrix we can obtain the kinetic energy which is $\frac{1}{2} I_1 (\dot{\theta}_2^2 + s_2^2 \dot{\theta}_1^2) + \frac{1}{2} I_3 \left(\dot{\theta}_3 + c_2 \dot{\theta}_1\right)^2$.

The potential energy for this spinning top is nothing but $mgRc_2$, $c_2$is this tilting angle. The Lagrangian can be obtained as KE - PE and in this case the 3 generalized coordinates are theta θ_1, θ_2 and θ_3. And then we can obtain the equations of motion by doing $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$. So, there are no external forces which are acting on this top.

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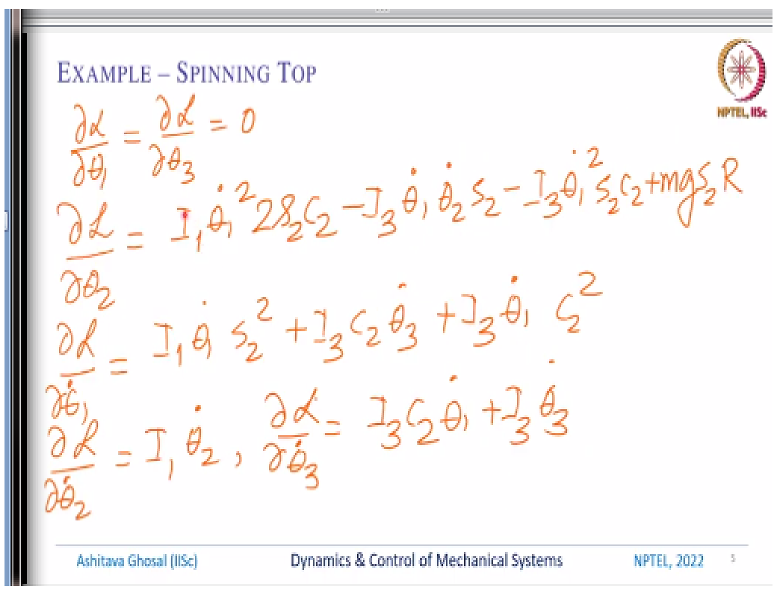

So, let us compute all these terms in the partial derivatives and ordinary derivatives. So, $\frac{\partial L}{\partial \theta_1}$ and $\frac{\partial L}{\partial \theta_3}$ is 0, there is no theta, only the Lagrangian is function of θ_2. The Lagrangian with respect to θ_2, $\frac{\partial L}{\partial \theta_2}$, you will get all these terms. So, you will get $I_1 \dot{\theta}_1^2\ 2\ s_2 c_2 - I_3 \left(\dot{\theta}_1\right) \dot{\theta}_2 s_2 - I_3\left(\dot{\theta}_2\right) s_2 c_2 + mgs_2 R$.

The partial of the $\frac{\partial L}{\partial \dot{\theta}_1}$ is $I_1 \dot{\theta}_1 s_2^2 + I_3 c_2 \dot{\theta}_3 + I_3 \dot{\theta}_1 c_2^2$. And the $\frac{\partial L}{\partial \dot{\theta}_1}$ is just $I_1 \dot{\theta}_2$, $\frac{\partial L}{\partial \dot{\theta}_3}$is $I_3 c_2 \dot{\theta}_1 + I_3 \dot{\theta}_3$.

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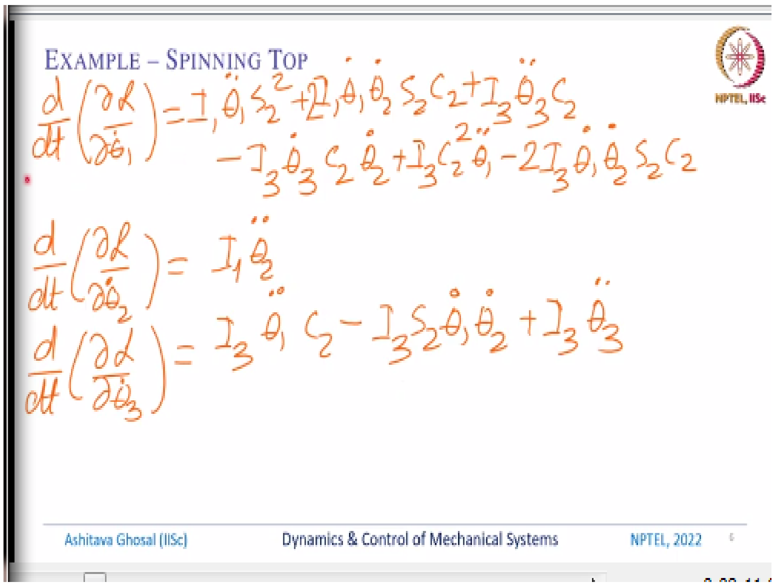

The $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_1})$ again we can compute, so I am writing this by hand but you can always do it using some computer algebra tool. So, it is $I_1 \ddot{\theta}_1 s_2^2 + 2I_1 \dot{\theta}_1 \dot{\theta}_2 s_2 c_2 + I_3 \ddot{\theta}_3 c_2 - I_3 \dot{\theta}_3 c_2 \dot{\theta}_2 + I_3 c_2^2 \ddot{\theta}_1 - 2 I_3 \dot{\theta}_1 \dot{\theta}_2 s_2 c_2$.

The time derivative of this Lagrangian partial of the $\frac{\partial L}{\partial \dot{\theta}_2} = I_1 \ddot{\theta}_2$. And finally the time derivative of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) = I_3 \ddot{\theta}_1 c_2 - I_3 s_2 \dot{\theta}_1 \dot{\theta}_2 + I_3 \ddot{\theta}_3$. So, we now have all the ingredients to derive the equation of motion, we have the time derivatives.

We also have the partial derivatives of the Lagrangian with respect to θ_1 and θ_2 and θ_3. So, remember only θ_2 is there, partial with respect to θ_1 is 0, partial with respect to θ_3 is also 0.

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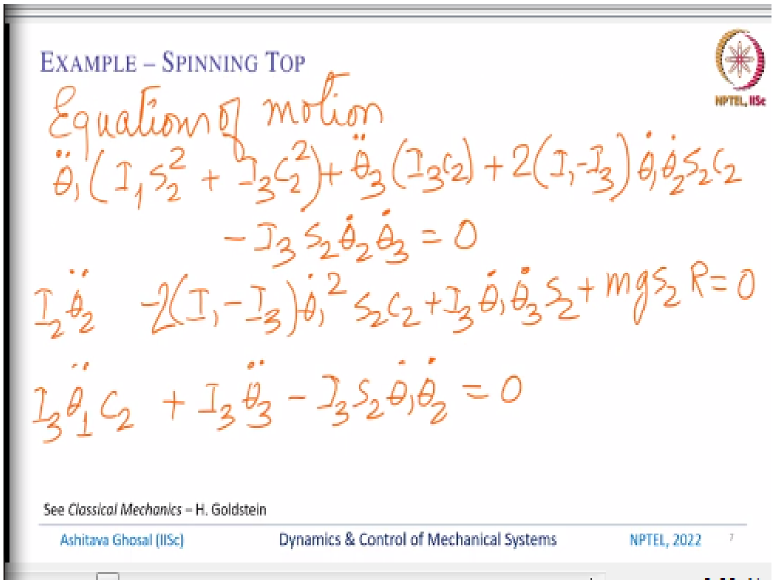

So, the equation of motion now can be assembled just by following the recipe of the Lagrangian formulation. So, we can write as $\ddot{\theta}_1(I_1 s_2^2 + I_3 c_2^2) + \ddot{\theta}_3 I_3 c_2 + 2(I_1 - I_3)\dot{\theta}_1\dot{\theta}_2 s_2 c_2 - I_3 s_2 \dot{\theta}_2 \dot{\theta}_3 = 0$, this is the first equation of motion. The second equation of motion is $I_2\ddot{\theta}_2 - 2\left(I_1 - I_3\right)\dot{\theta}_1^2 s_2 c_2 + I_3\dot{\theta}_1\dot{\theta}_3 s_2 + mgs_2 R = 0$.

And the last equation is $I_3\ddot{\theta}_1 c_2 + I_3\ddot{\theta}_3 - I_3 s_2 \dot{\theta}_1 \dot{\theta}_2 = 0$. So, as you can see the terms multiplying $\ddot{\theta}_1$ or $\ddot{\theta}_3$ or $\ddot{\theta}_2$ they are like the inertia term, so that is like $I_2\ddot{\theta}_2$. The terms which are quadratic $\dot{\theta}_1\ \dot{\theta}_2$ they are like the centripetal and coriolis term. So, we will have this $\dot{\theta}_1\ \dot{\theta}_2$ into some $\sin\theta_2 \cos\theta_2$.

Likewise we also have some $\dot{\theta}_2\ \dot{\theta}_3$ in the first equation. In the second equation we have also $\dot{\theta}_1^2$, remember this coriolis and centripetal term always have quadratic in Q dots, so that is what we see here. And then in the second equation we also have this gravity term, so this is the term which is coming from the gravity. And the last equation has $\ddot{\theta}_1$ or $\ddot{\theta}_3$ but also $\dot{\theta}_1\ \dot{\theta}_2$.

So, apparently this is a very complicated set of 3 nonlinear ordinary differential equations, so there are 3 and there is lot of nonlinearity. We have s_2^2; we have $\dot{\theta}_1\ \dot{\theta}_2\ s_2\ c_2$ and various terms. So, this is one example which has been looked by many, many researchers in the past, physicists and mechanics people and lots of people have looked at it. And it turns out this is one of the very few examples which can be solved we will discuss what we mean by solved.

But we can solve this set of 3 nonlinear ordinary differential equations. And let us proceed, let us see what we people have found out over maybe more than 100 years back people have found this out.

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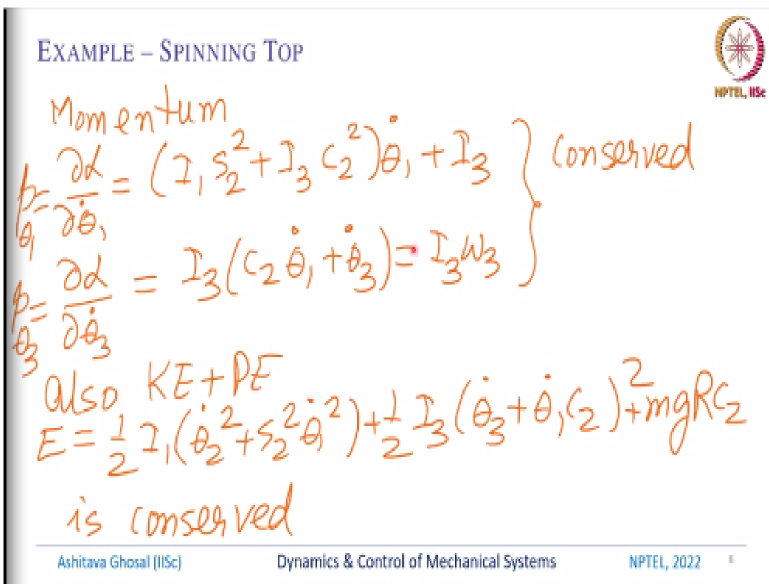

So first thing that you can see is this partial of this $\frac{\partial L}{\partial \dot{\theta}_1}$, let us denote that by p_{θ_1} is given by this. So, it is $(I_1 s_2^2 + I_3 c_2^2)\,\dot{\theta}_1 + I_3$. Similarly the partial of this $\frac{\partial L}{\partial \dot{\theta}_3} = I_3(c_2\,\dot{\theta}_1 + \dot{\theta}_3)$, so it turns out this p_{θ_1} and p_{θ_2}, so these are basically momentum corresponding to θ_1 and θ_3, so these 2 momentum are conserved.

So, what do we mean by conserved? This quantity remains constant when the top is spinning, so this complicated expression of $(I_1 s_2^2 + I_3 c_2^2)\,\dot{\theta}_1 + I_3$ is remaining constant when the top is spinning or even tilting or even processing. So, the in the complicated motion of the top this term is conserved, likewise $I_3(c_2\dot{\theta}_1 + \dot{\theta}_3)$, is also conserved and it turns out that this is nothing but ω_3.

So, remember we computed the $A\omega_B^b$ in the body fixed angular velocity vector, the Z component of that is this $(c_2\dot{\theta}_1 + \dot{\theta}_3)$, you can go back and see the slide. So, this is also conserved and finally like in many other conservative mechanical systems without any damping the total energy is also conserved. So, kinetic energy + potential energy is also conserved.

So, E which is the sum of the kinetic + potential energy, so this is kinetic energy I have shown you the example before, this is the potential energy, so this is also constant. So, there are 3 of these quantities which are constant and they are conserved.

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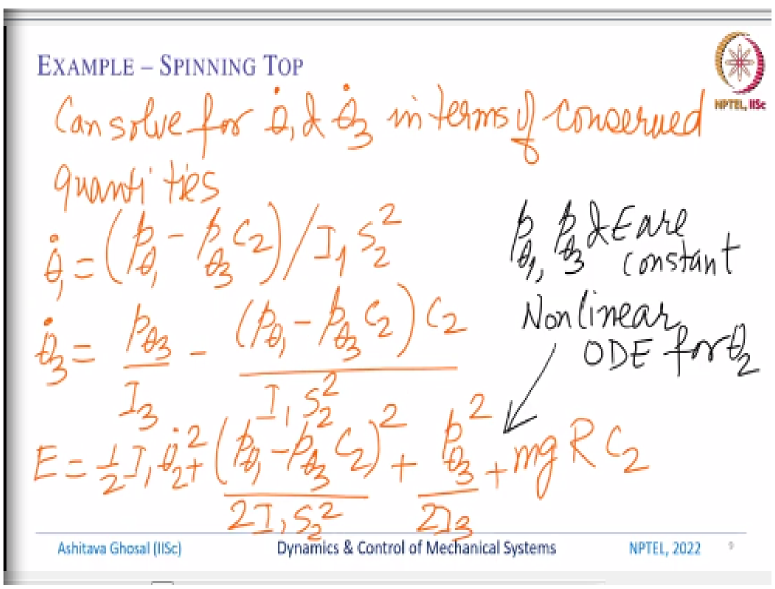

So, now if you look at this p_{θ_1} and p_{θ_3}, we can solve for $\dot{\theta}_1$ and $\dot{\theta}_3$ in terms of the conserved quantities. So, for example $\dot{\theta}_1 = \frac{p_{\theta_1} - p_{\theta_3}c_2}{I_1 s_2^2}$, so we can write this expression. Likewise $\dot{\theta}_3$ is given by p_{θ_3}, so which is the momentum corresponding to θ_3 and this is the momentum corresponding to θ_1. So, $\frac{p_{\theta_3}}{I_3} - \frac{(p\theta_1 - p\theta_3 c_2)c_2}{I_1 s_2^2}$.

So, we can write expressions for $\dot{\theta}_1$ and $\dot{\theta}_2$. And then we can substitute $\dot{\theta}_1$ $\dot{\theta}_2$ in the expression for the energy, remember energy had all these terms of $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$. So, then we can get rid of $\dot{\theta}_1$ and $\dot{\theta}_3$ in the expression for energy and write an expression of energy which is the sum of the kinetic + potential energy only in terms of $\dot{\theta}_2$ and θ_2.

So, what we will get is $\frac{1}{2} I_1 \dot{\theta}_2^2 + \frac{(p\theta_1 - p_{\theta_3} c_2)^2}{2I_1 s_2^2} + \frac{p_{\theta_3}^2}{2I_3} + mg\,Rc_2$. So, what have we done? So, basically E is some conserved quantity; it is a given number, so this right hand side is a nonlinear ODE for θ_2, why? Because p_{θ_1} *and* p_{θ_3} these are also constants, these are numbers which are constant during the motion of this top.

So, we have this very, very complicated nonlinear ODE only in terms of θ_2, it is still pretty bad, why? Because we have c_2^2 *divided by* s_2^2 and then we have this $\dot{\theta}_2^2$, it is not a simple $\dot{\theta}_2$, so it is a non-linear ODE in θ_2 and $\dot{\theta}_2^2$.

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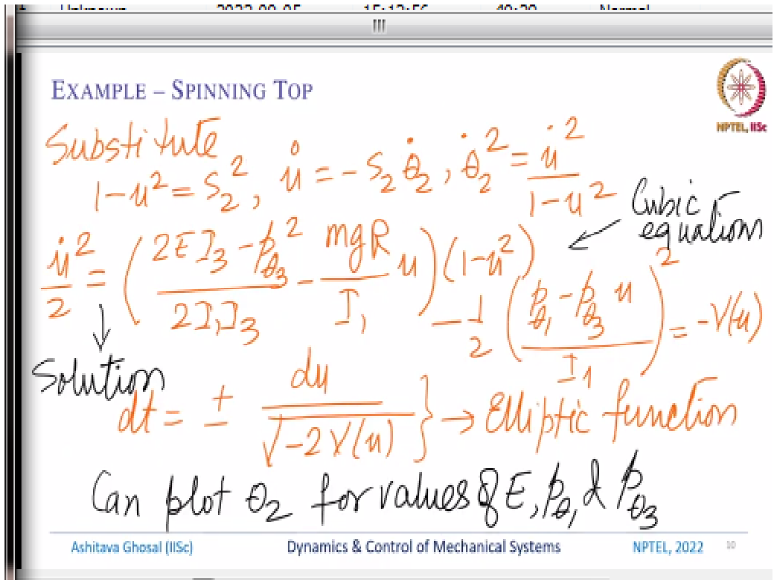

But it turns out and this is what people have found out after lot of effort and people have studied spinning tops extensively. That we can make this substitution let us call$l_1 - u^2 = s_2^2$, s_2 is what? Sine θ_2^2 then $\dot{u} = \; -s_2\dot{\theta}_2$, $\dot{\theta}_2^2 = \frac{\dot{u}^2}{1-u^2}$. So, we substitute what is $\dot{\theta}_2$ and what is s_2^2 and various other terms.And then we will get this expression which is $\frac{\dot{u}^2}{2} = \left(\frac{2EI_3 - p_{\theta_3}^2}{2I_1I_3} - \frac{mgR}{I_1}u\right)\left(1 - u^2\right) - \frac{1}{2}\left(\frac{\left(p\,\theta_1 - p_{\theta_3}u\right)}{I_1}\right)^2 =- V(u)$. So, what do we have here? That $\dot{u}^2$ this term is basically a cubic function of u, so how do we get cubic? Because you can see there is a u here and a u^2 here, so the maximum power of u on the right hand side is u cube.

So, let us denote this right hand side as $-V(u)$, so what is $\dot{u}$? $\frac{du}{dt}$. So, it turns out that we can now solve for dt not du. So, this is $\frac{\dot{u}^2}{2}$, sO_2 can go to this side, so we will have 2 into minus sum V which is a function of u, so $\dot{u}^2$ .So, we will get this expression which is $dt = \pm \frac{du}{\sqrt{-2V(u)}}$. So, what have we done?

We have obtained du by dt = some expression involving $-V(u)$which is a cubic function of u, so these are called elliptic functions. So, those of you who have seen or heard solutions involving elliptic functions these are very, very complex ways of giving solutions to a differential equation. So, although we cannot really write analytical expressions for elliptic function but nevertheless we can bring the solution of du by dt in terms of or we can solve t in terms of u as elliptic functions.

Importantly, we can plot θ_2 for values of E, p_{θ_1} and p_{θ_3}. So, although I said initially that this is the solution of the non-linear ODEs but this is not really a solution. Because these elliptic functions we do not know there are no analytical ways of finding these elliptic functions. So, if you have a quadratic.

So, I can write an expression for solution of a quadratic equation in terms of the coefficients of the quadratic, elliptic functions are not that simple. Nevertheless these are known, there are some things called elliptic functions and if you can write an expression in terms of elliptic functions then it is sort of like solution and then we can plot. So, this is one of the very few examples of non-linear ODEs or equations of motion which gives non-linear ordinary differential equations which can be at least written finally in terms of elliptic functions.

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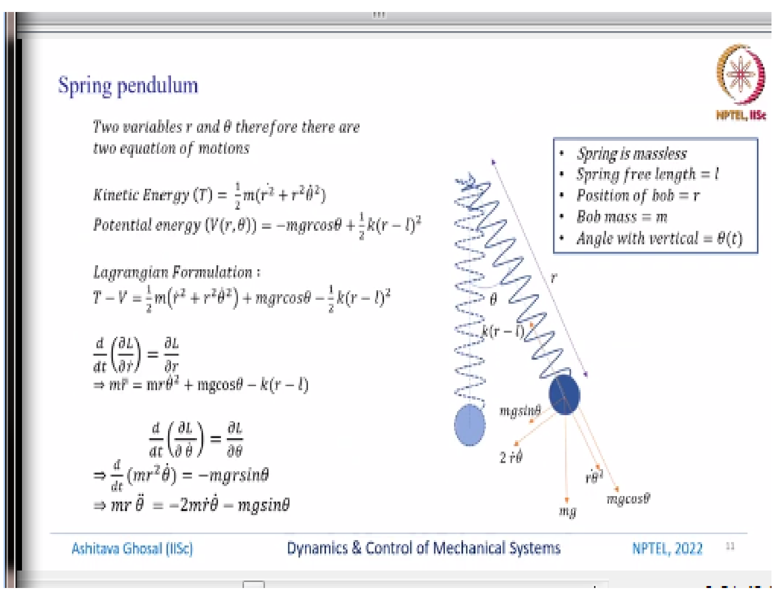

This example is a modification of a simple pendulum, the modification is as follows. So, we have a bob which is connected not by a wire or a rope but with the spring. So, as you can see there is gravity acting downwards, so this bob can move in some path which is sort of like an arc of a circle but it can also go up and down. So, there are 2 variables r and θ which need to be used to describe the motion of this bob.

So, this r here is the position of the bob along this direction, this free length of the spring is given by l, the mass of the bob is m and the angle from the vertical of this line which is sort of the center line of this spring is $\theta(t)$. So, the kinetic energy of this bob can be now written in terms of $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$. So, $\dot{r}^2$ is because the bob can go up and down and also it can have a velocity which is $r^2\dot{\theta}^2$.

The potential energy of this system is $-\ mgr\ cos\theta$ and also the energy which is stored in the spring which is $\frac{1}{2}\ k\ (r\ -\ l)^2$. So, note this is r, the free length is l, so the force which is acting due to the spring is $k\ (r\ -\ l)$ and of course there is gravity mg, that is $mgr\ sin\theta$ and $mgr\ cos\theta$ and $r^2\dot{\theta}^2$ which is acting in this direction. And we have $2\dot{r}\dot{\theta}$ which is acting in this direction.

So, we have the kinetic energy and we have the potential energy, the Lagrangian then is kinetic - potential energy in this case we are denoting kinetic energy with T and the potential energy with V. The potential energy is function of both r as well as θ, so the $L = \frac{1}{2}\ m(\dot{r}^2 +\ r^2\dot{\theta}^2) +\ mgr\ cos\,\theta\ - \frac{1}{2}\ k\ (r\ -\ l)^2$. So, once we have this Lagrangian we can obtain the equations of motion.

In this case there are no external forces, so we can write $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ or $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$.

So, if you do this then you will get $m\ddot{r} = mr\dot{\theta}^2 + mg\theta cos\theta - k(r - l)$. We can also derive the partial derivative with respect to $\dot{\theta}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$.

So, then the second differential equation that we will get is $mr\ddot{\theta} =- 2m\dot{r}\dot{\theta} - mgsin\theta$ So, we have these 2 equations of motion very, very easily derived from the Lagrangian formulation. And the Lagrangian formulation inherently you compute the kinetic energy, you compute the potential energy and then you find T - V. So, this is a nice and simple extension of a simple pendulum, where I can show how to accommodate things like a spring.

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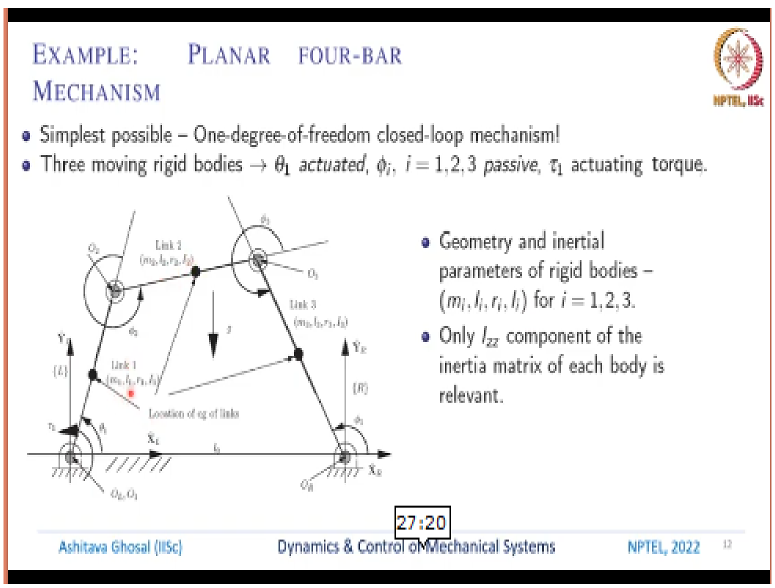

Let us continue this is an example of a planar four-bar echanism. This is the simplest possible one-degree of freedom closed loop mechanism. So, there is loop, so I can start from this fixed end which is either O_L and then go to this second joint which is O_2 then go to the third which is O_3 and then come back to what? But this is also fixed, both of these form single loop. So, there are 3 moving bodies link 1, link 2 and link 3, θ_1 is the actuated because this mechanism has only one degree of freedom.

So, we can have only one joint which is actuated, so the other$_3$ joints which is ϕ_2, ϕ_3 *and* ϕ_1 are passes. So, the geometry and inertial parameters of the rigid bodies are again denoted by mass, inertia, z component of the inertia because everything is planar, length l_i *and* I_i, so I is the z component of the inertia, so it is m_2, l_2, r_2, I_2 likewise m_2, l_1, r_1, I_1 and so on.

And then there is a gravity acting and we are going to assume that the CG of each link is located at r_1 starting from here, it is located at r_2 starting from here and it is located at r_3. So, it is along the link, these are just simplifying assumptions it does not make any difference whether this r_2 is somewhere with both an X and a Y coordinate.

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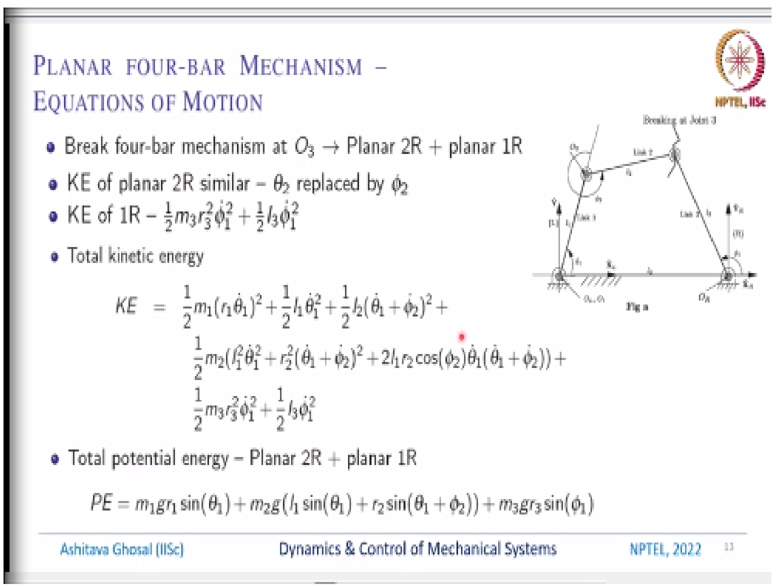

So, we break this four-bar at O_3, so we have a plana 2R and a planar 1R system. 2R mean*s* 2 links connected by 2 rotary joints and this other one is a planar 1R which is nothing like but similar to a pendulum. So, the kinetic energy of this 2R is very similar to what we have derived earlier, all we need to do is replace θ_2 by ϕ_2. So, in the place of the plana*r* 2R we have θ_1 θ_2, now we have θ_1 and ϕ_2.

The kinetic energy of the 1R is also nothing but a simple pendulum, so which is $\frac{1}{2}m_3 r_3^2\ \dot{\phi}_1^2 + \frac{1}{2}\ I_3\ \dot{\phi}_1^2$, so what is the total kinetic energy? It is the kinetic energy of this 2 moving links which is this 2R chain and then one moving link which is the 1R chain. The potential energy is also nothing but the potential energy of this 2link chain which we have derived earlier, except now we do not have θ_2 we have ϕ_2 .And also this one length chain which is nothing but $m_1 g r_1\ sin\theta_1$, $r\ sin\ \phi_1$ which is the last output angle.

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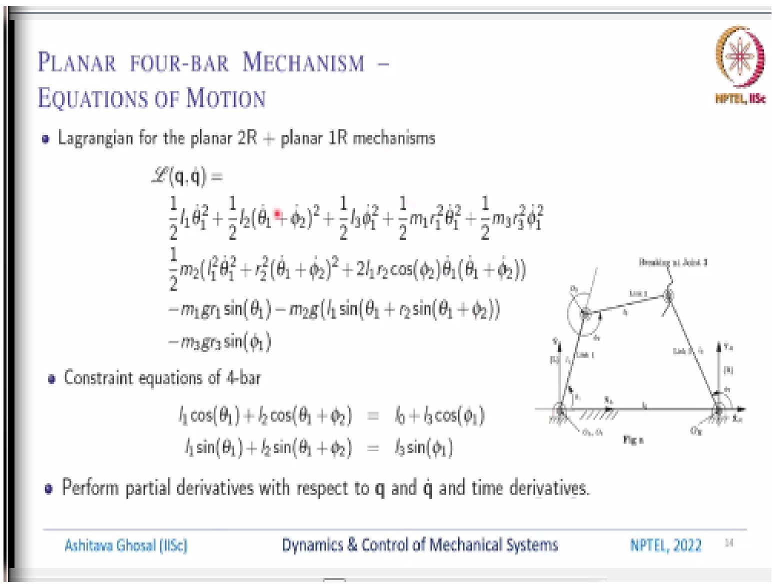

The Lagrangian for the planar 2 R + the planar 1 R mechanism is nothing but sum of all these kinetic energies. So, it is $\frac{1}{2}\ I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\left(\dot{\theta}_1 + \dot{\phi}_1\right)^2 + \frac{1}{2}\ I_3\ \dot{\phi}_1^2$. So, this is coming from the 1R chain or from the simple pendulum, these 2 are coming from the 2R chain. And then we have of course $\frac{1}{2}m_1 r_1\dot{\theta}_1^2 + \frac{1}{2}\ m_3\ r_3\dot{\phi}_1^2$, this is coming from that 1R chain.

And then m_1 and m_2 are coming from the planar 2R chain and the gravity terms are coming from the 2R chain m_1 and m_2 and m_3 is the last chain, 1R chain. For this four-bar we have 2 constraint equations we have seen this earlier. So, basically the x coordinate of the place where we are going to break is given by $l_1 cos\ \theta_1 +\ l_2\ cos\ (\theta_1 + \phi_2)\ =\ l_0\, l_3\ cos\ \phi_1$ and the y component is given by $l_1 sin\ \theta_1 +\ l_2 \sin sin\ (\theta_1 + \phi_2)\ =\ l_3 \sin sin\ \ \phi_1$.

So, we have the Lagrangian, we have the constraint equations and then we can perform the partial derivatives with respect to q and $\dot{q}$. So, what is q here? q is θ_1, ϕ_2 and ϕ_1 and $\dot{q}$ is the time derivatives of those.

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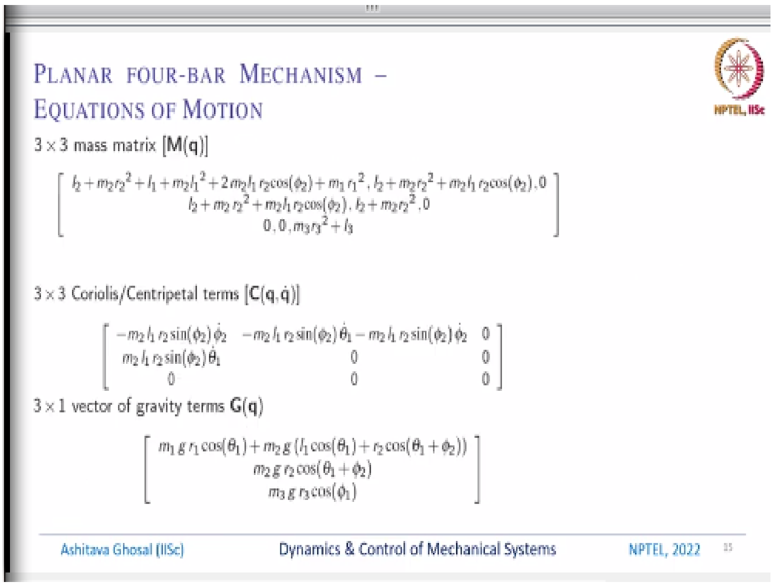

So, if you do all these work, lot of work and again I will show you a way to do it using a computer tool, we will get what a 3 by 3 mass matrix. So, again you can see that the mass matrix has some $I_2 + m_2 r_2^2 + I_1 + m_2 l_1^2$ and so on. So, these terms are exactly the same as what we obtained for the planar 2R chain except instead of θ_2 we have now ϕ_2. The 1, 2 elements is again the same as what we obtained for the planar 2R chain.

So, 2, 1 element is again what we have obtained from the planar 2R chain, this is $I_2 + m_2 r_2^2$. You can go back to your notes and see that these are exactly the same this 2, 3 and 4, these 4 elements are exactly the same as what we have obtained for the planar 2R chain. This last $0,\ 0,\ m_3 r_3^2 + I_3$ *and* this 0, 0 is the effect of the last link which we have broken.

So, the 3 by 3 coriolis and centripetal term can also be found out, this will now be again similar to these terms are very similar to the planar 2R chain and then this is the other part. So, we have again $\dot{\phi}_2$, $\dot{\theta}_1$ and so on, and the gravity vector is also here. So, we do not have $\dot{\phi}_1$ in the coriolis centripetal term because that is just a simple rotation and there is no $\omega \times V_{rel}$ in that term.

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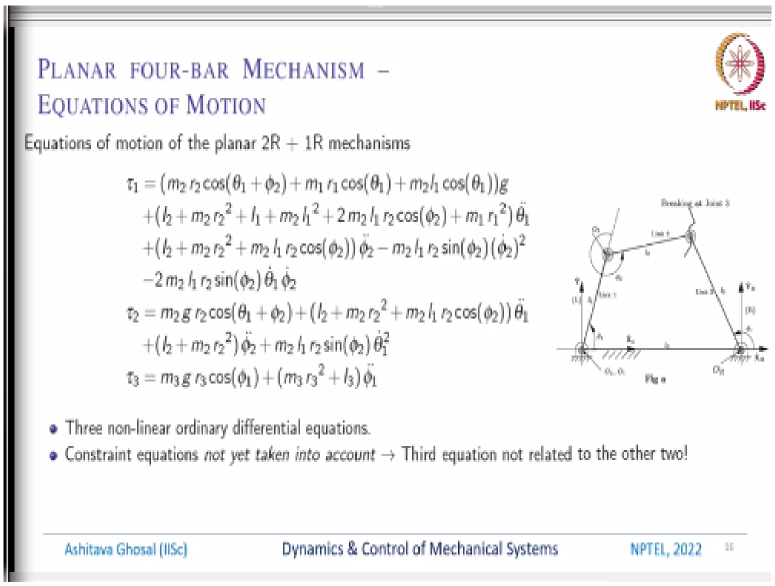

So, the equation of motion for the planar 2R and 1R mechanism can be written down. We just collect all the terms which contains $\ddot{\theta}_1$ which is this, collect all the terms which contains gravity, which is this, collect all the coriolis and centripetal term which are these, this one and this one and collect all the terms which contains $\ddot{\phi}_2$. So, this is from the 2R chain.

Likewise for τ_2 we have this gravity term, we have something into $\ddot{\theta}_1$ $I_2 + m_2 r_2^2$ and then this is the centripetal coriolis term. And then we have another equation which is simply the single link 1R chain or this 1R mechanism which is τ_3 is given by $m_3 g r_3 \cos\phi_1 + (m_3 r_3^2 + I_3)\ddot{\phi}_1$. So, these are 3 nonlinear ordinary differential equations, we have not yet taken the constraints into account and that is obvious.

You can see that the third equation is not yet coupled to these 2; we have broken the joint at O_3. So, when you break it at O_3 then we have just 2 individual serial chain systems, we have a 2R mechanism or a 2R serial chain and 1R mechanism. So, these equations will be now coupled when we go back and use the constraint equations.

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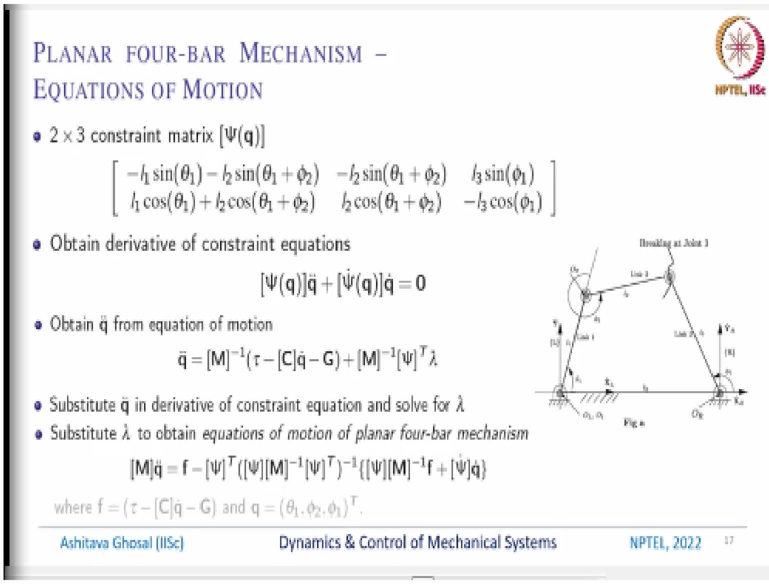

So, the constraint matrixes in this case are the derivatives of the loop closure constraint equations, we will get a 2 by 3 constraint matrix. So, remember it was eta j of q and then we take the partial of η_j with respect to q_i, so we will get all these terms, so it is a 2 by 3. The first term is this, so this is the η_1 and then this is the second term and this is the third term. To obtain this derivative of the constraint equations we take this $\psi \dot{q} = 0$, so we take the derivative, so we will get $\psi\ddot{q} + \dot{\psi}\dot{q} = 0$.

Then we obtained $\ddot{q}$ from the equations of motion, again very, very standard you can go back to the notes and see. So, we can write $\ddot{q}$ is some $M^{-1}\left(\tau - [C]\dot{q} - G\right) + M^{-1}[\psi]^T\lambda$. So, this is psi λ is the Lagrange multiplies, m is the mass matrix. So, we substitute $\ddot{q}$ in this expression here and solve for λ and then we substitute λ back to the equations of motion and we will get $[M]\ddot{q} = f - [\psi]^T\left([\psi]M^{-1}[\psi]^T\right)^{-1}$ and this.

So, we have done this earlier I am just repeating it once more for this four-bar mechanism. So, in the case of the four-bar mechanism ψ is 2 by 3, how about mass matrix? It is 3 by 3, there are 3 variables θ_1, ϕ_2 and ϕ_1 and so on. So, what will be this $[C]\dot{q}$? It will be a 3 by 1 vector. So, again this f is nothing but $\tau - [C]\dot{q} - G$ and the generalized coordinates for this four-bar mechanism is θ_1, ϕ_2 and ϕ_1.

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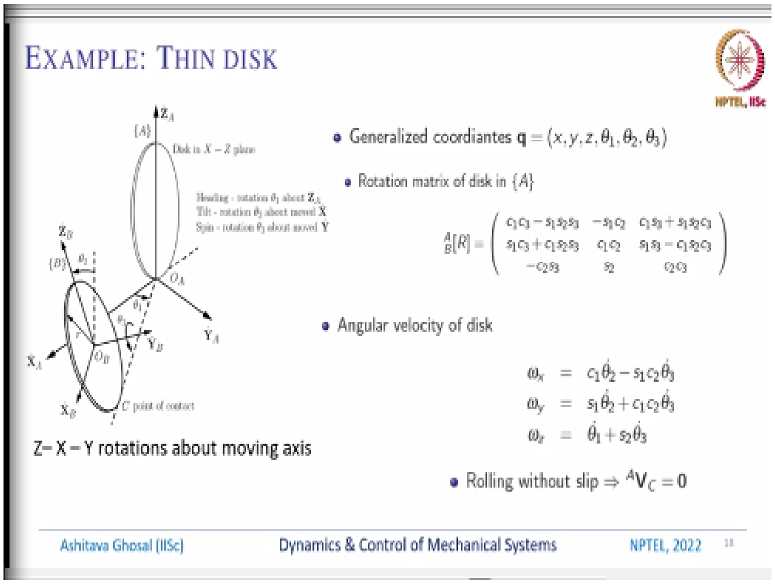

Let us take one more example. This is an interesting example of a thin disk which is rolling on the ground with no slip, this we have discussed earlier this is an example of non-holonomic constraint. So, we have this disc initially in the X-Z plane, so there is a heading angle which is θ_1 which is rotation about the Z-axis. Then there is a tilt rotation θ_2 above the moved X-axis and then there is a spin which is the rotation θ_3 about the moved Y axis.

So, the generalized coordinates for this problem is x,y,z which is the center of this disk. And then θ_1, θ_2 θ_3, so θ_1 which is this one here, θ_2 which is this tilt and θ_3 which is the rotation of the or the spin of this disk. So, if you have this kind of angles which is θ_1, θ_2 and θ_3 then the rotation matrix should contain Z-X-Y. So, out of this θ_1, θ_2, θ_3 if you want to actually obtain the rotation matrix we have to use Z-X-Y Euler angles.

And this rotation matrix which is $BA[R]$, so B is this coordinate system which is attached to the moving disc or the rolling disc and A is a reference coordinate system. $BA[R]$ can be written in terms of θ_1, θ_2 and θ_3 as sine and cosine of these angles. So, we have seen this when we did Euler angles. So, c_1 is nothing but cos θ_1, c_3 is nothing but cos θ_3, s_1 is sin θ_1 and so on.

So, we can obtain the rotation matrix in terms of Z-X-Y rotations about 3 distinct axis using the Euler angle idea and then we have this rotation matrix. The angular velocity of the disk can be obtained by $[\dot{R}]\ [R]^T$. So, I am skipping many steps, we can take the time derivative of each one of these elements. So, $c_1\ c_3$ will become $-\ s_1\ \dot{\theta}_1\ c_3\ +\ c_1\ (-\ s_3)\ \dot{\theta}_3$, we have to use chain rule and we have to find the derivatives of each one of these terms.

So, this is r_{11}, so $[\dot{R}]$ means $\dot{r}_{11}$ and then we do $[\dot{R}]\ [R]^T$ that will be a skew symmetric matrix from which we can extract the X, Y and the Z component of the angular velocity vector. The X, Y and Z component of the angular velocity vector are given like this. So, we will have $c_1\dot{\theta}_2 -\ s_1 c_2\left(\dot{\theta}_3\right)$ and $s_1\dot{\theta}_2$ $which\ is\ \omega_y\ and\ c_1 c_2 \dot{\theta}_3$ and $\omega_z\ is\ \dot{\theta}_1\ s_2\ \dot{\theta}_3$.

Again it is a lot of effort but then nowadays nobody does this manually. So, there are computer tools which we will discuss little later called Maple which can be used to perform not only the rotations about Z-X-Y and the multiplication of those matrices in that order but also do $[\dot{R}]\ [R]^T$ and find the angular velocity components. If you have rolling without slip then the velocity of this point of contact with respect to the A coordinate system should be 0.

So, there is no slip here. So, there is no translation velocity between a point on the ground and a coincident point on the disk, this is the condition for rolling without slip.

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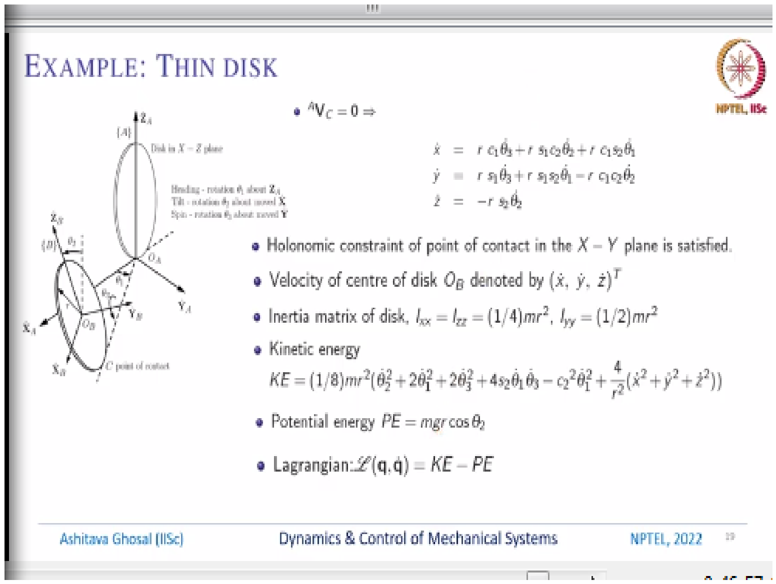

So, continuing with the example of a thin disk rolling without slipping on a horizontal plane. So, we have this disk which is rolling without slipping on this X-Y plane. The constraint that the velocity at the point of contact is 0 which is the constraint which implies it is rolling without slipping can be written in terms of $\dot{x},\ \dot{y},\ \dot{z}$ which is the coordinates X-Y-Z are the coordinates of the center of the disk.

And we can find out the velocity of this point from $\dot{x},\ \dot{y},\ \dot{z} + \omega\times r$, we know what is ω, from $[\dot{R}]\ [R]^T$, ω is in terms of θ_1, θ_2, θ_3 and also the time derivatives $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$. So, if you say that A, B, C is 0 we will get this $\dot{x} = r\cos\theta_1\ \dot{\theta}_3 + r\sin\theta_1\dot{\theta}_2 + r\cos\theta_1\sin\theta_2\,\dot{\theta}_1$.

Likewise $\dot{y} = r s_1\dot{\theta}_3 + r s_1 s_2\dot{\theta}_1 - r c_1 c_2\ \dot{\theta}_2$ and $\dot{z} = -r s_2\ \dot{\theta}_2$. Again θ_1 is the heading angle, it is the rotation about z axis of this disk, θ_2 is the tilt which is the rotation about the moved x axis and spin or θ_3 is the rotation about the y axis. So, there are 2 constraints one is the components of the velocity in this X-Y plane is 0, there is also another inherent constraint which is that this coin or this disk does not leave this X-Y plane, so the Z is constant.

And this holonomic constraint is automatically satisfied when you say the velocity of the point of contact all 3 components are 0. The velocity of the center of the disk O_B is denoted by $\dot{x}, \dot{y}, \dot{z}$ as I have mentioned earlier. The inertia matrix of the disk in the body fixed coordinate system can be easily obtained, this is the standard formulas which are available in many textbooks so the I_{xx} which is x axis and z which is this so X-Z plane is the plane of this disk both are equal to $\frac{1}{4}mr^2$.

And the I_{yy} which is perpendicular to this disk is $\frac{1}{2}mr^2$. The kinetic energy of this disk can be obtained by finding what is $\frac{1}{2}I\omega^2 + \frac{1}{2}m\dot{vc}^2$, so a $\frac{1}{2}mv$ of the center of disk square. So, that is $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and then this is $\frac{1}{2}I\omega^2$ and ω remember was $[\dot{R}]\ [R]^T$.

And we know what is R? It is Z-X-Y rotation matrix. So, the kinetic energy can be written in terms of $\left(\frac{1}{8}\right)mr^2(\dot{\theta}_2^2 + 2\dot{\theta}_1^2 + 2\dot{\theta}_3^2 + 4s_2\dot{\theta}_1\dot{\theta}_3 - c_2^2\dot{\theta}_1^2 + \frac{4}{r^2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2))$.

So, this actually is written such that it is nice compact form which will fit in the slide but you can see here this 4 and 8 will become half, this r^2 will go away. So, it is $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ which makes sense. The potential energy is $mgr\cos\theta_2$, so again if you see a little bit when it is vertical and it is tilting by θ_2, so then the height above this ground is $r\cos\theta_2$ and hence the potential energy is $mgr\cos\theta_2$. The Lagrangian can be obtained which is given by KE - PE.

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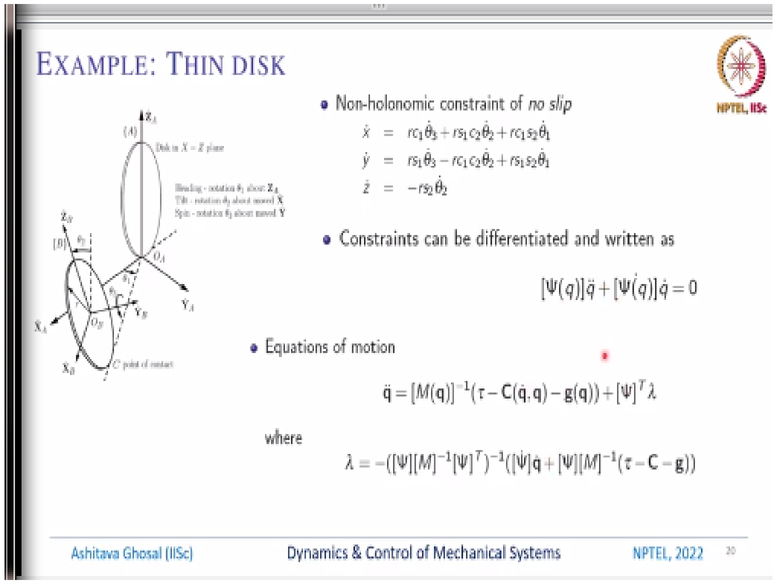

So, once the Lagrangian is obtained we can do the partial derivatives with respect to q *and* $\dot{q}$ and time derivative with respect to T and we can obtain the equations of motion. But we also have these non-holonomic constraints which are basically that there is no slip at the point of contact. So, as I said we obtained that by obtaining the velocity of the point of contact with $\dot{x}$, $\dot{y}$, $\dot{z}$ as the velocity of the origin + $\omega \times r$ and then we equate that to 0 and we will get these 3 expressions.

So, we can take the derivative of these non-holonomic constraints. So, first we need to write it in $\dot{\psi}\,\dot{q} = 0$and then we can take $\psi\ddot{q} + \dot{\psi}\,\dot{q} = 0$. So, basically we take the derivatives of these constraints. And then the equations of motion can be written as $\ddot{q}$ is some $M^{-1}\left(\tau - [C]\dot{q} - G\right) + [\psi]^T\lambda$ and where λ can be solved as this.

So, very standard way of deriving the equations of motion of this pure rolling of this thin disk subjected to these non-holonomic constraints of no slip. So, I am not going to write down all the terms because they are very big, they will not fit into the slide but later on I will show you what computer tools called Maple which can be used to obtain each one of these terms which go into the equations of motion.

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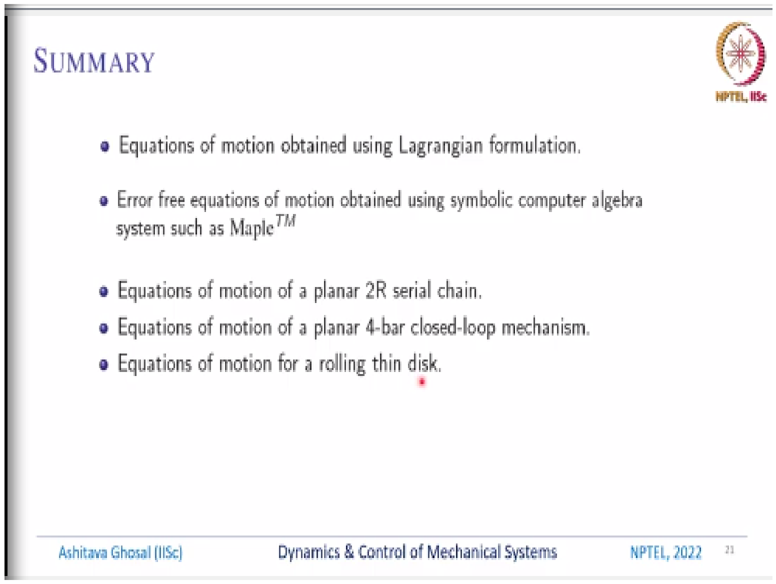

So, in summary I have shown you the equations of motion obtained using Lagrangian formulation. We can obtain these error free equations of motion using symbolic computer algebra systems such as Maple. Equation of motion for a planar 2R chain was derived, for a 4-bar mechanism was derived and also for the rolling of a thin disk was derived.