Dynamics and Control of Mechanical Systems

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Lecture-14

Lagrangian Formulation

In the last lecture we looked at the newton Euler formulation. The basic idea in the newton Euler formulation is propagation of velocities and accelerations from the base link to the free end and then we use the Newton's law and Euler's equation to compute $F = ma$ and $\tau = I\alpha + \omega \times I \times \omega$. And then we propagate these forces backwards to the base.

And to find the joint torque we take the z component of the moment. In this lecture we look at another way to derive the equations of motion. This is called the Lagrangian formulation.

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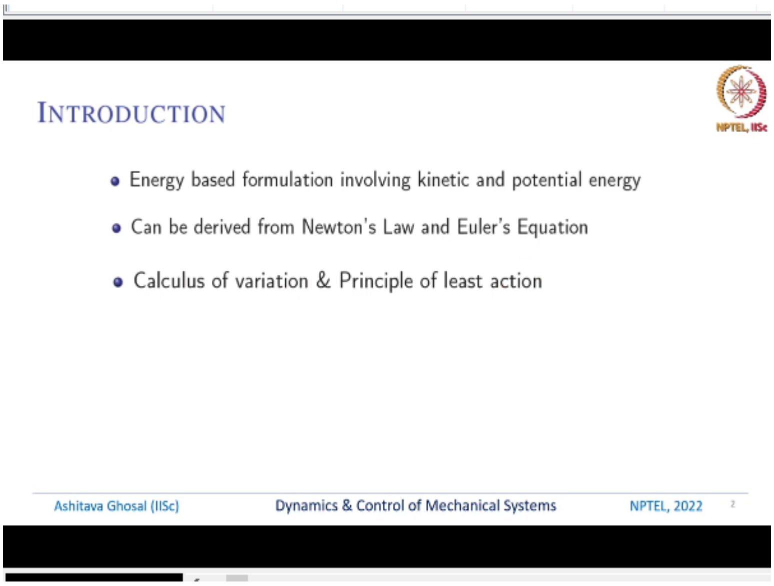

The Lagrangian formulation is an energy based formulation, it involves obtaining the kinetic and potential energy of the rigid body or a system of rigid bodies. We can derive the Lagrangian formulation from the Newton's law and Euler's equation. This involves advanced calculus of variation and something called as the principle of least action. We will not prove how to derive the Lagrangian formulation from Newton's law and Euler's equation.

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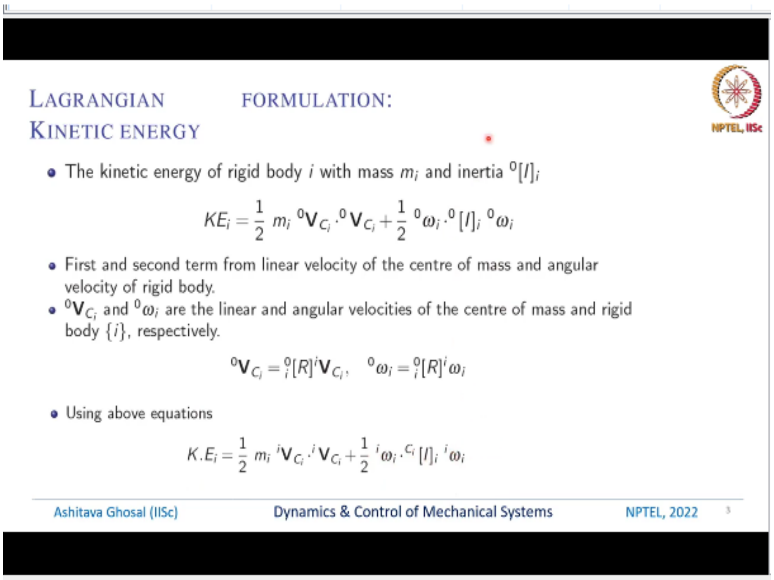

So, as I said the Lagrangian formulation involves evaluation of the kinetic energy and the potential energy. The kinetic energy of a rigid body i with mass m i and inertia I with respect to some fixed coordinate system can be written in this form .So, the first term is $\frac{1}{2} m_i \, 0V_{C_i} . \, 0V_{C_i}$; basically we find the velocity of the C g of any link of mass m_i and then $0V_{C_i} . \, 0V_{C_i}$. Likewise we find in the second term the $\frac{1}{2}$ $0\omega_i . \, 0[I]_i \, 0\omega_i$ of link I or rigid body I and then we multiply with inertia matrix.

So, $0\omega_i . \, 0[I]_i$ and then we take the dot product of this with $0\omega_i$. So, this is the expression for the kinetic energy of a rigid body i with mass m_i and inertia $0[I]_i$ with respect to $0V_{C_i}$ or a fixed coordinate system. The first and the second term from the linear velocity of the center of mass and angular velocity of the rigid body as I had explained. Again just to reiterate $0V_{C_i}$ and $0\omega_i$ are the linear and angular velocities of the center of mass and rigid body i respectively.

The $0V_{C_i}$ is nothing but the iV_{C_i} if you recall in the propagation equation we obtained iV_{C_i} but pre-multiplied by a rotation matrix. So, iV_{C_i} is the velocity of the center of mass described in its own coordinate system. Likewise the angular velocity of rigid body i in some fixed 0 coordinate system is also obtainable from $i\omega_i$ which was again if you recall was involved in the propagation equations.

So, you take $i\omega_i$ and pre-multiplied by $i0[R]$ and $i\omega_i$ was also nothing but the angular velocity of rigid body i described in its own coordinate system. So, using the above equations we can show that the kinetic energy of rigid body i instead of writing $0V_{C_i}$, we can write $iV_{C_i} . \, iV_{C_i}$ because this is basically a scalar which is the magnitude of the velocity.

This is also a scalar which is magnitude of the velocity of the center of mass. So, it does not really matter whether we are obtaining it from in the 0 coordinate system or in the ith coordinate system and the same story is instead of obtaining $\frac{1}{2}\ 0\omega_i .\ \ 0[I]_i\ \ 0\omega_i$, we can also obtain the kinetic energy as half $\frac{1}{2}\ 0\omega_i .\ \ C_i[I]_i\ \ i\omega_i$.

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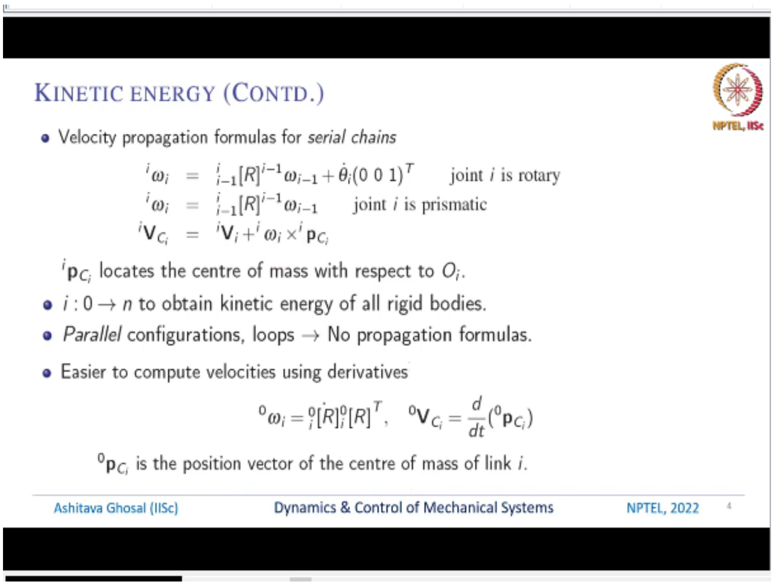

So, as I have mentioned we can use the velocity propagation formulas for serial chains. So, and recall that the angular velocity of the ith link written in its own coordinate system can be written in terms of the angular velocity of the i - 1th the previous link, will again written in its own coordinate system pre-multiplied by this rotation matrix and if it is a rotary joint then we have to add $\dot{\theta}_i$ along the z axis.

So, this is what the rotary joint is adding to the next link, the angular velocity of the next link is the angular velocity of the previous link plus what is happening at the rotary joint. If it is a prismatic joint then the angular velocity of the i - 1 at link is same as the angular velocity of the ith link. The velocity of the center of mass of the ith link can be obtained by the $iV_i +\ i\omega_i \times R,\ and\ here\ R\ means\ ip_{C_i}$.

So, this ip_{C_i} locates the center of mass with respect to the origin of the ith coordinate system or the origin of the ith rigid body. And we could do this from i = 0 through n and we can obtain the kinetic energy of all rigid bodies in a serial chain. So, all we need to do is $\frac{1}{2}\ m\ V^2_{C_i}\ \ + \frac{1}{2}\omega I \omega$. If you have parallel chains or configurations with loops then no such propagation formulas can be used.

So, in that case we go back to our basic definition of what is velocity which is basically in terms of derivatives of the position vector or derivatives of the rotation matrix and then we can obtain the angular velocity of any link in a parallel chain as $[\dot{R}]\ [R]^T$. This is 0 means it is with respect to the reference coordinate system or the fixed coordinate system.

Likewise we can obtain the velocity of the center of mass of any link in a parallel chain by taking the time derivative of the position vector of the center of mass and again this position vector of the centre of mass can be obtained by loop closure equations or by techniques of kinematics for parallel chains. So, $0p_{C_i}$ is the position vector of the center of mass of link i.

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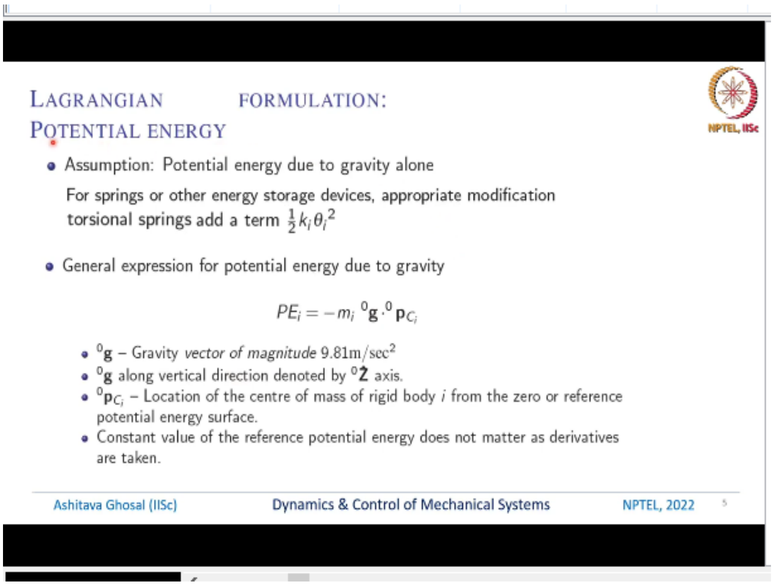

The other element in the Lagrangian formulation is the potential energy of a rigid body or the system of rigid bodies. So, when we want to compute the potential energy the main assumption is in this formulation. The potential energy is due to gravity alone, if you have springs or other energy storage devices we need to modify the potential energy term. So, for example if you have a spring which is a torsional spring then we have to add one term $\frac{1}{2}k_i\theta_i^2$, where k is the spring constant.

So, but most of the time or at least in this lecture we will assume the potential energy is due to gravity alone. The general expression for potential energy due to gravity is nothing but m g h. Now instead of writing simply $m\ g\ h$ for any link we write it in a mode formal form as dot product of the gravity vector with the vector locating the c g of the link .So, $0p_{C_i}$ locates the c g of link i, mi is the mass of the link i.

And then this gravity vector dot this will give you the potential energy, minus is because the gravity is normally acting in the downward direction. So, gravity is along vertical direction which is the z-axis but the gravity is in the opposite direction. The magnitude of this gravity vector is 9.81 meters per second square and as I said $0p_{C_i}$ is the location of the center of mass of rigid body i from the 0.

Or reference potential energy surface. In potential energy often this motion of a reference or a 0 potential energy surface is used to find the value of the potential energy. In this formulation we will see it really does not matter where you choose this 0 or potential energy surface because whatever it is it is a constant value and later on we will see in the Lagrangian formulation we have to take the derivatives of the potential energy.

So, derivatives of some constant quantity will go to 0 and so, we should not really be worried about what is the reference or the 0 potential energy surface you have chosen.

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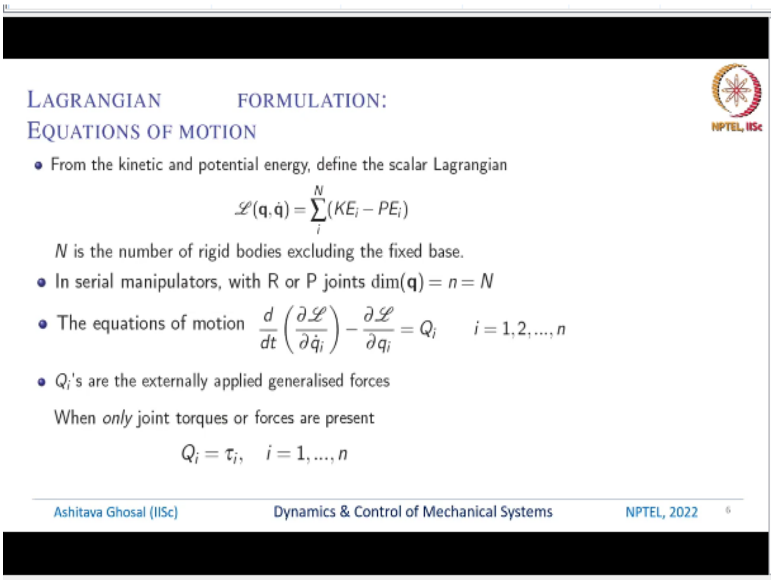

So, once we have the kinetic and potential energy we define a scalar quantity called the Lagrangian. So, this is the symbol for Lagrangian, this L, it will be a function of q and $\dot{q}$. So, $q's$ are the joint variables or any other generalized coordinates which you use to describe the serial or parallel chain, $\dot{q}$'s are the derivatives of the generalized coordinates and this Lagrangian is nothing but $KE - PE$.

So, for each link we find KE_i and potential energy of ith link. Then we subtract these two and then we sum over all the N links which are part of the rigid body chain. So, in this case this N does not include the fixed base. So, because the base is not moving. So, in serial manipulator with R or P joint the dimension of q is nothing but N which is the number of joints and this is also same as N in this example.

In kinematics if you recall this N was the number of links, but we have to add the fixed base. So, in the planar 3 degree of freedom robot we had 3 moving links but N was 4. Here we do not do that.

So, the equations of motion once you have this Lagrangian is given by $\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = Q_i$. So, let us go over this term by term.

So, this is $\frac{dL}{dq_i}$. So, the Lagrangian will be a function of both the generalized coordinates and also the derivatives of the generalized coordinates. Remember the potential energy will contain only some position vector. So, it will be a function of q_i, but the kinetic energy will be a function of $\dot{q}$s also. So, we also need to take the partial derivative of this Lagrangian with $\dot{q}_i$.

And then time derivative of the whole thing and this term in the left hand side is now equated to what is called the generalized forces. So, Q_i are the externally applied generalized forces and how many of these things do? We have to do we have to take i = 1 through n. So, where n in this case of serial robots are the numbers of moving links when only joint torques or forces are present at the links.

So, there is no other external force which is being applied. The Q_i will be nothing but the torque supplied by the motors at the joints.

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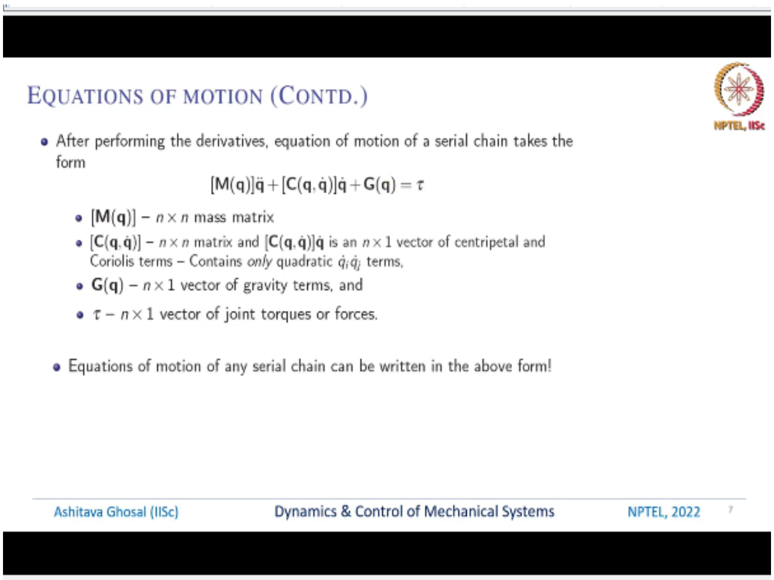

So, after performing the derivatives, the equation of motion of a serial chain takes this form. So, we find the Lagrangian we take the partial with respect to q_i then we take the partial with respect to $\dot{q}_i$ and then d /dt of that one and then if you do all these things for i = 1 through n and then simplify then we can rewrite the equations which you have got in this form. And this is a well-known form; we have seen this earlier also.

That we get in something called mass matrix $M(q)$. So, this mass matrix is only a function of the generalized coordinates q, it is an $n\times n$ matrix $\dot{q}$ is $n\times 1$ and then we have another term which is $C(q,\dot{q}) + G(q) = \tau$. $C(q,\dot{q})$, is also an $n\times n$ matrix and then this matrix into $\dot{q}$ is an $n\times 1$ vector of centripetal and coriolis terms. This coriolis term and centripetal term contains only quadratic terms.

So, you will have $\dot{q}_i \dot{q}_j$, you will not get $\dot{q}_i^3$ or $\dot{q}_i$ into $\dot{q}_2^2$. So, it is only quadratic terms. The right hand side is an $n\times 1$ vector of joint forces or torques. This term $G(q)$ what is called as the gravity term. This is only a function of the position of each link; it is not a function of the derivatives of the generalized coordinates. The equation of motion of any serial chain can be written in the above form.

This is a very, very important statement. If you give me any serial chain then it can be written in this form that there will be a mass matrix into $\ddot{q}$ then there will be a coriolis centripetal term, then there will be a gravity term and which is equated to the torques which are acting at the joints.

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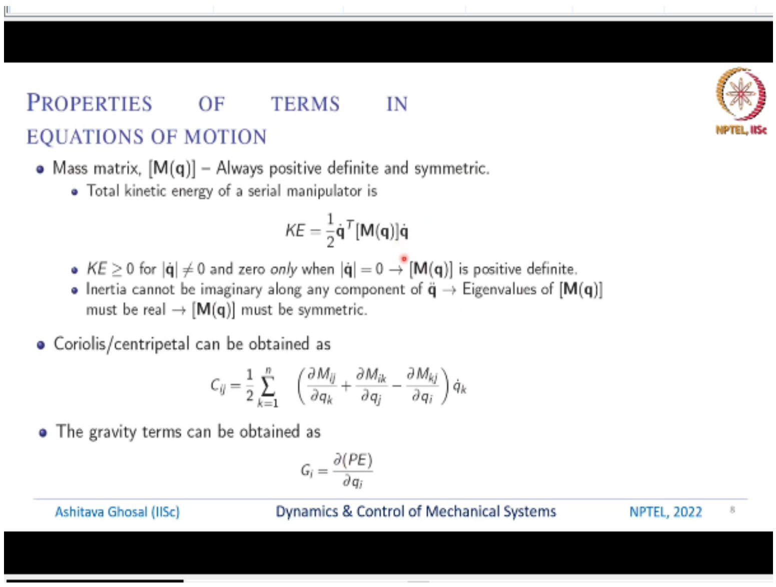

Let us look at some of the properties of the terms in the equations of motion. The most important property of this mass matrix is that it is always positive definite and symmetric and how do we prove that? You can show that the total kinetic energy of a serial chain or any serial robot is given by KE is $\frac{1}{2}\dot{q}^T [M(q)]\, \dot{q}$. So, this is a quadratic form, it is of the form $X^T A\, X$.

Kinetic energy is always greater than 0 for $\dot{q}$ not equal to 0 and 0 only when $\dot{q}$ is equal to 0, $M(q)$ positive definite. This implies that. Another way of looking at it is inertia cannot be imaginary along any component of $\ddot{q}$. So, basically Eigen values of mass matrix $M(q)$must be real and $M(q)$ must be symmetric. If it were not symmetric then the eigenvalues could be imaginary which it does not make sense.

It is also positive definite because the inertia can never be negative along in any component of $\ddot{q}$ or any direction. The coriolis and centripetal term can be obtained from this mass matrix which is M and we take $\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} + \frac{\partial M_{kj}}{\partial q_i}\right)\dot{q}_k$. So, we take all these elements of this mass matrix and take appropriate partial derivatives with respect to q's.

And then you $\frac{1}{2}\sum_{k=1}^{n}$, then before you sum it over you multiply by $\dot{q}_k$ and then you $\frac{1}{2}\sum_{k=1}^{n}$. So, where does this formula comes from? You can show that this is indeed true; this is sometimes in advance math it is called as a Christoffel symbol. So, it comes from that notion that you have something which is positive definite which is mass matrix.

So, it is related to some kind of a metric or a distance in some q space and then with that mass matrix or with that positive definite metric we can define something called Christoffel symbols and they are nothing but these components of this centripetal and coriolis terms. The gravity term can be obtained as partial of potential energy with respect to q_i. Remember gravity is only a function of the position, it is not a function of $\dot{q}$.

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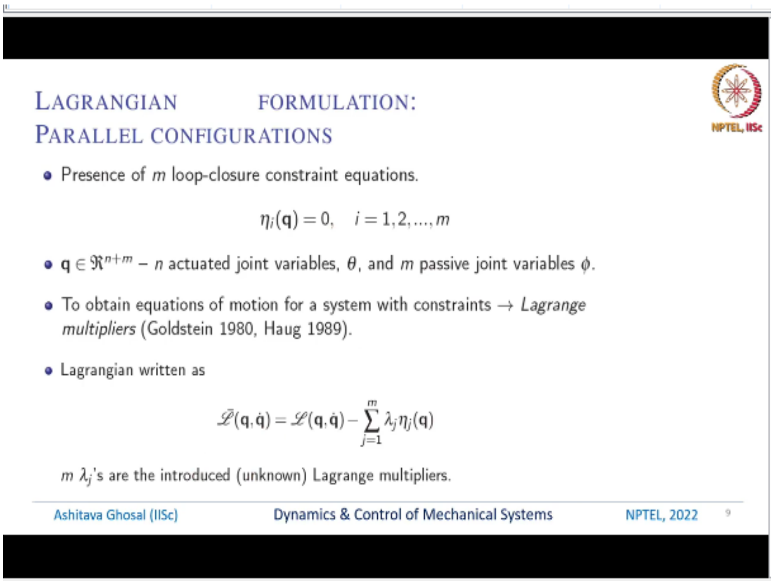

If you have loops as in parallel configurations then we can get this m loop-closure constraint equations. So, I have some loops, there are m such loops. So, our m loop closure constraint equations each loop can have more than one loop closure constraint equation. So, let us take the situation when in a parallel configuration we have m loop closure constraints equations. So, what are these?

These are some $\eta_i(q) = 0$ and $i = 1, 2,.. m$. So, in such a situation the dimension of the generalized coordinates is $n + m$. So, we have n actuated joint variables typically denoted by theta and m passive joint variables which we will denote by phi and to obtain the equations of motion for a system with constraints. So, basically we have kinetic energy, potential energy, but then we have these loop closure constraints for the generalized coordinates.

So, then we can use what is called as a Lagrange multiplier, this is very well known technique, it is available in many textbooks. So, for example in Goldstein or by Haug that we use this notion of Lagrange multipliers and then we can derive a new Lagrangian which is called L and notice this bar on top. So, this is a function of q and $\dot{q}$ which is same as the original Lagrangian which is kinetic minus potential energy minus this $\sum_{j=1}^{m}$.

So, there are m loop closure equations. So, n_j. So, this is like constrained into some λ_j . So, these are called as the Lagrange multipliers. So, those of you who have done any optimization course you can see that when you want to optimize an objective function with some constraints we introduce those Lagrange multipliers and this is the same idea. So, the Lagrangian we can introduce these Lagrange multipliers.

And with this loop closure constraints and we form a new Lagrangian and then we can go ahead and derive the equations of motion.

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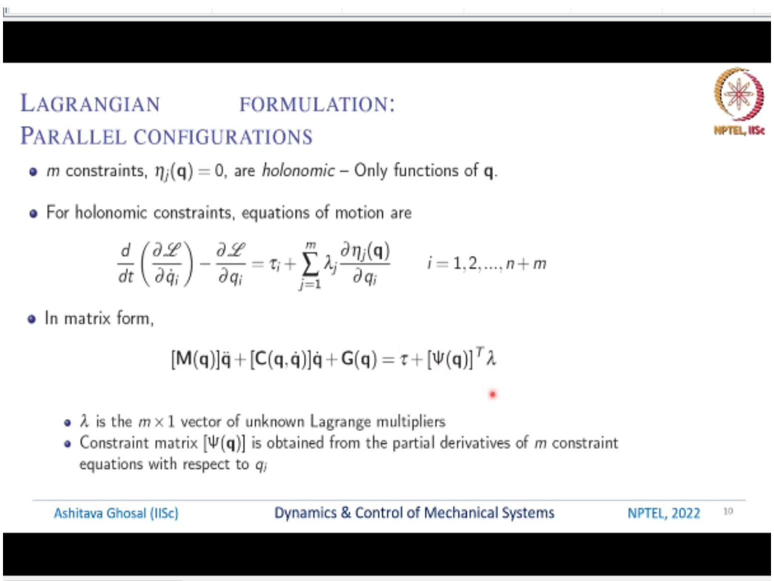

So, if these m constraints are holonomic which means only functions of q. Then the equations of motion are given in this form which is that you find the $\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = \tau_i + \sum_{j=1}^{m} \lambda_j \frac{\partial \eta_j(q)}{\partial q_i}$.

And now $i\,w = 1, 2, 3,..\ n + m$. So, remember there were theta actuated joint variables which we denote by theta and there are n of those and then there are phi's which are the passive joint variables which are not actuated and they were m of those. So, in matrix form these equations can be written in this form. Again, we have a $[M(q)]\ \ddot{q} + C\left(q, \dot{q}\right) + G(q)$.

This is same as the joint torques, but now we have another matrix which is $[\psi(q)]^T\ \lambda$. So, what is $\psi(q)$? $\psi(q)$ comes from here, partial derivatives of loop closure constraints with q_i. So, nj with qi we will get a matrix which is this psi and then this can be written as $[\psi(q)]^T\ \lambda$. So, in this equation we have λs which are the lagrange multiplier.

They are m of these Lagrange multipliers coming from the m constraint equations and then this constraint matrix psi of q_is obtained from the partial derivatives of the m constraint equations with respect to qi. So, if you do this and rewrite we can see that this could be written in this form.

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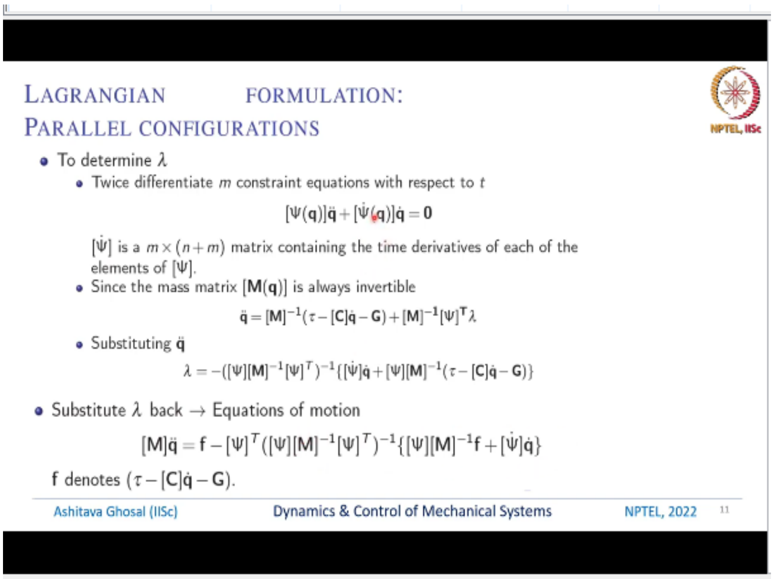

So, the λs are unknown as of now, we need to derive the λs in order to find the equations of motion. So, the λs have been introduced by us. So, we need to find what the λs are. So, to obtain the λs we twice differentiate the m constraint equations with respect to t. So, once if you differentiate $\eta_i(q) = 0$ you will get some$\psi(q)\ \dot{q} = 0$. Then if you differentiate that again then you will get $[\psi(q)]\ \ddot{q} + [\dot{\psi}(q)]\ \dot{q} = 0$.

Where $[\dot{\psi}(q)]$ is an $m\times(n + m)$ matrix containing the time derivatives of each of the elements of this constraint matrix $\psi(q)$. So, now since the mass matrix is always invertible. So, we can rewrite $\ddot{q} = M^{-1}\left(\tau - [C]\dot{q} - G\right) + M^{-1}[\psi]^T\lambda$. Where is this coming from? This is coming from the equations of motion.

The equations of motion where $[M]\ddot{q} = \left(\tau - [C]\dot{q} - G\right) + [\psi]^T\lambda$. So, we pre-multiplied both sides by M^{-1} and M^{-1} always exist, because the mass matrix is positive definite and symmetric and always invertible. So, once we have this $\ddot{q}$ we can substitute this $\ddot{q}$ back into this constraint equation which we have differentiated twice and then you can see that you will be left with only λ.

And if you do some simplification you will get λ in this complicated form which is some $-\left([\psi]M^{-1}[\psi]^T\right)^{-1}\{[\dot{\psi}(q)]\dot{q} + [\psi]M^{-1}\left(\tau - [C]\dot{q} - G\right)\}$ with the - sign and finally we can substitute this λ back into this equation and we can get equations of motion.

The form of the equation of motion is like this we have $[M]\ddot{q} = f - [\psi]^T\left([\psi]M^{-1}[\psi]^T\right)^{-1}\{[\psi]M^{-1}f + [\dot{\psi}(q)]\dot{q}\}$ quite complicated, but if you do and sit down and write it down you will get this because what is happening we can substitute λ here. So, you will get some $M^{-1}[\psi]^T$ this whole thing which is this minus term and so on.

It is not very hard and then we have this another term which is $\{[\psi]M^{-1}f + [\dot{\psi}(q)]\dot{q}$. So, $[\dot{\psi}(q)]\dot{q}$ is coming from here and then $[\psi]M^{-1}f$, this is f, f is this $\left(\tau - [C]\dot{q} - G\right)$. We already have some M^{-1}So, we will get this term $[\psi]M^{-1}f$ and where f denotes $\left(\tau - [C]\dot{q} - G\right)$. So, let us see what has happened. So, we have done lot of algebra and lot of simplification to arrive at an equation of motion which is of the form $[M]\ddot{q} = (\tau - [C]\dot{q} - G)$.

Plus this whole big term which comes from the constraint matrix. So, intuitively you can see that if we did not have loop closure equations, if it was a serial chain we did not have to bother about this Lagrange multipliers, we did not have to bother about this constraint matrix, then this part will not be there. So, you will be left with $[M]\ddot{q} = (\tau - [C]\dot{q} - G)$which is the same as what we obtained for the serial chain.

But because of the presence of loop closure constraint equations and constraint matrices psi we will get this complicated term, we will see some examples of this later on how we can use these loop closure constraint equations and obtain this constraint matrix and then obtain this equations of motion.

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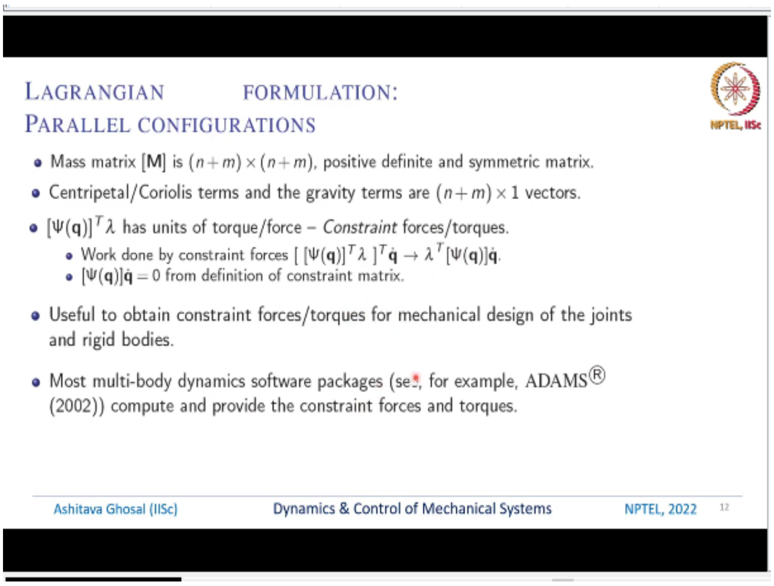

So, let us look at some of the properties of these equations of motion. First thing is that the mass matrix M is now $(n + m) \times (n + m)$. So, n at the actuated joints which is theta, m are the passive joints which comes from the m loop closure constraint equations, even then it is a positive definite and symmetric matrix. The centripetal coriolis terms and the gravity terms are $(n + m) \times 1$

In the case of serial robots or serial chains we had $n \times n$ mass matrix, the centripetal coriolis term was $n \times 1$, but now we have to take into account the m loop closure constraint equation. So, hence the dimensions get modified in both of these terms. The terms $[\psi(q)]^T \lambda$ as units of torque or force. So, these are called also constrained forces and torques.

Why have these units of torque? Because if you just go back one more slide you can see that it was $\tau + [\psi(q)]^T \lambda$. So, tau has units of torque or force. So, hence this will also be similar units. This constraint forces one of the interesting property is that the work done by the constraint forces is 0. So, how do I find out what is the work done by this constraint forces?

So, it is nothing but some force times velocity. So, we have $\left[[\psi(q)]^T \lambda \right]^T \dot{q}$ that is the work done by these constraint forces and that we can simplify. So, a into b whole transpose is b transpose a transpose. So, which is $\lambda^T [\psi(q)] \dot{q}$, but we know this $[\psi(q)] \dot{q}$ =0 because $[\psi(q)] \dot{q}$ comes from taking the first derivative of the loop closure constraint equation.

So, since if this is 0 then the work done whole thing is also 0. It is useful to obtain constrained forces and torques for mechanical design of joints and rigid bodies. So, although they do not appear to add to the work done or to the kinetic energy of the system but we still need to know what are these constraint forces. The whole idea is similar to what we had looked at in the newton Euler formulation.

Not only we need what is the torque due to the motor but we also need to know what are the reaction torques. In this Lagrange formulation we the same reaction torques and forces appear as constrained forces and torques not exactly the same numerically because in the newton Euler formulation the constraints forces were along x, y, z. So, remember f_x, f_y, f_z and n_x, n_y, n_z.

Here the constraint forces are along the generalized coordinate's q, but nevertheless the q's and the cartesian coordinates and orientation can be related. So, most multi-body dynamics software packages for example ADAMS, compute and provide the constrained forces and torques. So, this ADAMS is 2002, but there are now newer versions of ADAMS and we will see one such newer version later on. But the feature in ADAMS or any other multi-body dynamics software packages you can obtain the constraint forces and torques.

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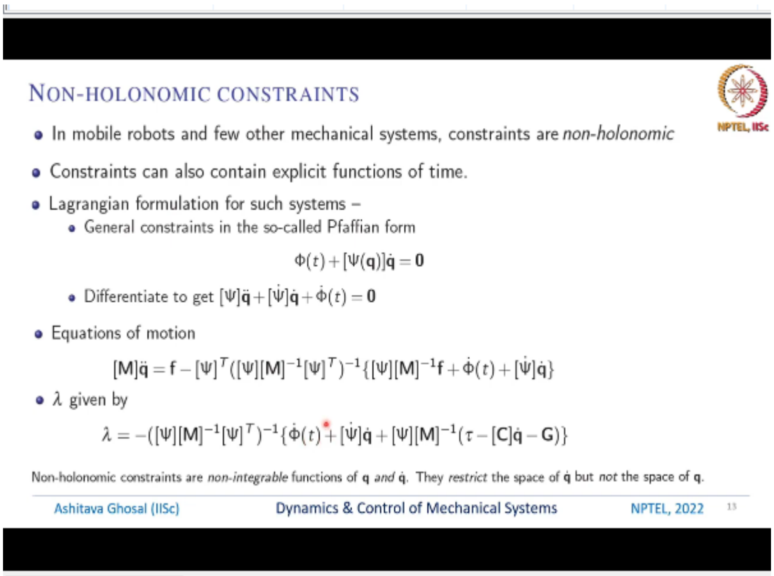

In mobile robots and few other mechanical systems, sometimes the constraints are non-holonomic. So, what do we mean by non-holonomic? They are not only functions of the generalized coordinates q, they could be function of $\dot{q}$ and we cannot get rid of the $\dot{q}$. So, they are non-integrable. So, in the example of that rolling disk thin disk I showed you that the constraints are in terms of the velocity at the point of contact being 0.

And hence the constraint contains both q and $\dot{q}$ and we could not get rid of the $\dot{q}$ by integration, sometimes the constraints can also be explicit functions of time. So, in such systems we can also use the Lagrangian formulation. So, the general constraints in what is called as the Pfaffian form is $\phi\,(t) = [\psi(q)]\,\dot{q} = 0$. So, $[\psi(q)]\,\dot{q}$ was similar to what we obtained by taking the derivatives of the loop closure equations.

So, sometimes the constraints could be directly in the form $[\psi(q)]\,\dot{q}$ and non-integrable and this term is showing the explicit function of time. So, we can have constraints which have as a explicit function of time and then involves q and $\dot{q}$. So, we can differentiate this constraint to get $[\psi]\ddot{q} + [\dot{\psi}]\,\dot{q}$ which is coming from this term using chain rule and $\dot{\phi}\,(t) \;=\; 0$.

Then the equation of motion can be modified and we can obtain the equation of motion if the constraints are given in this Pfaffian form. That it is given by $[M]\ddot{q} = f - [\psi]^T\,([\psi]M^{-1}[\psi]^T)^{-1}\{[\psi]M^{-1}f + [\dot{\psi}(q)]\,\dot{q}\}$again this whole complicated term but now we have terms which are $\dot{\phi}\,(t)$ and then this $[\psi(q)]\;\dot{q}$ was earlier also there.

And λ is also now can be obtained in to include the effect of $\phi\,(t)$. So, λ will now contain $\dot{\phi}\,(t)$ previously this term was not there. So, as I said these are non-holonomic constraints, they are non-integrable functions of q and $\dot{q}$, they restrict the space of $\dot{q}$ but not the space of q.

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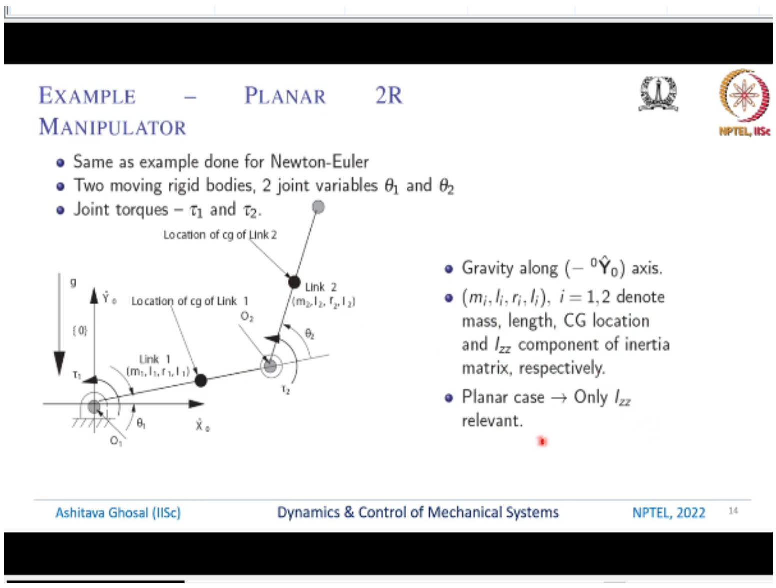

Let us now look at an example, we want to derive the equations of motion of a planar 2R manipulator. We have done this earlier, we have looked at these system two rigid bodies connected by 2 rotary joints, we used a long time back how to obtain using Euler's equation and Newton's law. Then we also derive these equations of motions for this system using the newton Euler formulation.

Now I want to show you that we can derive the equations of motion of this 2R system using the Lagrangian formulation. So, it is the same as the example done for Newton Euler formulation, we have two moving bodies, two joint variables θ_1 and θ_2, two joint torques τ_1 and τ_2 which are acting in this form counter clockwise positive. The first link has $m_1,\ I_1,\ r_1\ and\ l_1$.

So, m is the mass, I is the z component of the inertia, r locates the CG of this link 1 and l_1 is the length of this link and the same story with the second link. There is a gravity which is acting in the - y direction and m I, r these are denote the mass length CG location and the z component of the inertia matrix. This is a planar example. So, only I_{zz} is relevant.

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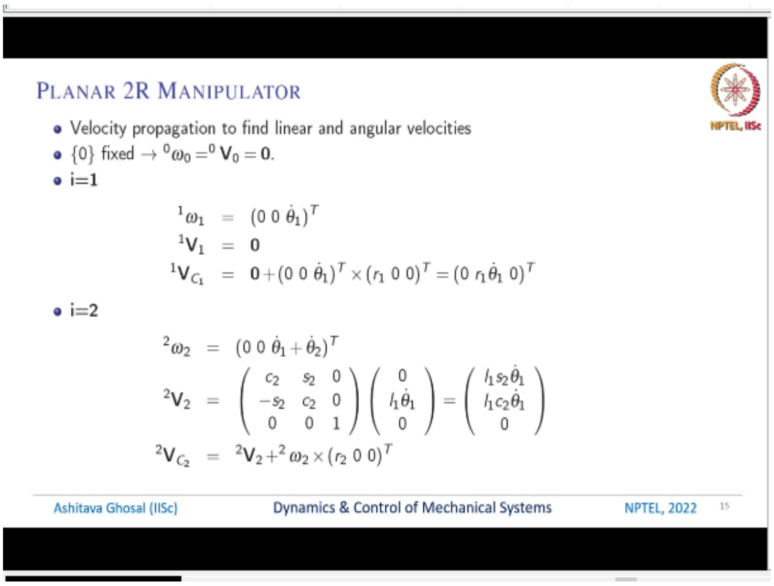

So, we can use the propagation formulas to find the linear and angular velocities. So, because it is valid for this serial chain also. So, with 0 as a fixed coordinate system and that is fixed. So, hence the angular velocity and the linear velocity is 0. So, for $i = 1$ we can find $1\omega_1$ which is 0 0 $\dot{\theta}_1$, $1V_1$ =0 and the linear velocity of the centre of mass is given by sum(0 r_1 $\dot{\theta}_1$ 0). So, this is quite obvious.

So, the first link is rotated mean θ_1. So, the CG is located at a distance r_1. So, that is $r\dot{\theta}_1$ and it will be along the y direction which is what is shown here. For $i = 2$ we can show that the angular velocity is $\dot{\theta}_1 + \dot{\theta}_2$ along the z axis. The velocity of the origin of the second link is given by $l_1 s_2\dot{\theta}_1$, $l_1 c_2 \dot{\theta}_1$, 0. This is nothing new, all of these were done when we did the Newton Euler formulation.

And the velocity of the center of mass is the $2V_2 + 2\omega_2 \times$ (r_2 0 0), we have assumed that the center of mass is along the link along the local x axis.

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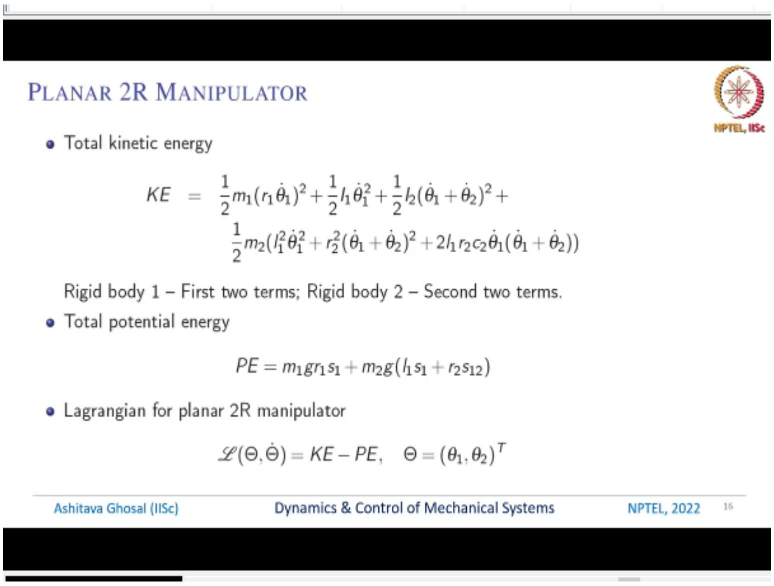

So, the total kinetic energy is nothing but $\frac{1}{2} m V^2$. So, one term from V^2 will give this. $I\_1 \dot{\theta}_1^2$, then $I_2(\dot{\theta}_1 + \dot{\theta}_2^2)$. So, this is the kinetic energy due to the rotation of the first link. This is the kinetic energy due to the rotation of the second link and then we also have this kinetic energy of the second link $V_{C_2}^2$. So, m_2 into velocity of the center of mass of the second link square of that.

And if you simplify this and we have done this earlier you will get one term which is $l_1^2 \dot{\theta}_1^2$, there is also $r_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2\, l_1\, r_2\, c_2\, \dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)$. So, we have done this earlier this is nothing new and it is very simple conceptually. We are doing the kinetic energy of the first link plus the kinetic energy of the second link.

And we know what is the angular velocity of first link and we know what is the angular velocity of the second link and similarly what is happening to the linear velocity of the center of mass of first link and the linear velocity of the center of mass of the second link. So, for rigid body 1 we have these 2 terms. For rigid body 2 we have these 2 terms the second, third and the fourth term.

The total potential energy is nothing but $m\, g\, h$ of the first link and again if you go back and see what is the height from the reference which is the x axis, it is r_1 into sin $sin\, \theta_1$. So, we have $m_1\, g\, r_1$ sin $sin\, \theta_1$. The second link the distance from again the horizontal is $l_1\, s_1 + r_2\, s_{12}$. So, the Lagrangian for the planar 2R manipulated is given by kinetic minus potential energy. In this case I am using theta to denote the generalized coordinates.

The generalized coordinates are θ_1 and θ_2. So, this Lagrangian is a function of θ_1, θ_2 and $\dot{\theta}_1$ $\dot{\theta}_2$. So, you can see $\dot{\theta}_1$, $\dot{\theta}_2$. Similarly there is a θ_1 here, this is sin sin θ_1 , this is θ_1 + θ_2).

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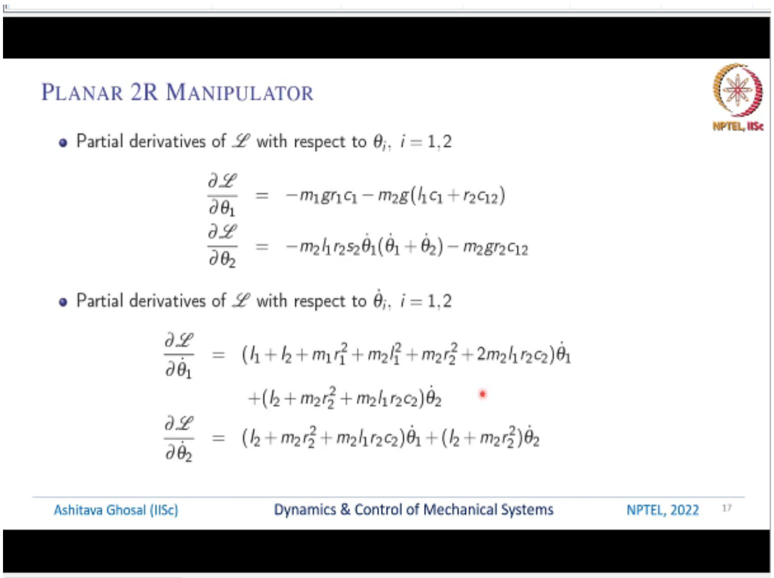

So, the partial derivatives of this Lagrangian with respect to theta i can be very easily obtained, you take the partial derivatives and you will get one term which is - $m_1 g\, r_1 c_1 - m_2 g\,(l_1 c_1 + r_2 c_{12})$. The partial derivatives with respect to θ_2 will be a little bit more complicated. We will of course have this gravity part which is $m_2\, g\, r_2\, c_{12}$, but we also have terms which are coming from c_2 into $\dot{\theta}_1$ and then ($\dot{\theta}_1$ + $\dot{\theta}_2$).

So, in the kinetic energy there was a term which contained c_2 into $\dot{\theta}_1$ whole multiplying $\dot{\theta}_1$ + $\dot{\theta}_2$. So, when you take the partial derivatives of cosine you will end up with – s_2. So, this term will also come. The partial derivative of this Lagrangian with respect to $\dot{\theta}_1$ and $\dot{\theta}_2$ can also be obtained. So, the partial with respect to $\dot{\theta}_1$ will contain $I_1 + I_2 + m_1\, r_1^2$ and so on.

And then into $\dot{\theta}_1$, there will be a term as I said which contains $m_2\, l_1\, r_2\, c_2$ and then of course there is this other term when you take partial with respect to $\dot{\theta}_1$ which is $I_2 + m_2\, r_2^2$ and so on into $\dot{\theta}_2$. The partial of Lagrangian with respect to $\dot{\theta}_2$ is little bit simpler, but nevertheless we have a term which is and then of course($l_2 + m_2\, r_2^2 + m_2\, l_1\, r_2\, c_2$) $\dot{\theta}_1$.

And($I_2 + m_2 r_2^2$) $\dot{\theta}_2$. So, this is very, very mechanical. So, once I know what is the linear velocity of the center of mass of each link and the angular velocity of each link I can find the kinetic energy and then if I know where is the center of mass with respect to a reference. So, I know the height. So, I can find out like mgh and I know the potential energy then I combine both of them as $KE - PE$ which is the Lagrangian.

And then mechanically take these 4 partial derivatives one with respect to θ_1, second with respect to θ_2 then with respect to $\dot{\theta}_1$ and then with respect to $\dot{\theta}_2$ and then we have to take the time derivatives of these two quantities which is what the Lagrangian formulation says you have to do.

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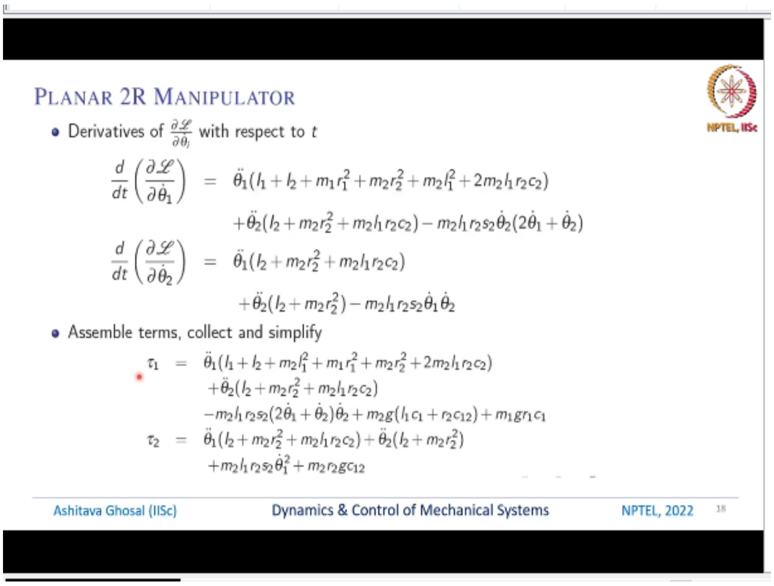

So, we take the time derivative of this Lagrangian partial Lagrangian with respect to $\dot{\theta}_1$ and when you take the time derivatives you will get some terms which are containing $\ddot{\theta}_1$, but then we also have time derivatives of c_2. So, you will again get some $\dot{\theta}_2$ into sine θ_2. So, which is what these terms are. So, the time derivatives of $\ddot{\theta}_1$ is this plus into this $\ddot{\theta}_2$ into this and then we have some $m_2 l_1 r_2 s_2 \dot{\theta}_2$ (2 $\dot{\theta}_1 + \dot{\theta}_2$).

So, there is some terms which are like $\ddot{\theta}_1$ into something and $\ddot{\theta}_2$ into something, but then you will get some terms which is 2 $\dot{\theta}_1 \dot{\theta}_2$ or $\dot{\theta}_2^2$. The time derivative of the second part which is Lagrangian with respect to $\dot{\theta}_2$, partial Lagrangian with respect to $\dot{\theta}_2$. Again you will see some terms which are $\ddot{\theta}_1$ into something.

$\ddot{\theta}_2$ into something, but then you will also have a term which is $\dot{\theta}_1 \dot{\theta}_2$. So, then we collect this term, this term and that the partial of Lagrangian with respect to θ_1, partial of Lagrangian with respect to θ_2, collect all of these terms according to the recipe given in the Lagrangian formulation and simplify. And once you do all that I have skipping a few steps.

You can show that the equation of motion is τ_1 is $\ddot{\theta}_1$ into something + $\ddot{\theta}_2$ into something and then you will get this $m_2 l_1 r_2 s_2 \dot{\theta}_2^2$ + 2 $\dot{\theta}_1 \dot{\theta}_2$ and then we have this gravity terms which are coming from the partial of the potential energy with respect to θ_1 and θ_2. So, we will get one term here gravity, another term here.

And for the second equation again we will have $\ddot{\theta}_1$ into something + $\ddot{\theta}_2$ into something and then $\dot{\theta}_1^2$ and $m_2 r_2 g c_{12}$. So, if you go back and open your notes and see what we did for the Newton Euler formulation equations are exactly same. So, there is absolutely no difference between the equations of motion which you obtained from Newton Euler or from this Lagrangian.

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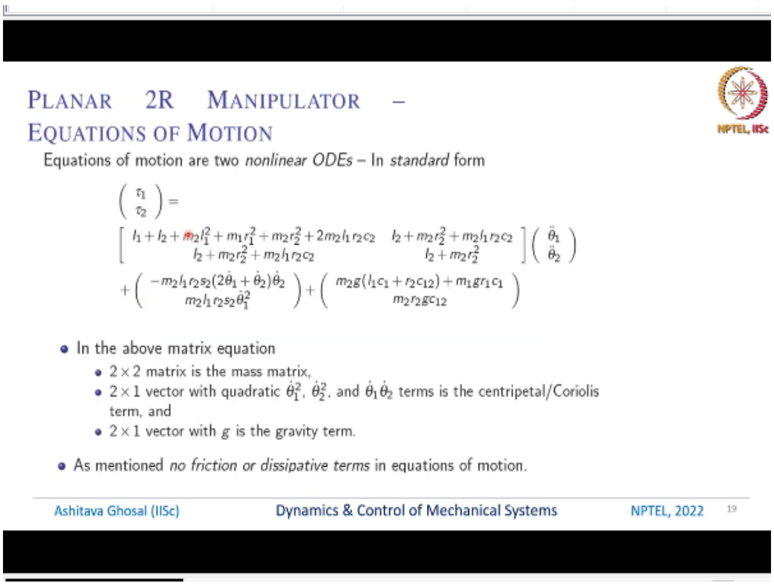

So, if you rewrite in the standard form you will get τ_1 τ_2 which is a vectorr_2 crosss_1 vector is some matrix into $\ddot{\theta}_1$ into $\ddot{\theta}_2$ plus this coriolis centripetal term plus this gravity term. So, in this equation this 2 by 2 matrix is called the mass matrix as before this 2 by 1 vector which is quadratic in $\dot{\theta}_1^2$,$\dot{\theta}_2^2$ and $\dot{\theta}_1 \dot{\theta}_2$ is called the centripetal and coriolis term.

And the last term which is a function of only θ_1 and θ_2 is called the gravity term. The Lagrangian formulation does not include any friction or dissipative term. The Lagrangian is for only conservative system, historically Lagrange was a mathematician in the 1700s and he was studying the motion of the planets. So, hence he derived a way to derive the equations of motion but there is no question of adding friction. So, because this Lagrangian formulation inherently is only for conservative systems.

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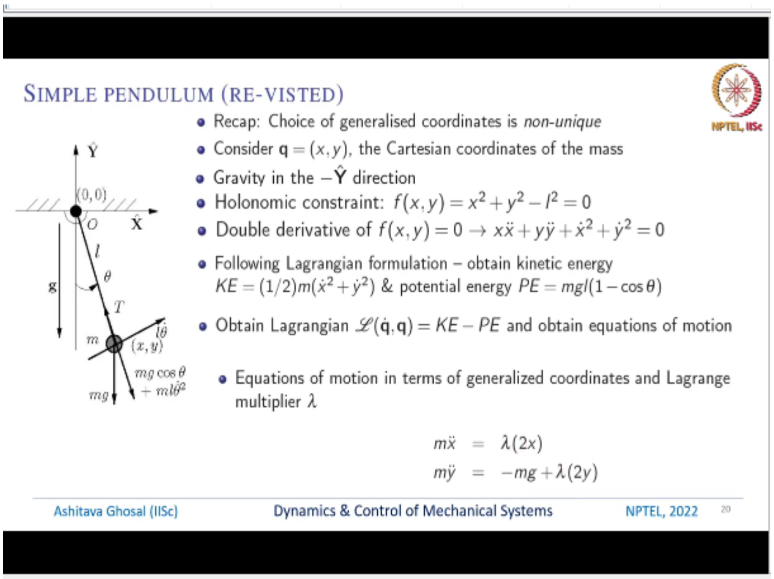

So, let us take some example. So, we have this very well-known example of a simple pendulum, we have looked at this earlier. So, we have a gravity vector like this there is an x and y axis, this is the origin and this is this link and a mass with end point x, y. We will see what this $l\dot{\theta}$ and other things come and we have this gravity which is mg. So, we recap that the choice of generalized coordinate is non-unique.

So, we could consider q which is the generalized coordinates as x, y which is the cartesian coordinates of the mass of the bob, the gravity is in the - Y direction. But if you choose x, y as your generalized coordinates then inherently there is a constraint which is what that the distance of this point x, y from the origin is governed by $x^2 + y^2 - l^2 = 0$. L is the length, this length here.

If you take the double derivative of this function $f(x, y) = 0$, you will get $x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0$, it is not at all very hard. So, the first derivative will be $2x\dot{x} + 2y\dot{y} = 0$. If you take the derivative again you will get this again using chain rule, the kinetic energy of this mass is nothing but $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$.

So, this is a massless rod of length l. So, the kinetic energy is only $\frac{1}{2}mV^2$, the potential energy is $mgl(1 - \cos\cos\theta)$. So, we take some reference here. So, this is $mgl(1 - \cos\cos\theta)$

whatever is the height above that reference. The Lagrangian is nothing but kinetic minus potential energy and just by following the Lagrangian formulation we can obtain the equations of motion of this simple pendulum.

And what are the equations of motion, sort of obvious we have $m\ddot{x} = \lambda\,(2x)$ and $m\ddot{y} =- mg + \lambda\,(2y)$. Remember there are no forces or any torques which are acting the only external force is due to gravity. So, hence and this is acting along the y direction. So, we will have some component of this external force gravity in the y direction and this $\lambda\,(2x)$ and $\lambda\,(2y)$ comes from the derivative of this.

So, as I said you will get $2\,x\dot{x} + 2\,y\dot{y}$. So, that $2\,x\ and\ 2\,y$ we need to multiply by λ because remember the constraints are $[\psi(q)]^T\,\lambda$. So, we will get these two terms.

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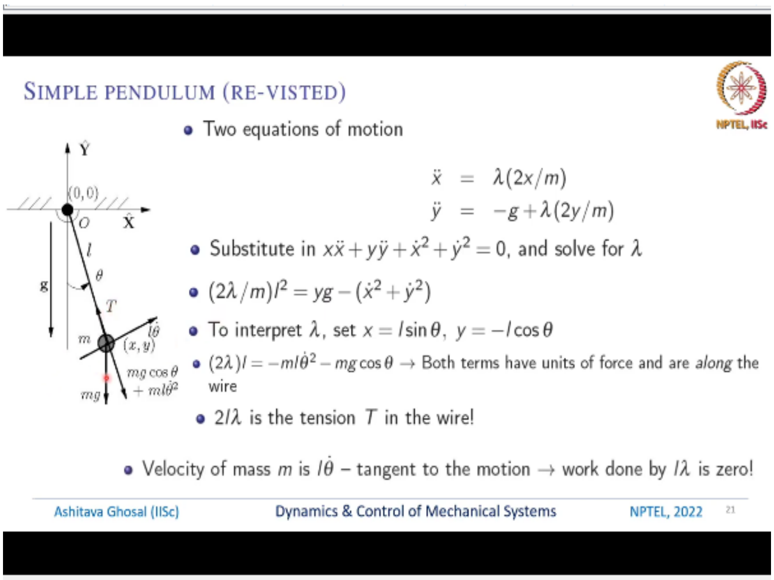

So, let us continue. So, I have two equations of motion one of them is $\ddot{x} = \lambda\,(2x/m)$ and the other one is $\ddot{y} =- g + \lambda\,(2y/m)$. So, let us substitute this $\ddot{x}$ and $\ddot{y}$ into this and we solve for λ . Basically similar to what I showed you what we do when we have loop closure constraints. In this case the constraints are double $x^2 + y^2 - l^2 = 0$ and this is the second derivative of the constraints.

So, if you substitute $\lambda\,(\frac{2x}{m})$ and $-\ g$ and so on into this expression, we will get $(2\,\lambda/m)l^2 = y\,g - (\dot{x}^2 + \dot{y}^2)$. So, this is what you will get. So, to interpret λ what does this λ mean in this problem or in this case of this simple pendulum. So, we can see here that we can also define this x and y by using this l length θ.

So, we can set x is some $l \sin sin\,\theta$ this is x coordinate which is this perpendicular distance that is $l \sin sin\,\theta$ and then the y component which is in the negative direction is - $l \cos cos\,\theta$. So, we can set these two by introducing a variable θ which is from the vertical and this length l. So, if you substitute x and y and then $\dot{x}$ and $\dot{y}$ and everything into this expression here.

You will get (2 λ) l= - $ml\,\dot{\theta}^2 - mg \cos cos\,\theta$. So, $\left(\dot{x}^2 + \dot{y}^2\right)$ will be what? It will be something like. So, $l^2\,\dot{\theta}^2$ and then if you simplify and then if you do some simple algebra you will get this expression which is (2 λ) l= - $ml\,\dot{\theta}^2 - mg \cos cos\,\theta$. So, this is what is shown in this figure here.

So, we have this is mg, the direction of motion of this mass is $l\dot{\theta}$. So, this is like the tangential velocity, this is the velocity of the mass instantaneously. Due to this velocity we have some $ml\,\dot{\theta}^2$ which is acting outwards and also $mg \cos cos\,\theta$ which is acting this. So, those of you who remember your basic mechanics you can see that this $mg \cos cos\,\theta$ and $ml\,\dot{\theta}^2$ is nothing but the tension in this rope or wire.

So, both of these terms have units of force and they are along the wire that is important. So, hence this 2 l λ is the tension T in the wire. So, in Newton Euler formulation we could have found this tension by seeing that there is an mg here, there is an $mg \cos cos\,\theta$ here and then there is something else and then there are all these $mg \sin sin\,\theta$ and so on. Here it is a little bit more complicated because we need to find what is the Lagrange multiplier.

And then this Lagrange multiplier is related to this tension in the wire. So, this is like a constraint force and again remember tension in the wire is an internal force it is not doing any work it is a constraint force but we can still find out what is this constraint force. So, the purpose of this example is that I could have chosen a different generalized coordinates in this case x, y for this pendulum.

And then we have this constraint which is $x^2 + y^2 = l^2$ and then due to that constraint just by following the Lagrangian formulation we can find an expression for the Lagrange multiplier and that Lagrange multiplier is related to the constraint forces. In this case there is a nice interpretation of what exactly is the constraint force; it is nothing but the tension in the wire.

So, the velocity of the mass is in this direction $l\,\dot{\theta}$, it is tangent to the motion and hence the work done by this constraint tension in the wire is also 0 because these two are always perpendicular to each other. Again everything is consistent with whatever I showed you mathematically earlier. I showed you that $[\psi(q)]^T$ λ that part the work done by $[\psi(q)]^T$ λ was 0.

Here also we know for a simple pendulum we have studied this in many places that there is something called the tension in the wire and that is perpendicular to the direction of motion and hence the work done by this force is 0.

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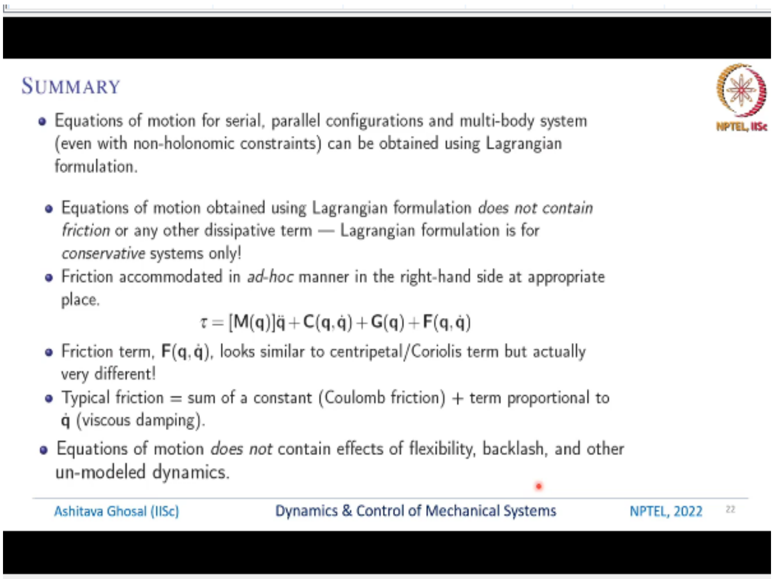

So, in summary the equations of motion for a serial or a parallel configuration and multi-body system even with non-holonomic constraints can be obtained using the Lagrangian formulation. The equations of motion obtained using the gradient formulation does not contain friction or any other dissipative term. The Lagrangian formulation is for conservative systems only.

However in any mechanical systems we have friction and how do you accommodate that we just add it in an ad-hoc manner. So, originally we have tau is $[M(q)]\ \ddot{q} + C\left(q, \dot{q}\right) + \ G(q)$, we just add a friction term. So, this friction could be a function of q and $\dot{q}$. So, if you just write it like this then you can think that this centripetal coriolis term and this friction term looks more or less similar but that is not true.

This is quadratic, this friction could be quadratic but it is very different from the centripetal coriolis term, this is a dissipative term. Typically this friction term is some constant times some function of $\dot{q}$, most of the time linear function of $\dot{q}$. So, this friction for τ_1 will be($c_1 + c_2$) $\dot{q}$ or $\dot{\theta}_1$. So, one part which is called Coulomb friction, one part which is called viscous damping.

This is a typical model for friction and note that this friction, this is quadratic this is typically not quadratic. The other disadvantage of Lagrangian formulation is that this equation of motion does not contain the effect of flexibility, backlash and other un-modeled dynamics. So, people are worried about what is happening to the energy which is going into the flexibility.

So, if there is a deformation. So, something called strain energy might be happening, we can have backlash between the gears which are there in the motors. So, we cannot model all these effects in Lagrangian formulation.