Dynamics and Control of Mechanical Systems
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Lecture-13
Newton-Euler Formulation for Serial Chains

In the last lecture we use the Newton's law and Euler's equation to derive the equations of motion of a two link robot or a 2R planar manipulator or also a double pendulum. In this lecture we take a more formal look at so, called newton Euler formulation for serial chains. So, we will describe an algorithm which uses the Newton's laws and Euler's equation and we can get the equations of motion for any serial chain.

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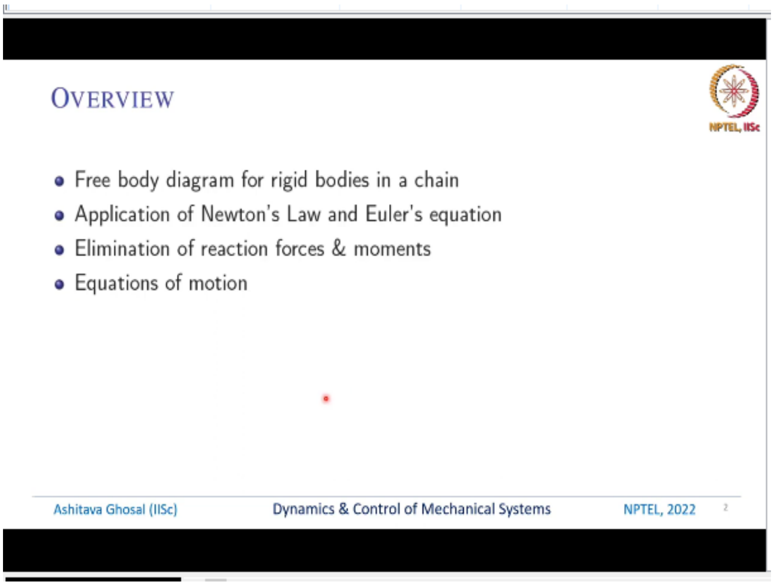

So, in overview of this Newton Euler formulation we will look at the free body diagram for rigid bodies in a chain, we will apply Newton's law and Euler's equation and then we will see how we can eliminate the reaction forces and moments and then finally obtain the equation of motion.

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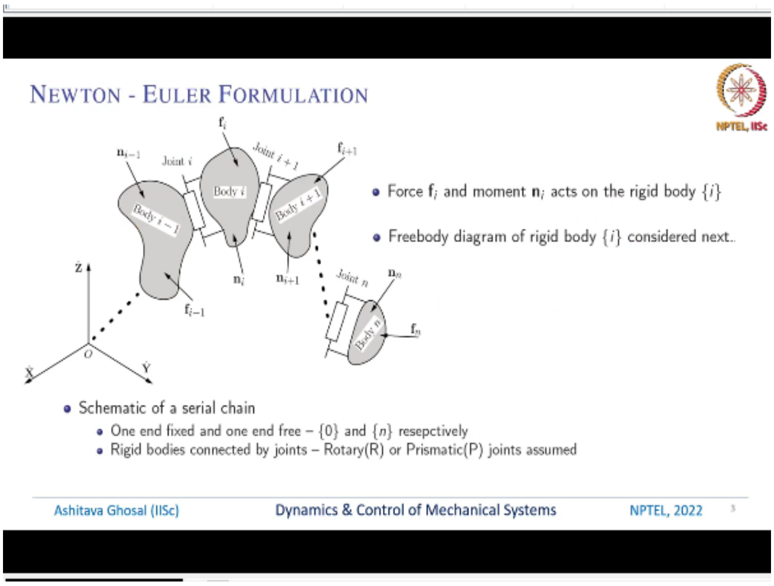

So, let us look at where Newton-Euler formulation can be applied. So, this is the schematic of a serial chain. So, we have rigid body i – 1, rigid body i, rigid body i + 1 and all the way to which is the rigid body end n. So, this is the last rigid body in this chain which starts from rigid body 0 which is fixed to x, y and z. So, this is the reference coordinate system. The first rigid body is fixed to this fixed base or the fixed reference coordinate system.

In between any two rigid bodies we have a joint. So, between i - 1 and rigid body i you have a joint i. Likewise there is a joint i + 1 between rigid body i and i + 1 and we also assume that this is rigid body i - 1 there is an external force f_{i-1} which is acting on this. Similarly there can be a external moment n_{i-1}which is acting on this rigid body i - 1. So, these are 3 by 1 vectors.

Likewise for rigid body i we have a moment n_i and a force f_i which is acting. So, in this schematic of a serial chain one end is fixed, one in this free. As I mentioned the fixed end is with this x, y, z coordinate system, the last rigid body is the n^{th} rigid body it is shown by this curly bracket n and these rigid bodies are connected by joints. So, in the Newton-Euler formulation we will assume that the joints are rotary or prismatic joints.

So, rotary allows rotation between 2 connected rigid bodies and prismatic joint allows relative translation between the 2 connected rigid bodies. The force f_i and moment n_i acts on the rigid body i as I have mentioned and we will look at the free body diagram of rigid body i.

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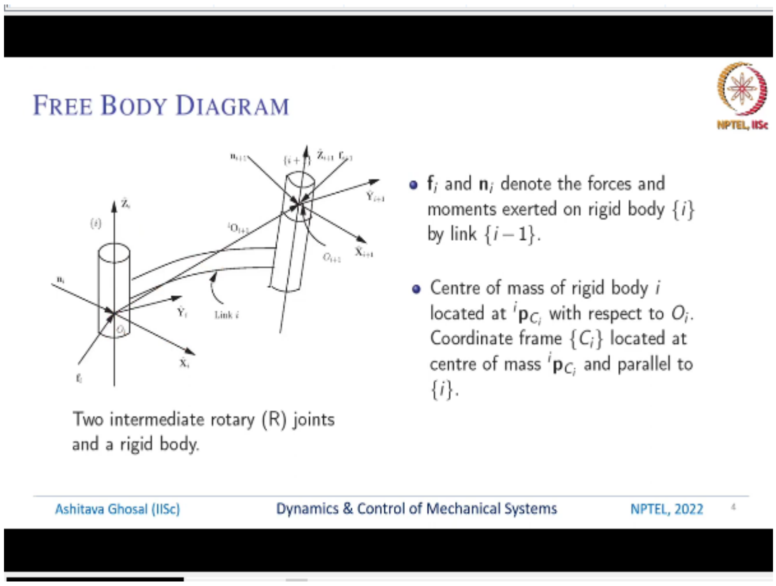

So, in this example schematic I am showing you one rigid body which is link i, one end there is one joint and the other end there is another joint. So, between 2 rigid bodies that is joined or at the 2 ends of a single rigid body there are 2 joints both mean the same thing and we will assume that the rotary joint is here or the translatory joint is on this joint and then this $\hat{Z}_i$ axis is along the joint axis.

So, $\hat{X}_i$ is one coordinates axis $\hat{Y}_i$ is the other and $\hat{Z}_i$ is along the joint axis and we have this force f_i and n_i which is acting at this O_i the origin of the ith coordinate system. Likewise we have a joint axis $\hat{Z}_{i+1}$, this is the rigid body i + 1 is after this joint and then we also have forces f_{i+1} and moment n_{i+1} which is acting at the origin of the i + 1 coordinate system.

So, this is O_{i+1}, the origin of the i + 1 coordinate system is located with respect to the ith coordinate system by this vector O_{i+1}. So, this is a vector from one origin to the next origin and again we have one x axis, y axis and a z axis for the rigid body i + 1. So, in this example we are showing the 2 intermediary rotary joints. These joints could have been also translatory joints and as I mentioned f_i and n_i denote the forces and moments exerted on rigid body i by link i - 1.

So, the i - 1 link is before this joint. The center of mass of rigid body i is located with some vector ip_{C_i} with respect to O_i. So, some point here, a coordinate system is attached to the center of mass which is denoted by C_i, it is located at the center of mass and it is parallel to $\hat{X}_i$ $\hat{Y}_i$ $\hat{Z}_i$. So, basically we take this coordinate system $\hat{X}_i$ $\hat{Y}_i$ $\hat{Z}_i$ with Y and shift it in a parallel way to the center of mass.

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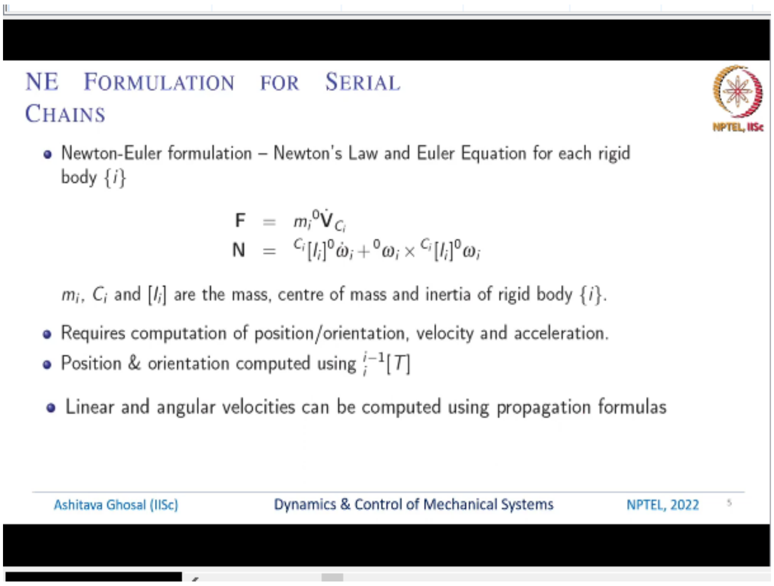

So, the Newton-Euler formulation goes like this we apply Newton's law and Euler 's equation for each rigid body i and the Newton's law is given by force is mass times acceleration .So, m i is the mass of the i th rigid body, F is the external force and $0\dot{V}_{C_i}$ is the acceleration of the center of mass of the ith rigid body with respect to the zero coordinate system or the reference coordinate system.

Likewise the external moment n can be written as $C_i[I_i]$ $0\dot{\omega}_i$ + $0\omega_i \times C_i[I_i]$ $0\omega_i$ and in both of these you can see that the acceleration and the angular velocity they are with respect to the zero coordinate system, but the fixed reference coordinate system and the inertia is with respect to the center of mass which is what the C_i is showing with respect to a coordinate system which is also fixed at the center of mass.

So, in order to use these right equations or we want to substitute what is acceleration and angular velocity and angular acceleration we have to start from position and orientation then we have to go to velocity of each link and then we have to go to acceleration of each link. To obtain the position and orientation of any link with respect to the zero or the fixed reference coordinate system we go back and use this 4 by 4 homogeneous transformation matrix.

We had discussed this. So, this homogeneous transformation matrix the top 3 by 3 contains the orientation of rigid body i with respect to i - 1 and the last column the 3 by 1 vector contains the position of the coordinate system attached to rigid body i with respect to the rigid body i - 1. So, this we can obtain. The linear and angular velocity can be computed using propagation formula. This was also discussed last week.

So, basically what we have is we start with the fixed base which we know zero velocity and zero angular velocity and then we go to the next link and if there is a rotary joint then $\theta\dot{\theta}$ is added if it is a prismatic joint then the linear velocity is changed with $\dot{d}_i$ So, we had derived this propagation formulas starting from a fixed base we could find the linear and angular velocity of each link as we go along the chain. So, we will use those propagation formulas.

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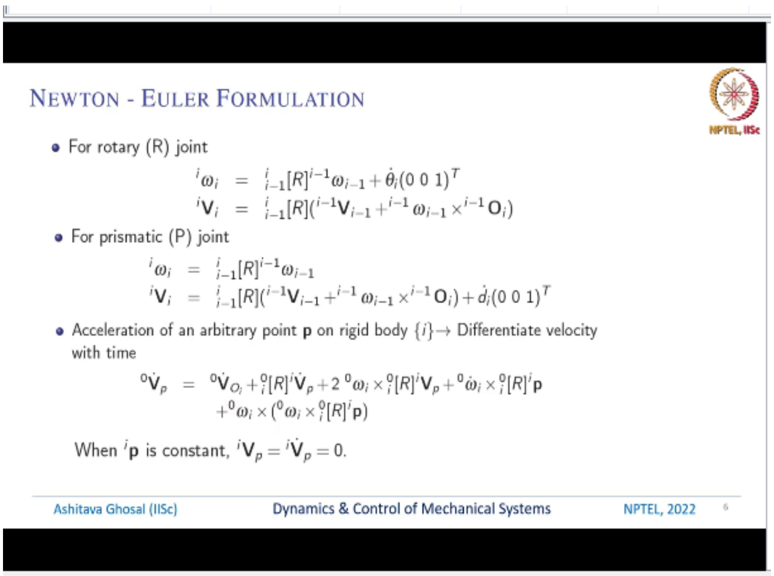

So, just to recapitulate for a rotary joint the propagation formula was the following. So, if you have the angular velocity of i - 1 rigid body with respect to its own coordinate system then we can find the angular velocity of the ith rigid body the next rigid body in its own coordinate system by this formula. It tells you that we have to pre-multiply this angular velocity with the rotation matrix i to i – 1.

And then we have to add this angular velocity at the rotary joint which is $\dot{\theta}$ (0 0 1). Remember the joint axis is along the z axis. So, hence it is $\dot{\theta}$ (0 0 1). The linear velocity of the origin of the ith coordinate system or the linear velocity of the point in the rigid body i at the origin of the coordinate system is again the linear velocity of the i - 1 rigid body plus some $i - 1\omega_{i-1}$ ×R.

So, $i - 1O_i$ locates the origin of the next rigid body with respect to the the origin of the coordinate system attached to the previous rigid body and this is the angular velocity. And again in order the propagation formula we pre-multiplied by some rotation matrix. So, it looks like this iterative form. So, what we will do is we will put some i equals let us say 1 which will be $0\omega_0$ and we find out $1\omega_1$.

Similarly i = 0 will give me $1V_1$ and so on then i = 2 then 3 and so on, all the way to the last rigid body n. So, for prismatic joint we have omega is same. So, the previous link $i - \omega_{i-1}$ will be same as the $i\omega_i$ of the next link because the prismatic joint does not allow relative rotation, it only allows relative translation which is what is shown here.

The linear velocity of the origin of the next link will be the linear velocity of the origin of the previous link plus some $\omega \times R$ plus what is happening at the translatory joint or the prismatic joint which is $\dot{d}_i$ (0 0 1) and again the joint axis is along the Z axis. So, that is why we have this 0 0 1. The acceleration of an arbitrary point on the rigid body i we can differentiate the velocity with respect to time.

So, if it were a rotary joint we could differentiate this and then what we will see is that the velocity of this point p when you differentiate and you will get acceleration of this point p, it is nothing but the acceleration of the origin plus the relative acceleration plus some 2 omega cross V relative plus alpha cross R and then omega cross omega cross R. So, we had looked at this equation earlier.

So, these are the five terms typically in the acceleration of a point in a rigid body which is both translating and rotating with respect to a previous fixed reference. So, this we have seen, this is acceleration of the origin, this is like similar to the ant problem we have discussed earlier this is the relative acceleration of the end then this is the $2\,\omega \times V_{rel}$ which is the coriolis term.

This is the tangential acceleration and this is the centripetal acceleration. So, we can obtain this, we have seen it earlier, all we are doing it we are writing it in a formal way, if the point on the ith rigid body is stationary or this ip vector is constant then there is no relative acceleration or relative velocity. So, we have both these terms which is this relative acceleration term and this coriolis term 0.

So, we will only have the acceleration of the origin plus some alpha cross R and then the centripetal term.

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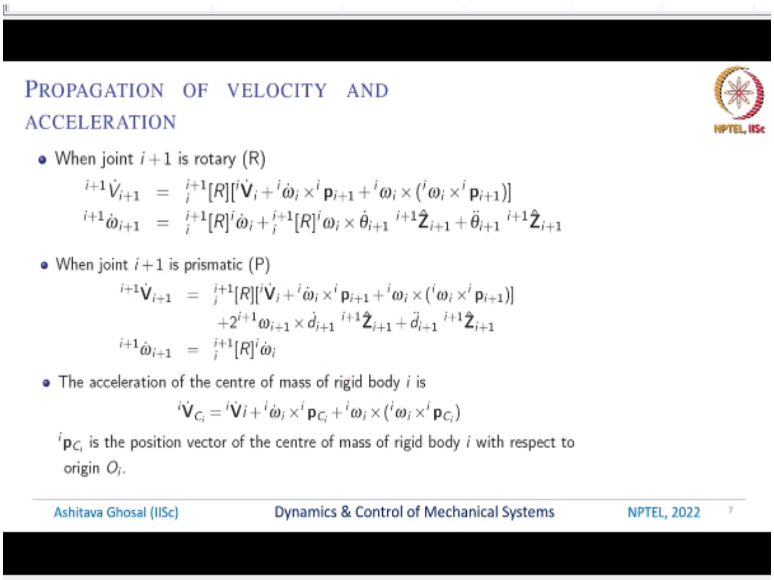

So, continue with this idea of propagation of velocity and acceleration. If the joint i + 1 is a rotary joint which allows relative rotation between the two rigid bodies then the acceleration of the i + 1 is rigid body written in its own coordinate system will have 3 terms, rotary joint there is no relative velocity or relative acceleration. So, we have the acceleration of the origin plus alpha times R plus this centripetal term.

The angular acceleration of the i + 1th link is again nothing but the angular acceleration of the previous link then what is happening at the joint which is $\ddot{\theta}_{i+1}$ along the Z axis, but interestingly we will also have one additional term which is $i\omega_i \times \dot{\theta}_{i+1}$ and this is also along the Z axis. So, it turns out that we will get alpha and then $\ddot{\theta}_{i+1}$ plus something like $0\omega_i \times \dot{\theta}$

So, this is what appears when the joint i + 1 is rotary and all of these can be derived from first principles if you are interested. When joined i + 1 is prismatic we will look at the simple case first which is that the angular acceleration of the i + 1 at link is same as the angular acceleration of the ith link because there is only relative translation between the two links.

However, the linear acceleration of the i + 1 at rigid body; will now have all the 5 terms which are there in the expression for acceleration. So, we have the acceleration of the origin then alpha times R, this is the tangential acceleration, this is the centripetal term, then this is the coriolis term and this is the term which is showing the relative acceleration of the point in $\ddot{d}_{i+1}$.

So, we have these generic expressions for propagating both linear velocity, the angular velocity, the linear acceleration and the angular acceleration from one link to the next link. The acceleration of the center of mass is nothing but the acceleration of the origin plus alpha times R plus the centripetal term. So, we will need the acceleration of the center of mass because we would like to apply Newton's law which is F equals mass times acceleration of the center of mass.

So, the acceleration of the center of mass of any link can be obtained by this acceleration of the origin plus alpha times the distance from the origin to the center of mass plus omega cross omega cross R. So, ip_{c_i} is the position vector of the centre of mass of rigid body i with respect to the origin O_i.

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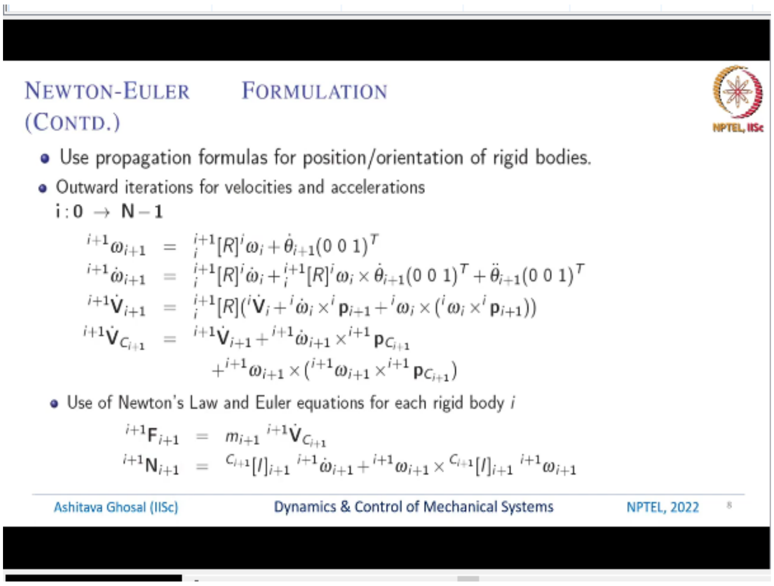

So, let us continue with this Newton-Euler formulation, we have the propagation formulas for position and orientation of rigid body basically using 4 by 4 homogeneous transformation matrices and then we need the outward iterations for velocities and acceleration which is given in this algorithmic form. So, we start from i = 0 and we end at n - 1 because for n - 1 the left hand side will give the nth position linear and angular velocities and acceleration.

So, what do we have and this is for rotary joint we are showing you this examples, we need to change these formulas if there is a prismatic joint in between, but nevertheless the angular velocity of the i + 1 at link can be written in terms of the angular velocity of the ith link plus the rotation or angular rate at the rotary joint. The angular acceleration of the i + 1 at link can be written in terms of the angular acceleration of the ith link.

Then this term of $\omega \times \dot{\theta}$ and then $\ddot{\theta}_{i+1}$, the acceleration of the origin of the i + 1 at link the linear acceleration is again 3 terms which is the linear acceleration of the origin plus the tangential term which is alpha across R and then this is the centripetal term. The acceleration of the center of mass of the i + 1 it link can be written in terms of the acceleration of the origin plus $\omega \times R + \omega \times (\omega \times r)$

So, with these formulas I can start with some i and if you substitute what is happening on the right hand side we will get the terms in the left hand side. Once we have this linear acceleration of the center of mass then the angular velocity of the ith link and alpha of the higher link or the angular acceleration of the ith link then we can apply Newton's law. So, what the Newton's law says is that the external forces are nothing but mass times acceleration.

This is what is written here and we are writing it with all this superscript and subscript because it is in the form of an outward iteration. So, I can start with i = 0 and end with i = n - 1 actually. So, we end with i = n – 1. Similarly the external moment is nothing but $I\alpha + \omega \times I\omega$. So, this is Newton's law and this is Euler's equation.

So, we can apply both of these, we know the right hand side because from this outward iteration I know what is omega dot, I know what is omega, I know what is the acceleration of the center of mass. So, I know what is the right hand side here, we know of course what is the mass and what is the inertia matrix. So, if I substitute all these things on the right hand side I will get the force and the moment which is acting on the link i + 1.

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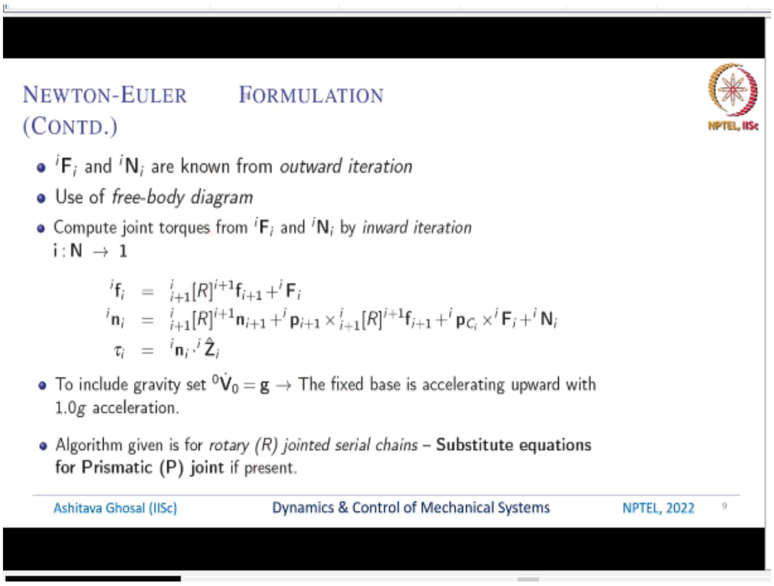

Once I know what is this force and the moment which is acting on a link, this is called the outward iteration. So, we start with position, then velocity, then acceleration and then we compute the forces and moments using Newton's law and using Euler's equation. So, we have used basically some kind of a free body diagram in addition to what are these propagation equations.

So, once we have these forces and moments which are acting on the ith link we go backwards. So, this is called inward iteration. So, we start with i = N and then we go backwards. So, what is i at for nth rigid body. So, the if_i will be nothing but what is the external force which is acting on the link N plus what was the result of the application of Newton's law. So, we have this last rigid body which is rotating and accelerating and so on.

So, based on Newton's law we can find the F = m a. So, we know ma, similarly Euler's equation moment is equal to $I\,\alpha + \omega \times I\,\omega$. So, I know the right hand side which is F_i and N_i and then we can find what is the external force which is acting at the end. So, if the robot is pushing something or if there is a some external force which is acting on the last link then we substitute it here and then we find what is the force acting on the ith link.

Similarly we find the moment which is acting on the ith link based on what is the external moment obtained from Euler's equation then the $ip_{c_i} \times iF$ which is the moment of this external if_i which is coming from here from Newton's law and then we have these two terms which is the external moment which may be acting on the last link and the external force which may be acting on the last link.

So, once we have the right hand side from the outward iteration we can use these expressions to obtain what is the force at the moment which is acting at the origin of the ith link. So, for i equals n we know what is the force and moment which is acting at the coordinate system which is fixed at the nth link. The last step in this Newton-Euler formulation is to find the torque which is acting on the ith link.

So, the last link there is the joint between the nth link and the n – 1th link that is a joint, we assume that for the moment that joint is a rotary joint. So, if it is a rotary joint the z component of the rotary joint is given by some motor or some torque. So, the torque which is acting about the z-axis is the z-component of this in_i which we have computed. If you want to include gravity all we need to do is we need to set the zeroth coordinate system or the fixed coordinate system is accelerating upwards by g.

So, if it is accelerating upward by g all the outward propagation formulas for acceleration we now see this g. So, hence all the links will see the effect of gravity. So, this algorithm which I showed you that first we position velocity, then acceleration and we go outwards which was the outward iteration and then we use Newton's law and then we use Euler's equation to find this iF_i and iN_i and then we use free body diagram and we go backwards.

So, we find what is the net force which is acting at the; origin of the ith link and the moment which is acting at the origin of the ith link and then we say that the torque which this rotary joint is giving by means of a motor or some other means is the z component of the moment. So, all this formula was for rotary joint. If it is a prismatic joint we need to change these outward equations.

So, wherever there was velocity we would now have $\dot{d}_i$, wherever there was acceleration we would have all the 5 terms in the acceleration and this instead of torque we would have some force which is acting along the z axis or along the z direction. So, basically we carefully substitute the equations for a prismatic joint in both these outward iterations and inward iteration.

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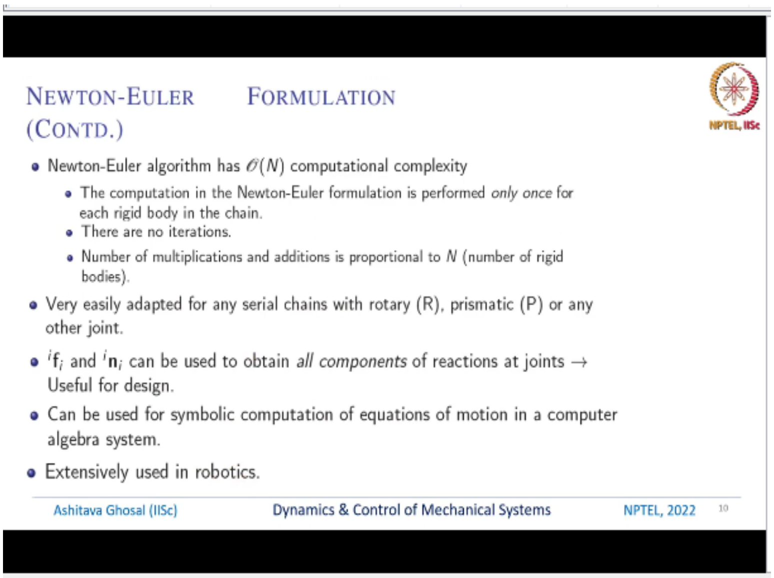

So, this Newton-Euler algorithm is very well known, it is very, very efficient and it has been used extensively in many fields. So, for example in robotics. The reason is you can show that this Newton-Euler algorithm as O (N) complexity. So, this means that this is a computer science term terminology which means that the complexity increases linearly with N, where N is the number of links.

So, and you can think of it this way that the computation in the Newton-Euler formulation is performed only once for each rigid body in the chain. So, you are going outward once and then you are coming back, there are no loops and there is no iteration for each rigid body. If there were iterations, then it will not be linearly increasing with number of links. So, the number of multiplications, addition is proportional to N which N here stands for the rigid body.

So, if you have a serial chain with let us say 6 rigid bodies and you get so many multiplications and additions. If the number of links goes to 12 then the number of multiplication and additions will become twice, it will not become N square, it will not become 4 times or N cube. So, it is linear complexity. It is very, very easily adapted for any serial chains with rotary R and prismatic joint.

In fact there have been extensions you can apply it for any joint. So, if it was a 3 degree of freedom spherical joint then you have to replace this spherical joint with 3 rotary joints, but originally the Newton-Euler formulation was mostly for rotary and prismatic joints and it can be very easily adapted to find the equations of motion for any serial chain.

The other very, very useful thing in this Newton-Euler formulation is we can compute the net force if_i and the net moment in_i which is acting at the origin of the ith link. So, this if_i has 3 components $f_x f_y f_z$. Similarly n i has 3 components $n_x n_y n_z$. So, the z component of n was the motor torque, but we can also find what is n_x and n_y. We can also find out what is f_x and f_y.

So, we can find out all the components of reactions at the joints and this is very useful for design. So, if I want to find what is the bearing which I should use for any such serial chain I need to know all the components, I need to know f_x and f_y sometimes even f_z or I need to know the other bearing. So, then I can go about choosing what is the bearing at each joint, because the bearing must overcome the friction.

And also it should overcome the reaction forces. So, we can do some kind of calculations to choose the forces which are acting at the bearing. So, remember I said friction, because friction also will depend on what are the reaction forces. So, if there is a lot of reaction large reaction then the friction might be more. The other very nice feature of this Newton-Euler formulation is it is very, very mechanical.

So, you go from position, velocity, acceleration for one link then you go to the next link and so on and then at the end of position velocity acceleration you calculate the f =m a and n which is $I\,\alpha + \omega \times I\,\omega$, we use Newton's law and Euler's equation and then we go backwards and we find what are the joint torques of the joint forces. So, this is very, very mechanical and it can be easily done by symbolic computation.

We do not need to sit down and work all these steps by hand. In fact there are computer algebra systems and I will show you one which is called maple which can be very easily used to compute all the steps one by one symbolically not using numbers and eventually we can end up with the equations of motion and it is extensively used in robotics, it is extensively used in many other places which have serial chains.

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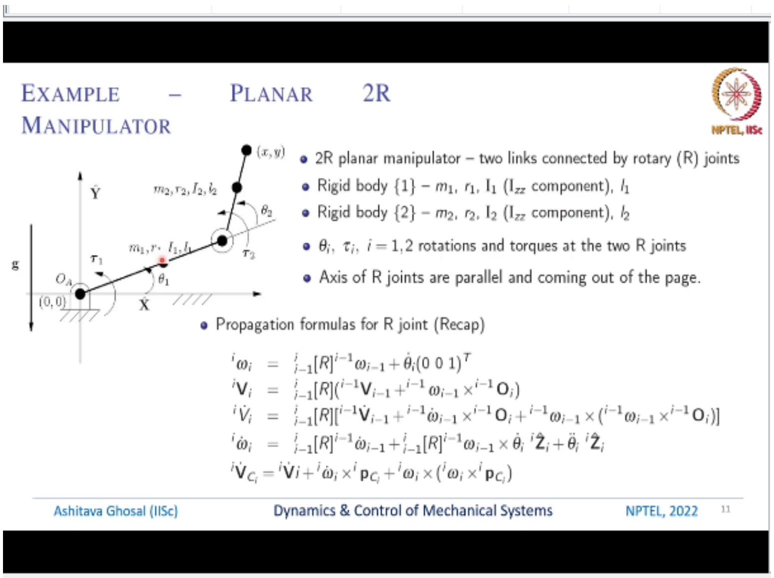

So, let us look at an example. So, we will take our simplest possible example of a planar 2R manipulator. Remember we had worked out the equations of motion using very basic mechanics which was Newton's law and Euler's equation. So, we looked at some of the external moment was $I\,\alpha$ and so on. And then we could eliminate the reaction forces at these joints. And then we derive the equations of motion.

Now I want to take the same example and go through the steps of this Newton-Euler formulation and I will show you obviously that the equations of motion are same. So, what do we have? We have 2 degrees of freedom, we have 2 links and there are 2 rotary joints. So, this is the fixed base with 0, 0, t his is the origin of the fixed coordinate system X and Y. Then we have one link and again θ_1 and θ_2 are the rotations at the rotary joints.

And just like before we have m_2 is the mass of second link, r_2 is the location of the C g, i_2 is the z component of the inertia matrix, l_2 is this length and so on. So, for rigid body 1 we have $m_1,\ r_1, I_1,\ l_1$ which is the z component and l_1 and for rigid body 2 which is $m_1,\ r_1, I_1\ and\ l_1$. Theta is the rotation of the first one 2 joints θ_1 and θ_2 and τ_i i = 1 and 2 where the torques which are being given at these joints by motors let us say.

The axis of these R joint which is here and here they are parallel and they are coming out of the page. So, this is the z-axis, this is X, Y and if you use a right-hand rule the z-axis is coming towards you. So, let us do the recap of the propagation formulas. So, we have angular velocity of the ith link is related to the angular velocity of the i - 1 at link plus some $\dot{\theta}$. Similarly, linear velocity of the ith link which means linear velocity of the origin of the ith link is nothing but the linear velocity of the i - 1 at link plus $\omega\times r$.

The acceleration is similarly acceleration of the origin plus $\alpha r + \omega\times\,\omega\times r$ and angular acceleration, this angular acceleration of the previous link plus this $\omega\times\dot{\theta}$ and then $\ddot{\theta}$ along the z axis. So, these are the four propagation equation which we will use and we also have the acceleration of the center of mass of each link which is nothing. But the acceleration of the origin of the link plus $\omega\times r + \omega\times\,\omega\times r + \alpha r$.

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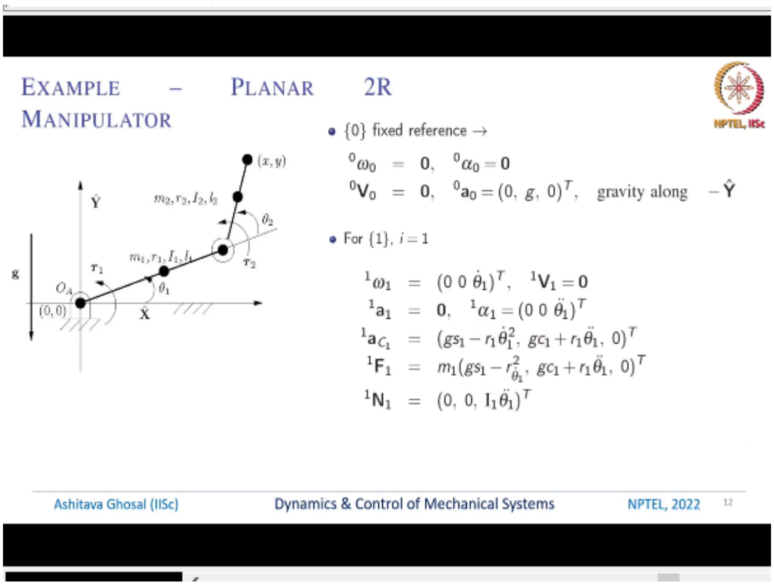

So, let us continue we have this to planar 2R manipulator, 0 is the fixed reference and if you have 0 as the fixed reference nothing is moving. So, the angular velocity of the 0th coordinate system is 0, angular acceleration is 0, the linear velocity of the origin of the 0th coordinate system which is this point is 0 and the acceleration of the origin of the 0th coordinate system is g.

Remember we want to also capture the effect of gravity and if you want to capture the effect of gravity we have this base accelerating upwards by g, g here is 9.81 meters per second and it is along the y axis. So, for i = 1 we use this propagation formulas which was $1\omega_1$ will give me 0 0 $\dot{\theta}_1$, $1V_1$ will be 0, 1 $1a_1$ will still be 0, $1\alpha_1$ is 0 0 $\ddot{\theta}_1$.

The acceleration of the center of mass of the first link is $gs_1 - r_1\dot{\theta}_1^2$. The y component is $(gc_1 + r_1\ddot{\theta}_1$, 0). Here we can see that it looks right that the center of mass which is here, there will be a coriolis term which is $r_1\dot{\theta}_1^2$ which is going towards and then there will be a tangential term which is $r_1\ddot{\theta}_1$ and then of course we have g which is coming down.

But now we have this sin θ_1 and cos θ_1 because we are rewriting all these terms in its own coordinate system. The Newton's law tells me that the force $1F_1$ will be mass times this acceleration and that is exactly what is written here. So, we take this acceleration vector and multiply by m_1. Euler's equation tells me that the external moment is nothing but I $\ddot{\theta}_1$, I times α because remember $\omega \times I\,\omega$ is again 0.

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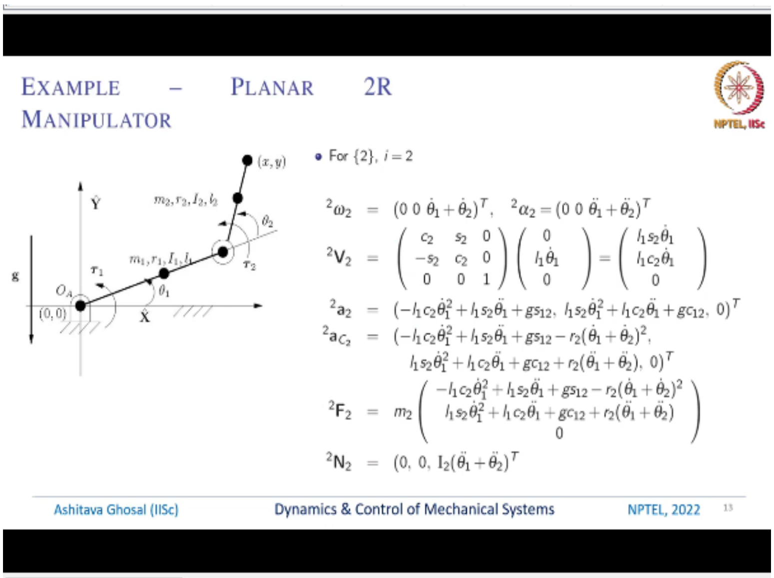

So, let us continue. So, for i = 2 for second link we substitute what is $1\omega_1$, $1V_1$, $1\alpha_1$ and so on in those propagation equation and we can find $2\omega_2$ =(0 0 $\dot{\theta}_1 + \dot{\theta}_2$) that makes sense right that the angular velocity of the second link will be the sum of $\dot{\theta}_1$ and $\dot{\theta}_2$. θ_2 is relative rotation. So, the angular velocities will be summation of $\dot{\theta}_1$ and $\dot{\theta}_2$.

The acceleration of derivative of this. So, you will get $\ddot{\theta}_1 + \ddot{\theta}_2$. The linear velocity of the origin of this second link which is basically this term will be some matrix times 0 $l_1\dot{\theta}$ 0. So, where is this $l_1\ \dot{\theta}_1$ coming from? So, this is l_1, there is a $\dot{\theta}_1$ here. So, there is a velocity component in this. So, this is the local x direction, this is the local y direction.

So, that is why we have 0 $l_1\dot{\theta}_1$ and then we pre-multiply by a rotation matrix. So, hence the linear velocity of this origin of the second link written in its own coordinate system is ($l_1 s_2\dot{\theta}_1$, $l_1 c_2\dot{\theta}_1$ 0). So, there is nothing very confusing about this whole thing except that we are mechanically using those propagation equations. So, if you write down the propagation equation on a sheet of paper and then you just substitute you will get these terms.

Likewise the acceleration of the origin of the second link will have 3 terms, there will be a gravity, then there will be a $\ddot{\theta}_1$ and there will be a $\dot{\theta}_1^2$. That is what you will get. This is the origin of this point. So, this one there will be a $\ddot{\theta}_1$ which is this direction, $\dot{\theta}_1^2$ which is this direction and then there is this gravity term.

Similarly for y component we will have $\dot{\theta}_1^2$ one term with $\ddot{\theta}_1$ and again gravity. The z component in all these expressions are 0, both velocity as well as acceleration because this is the planar system, all the motion is happening in the X, Y plane. The $C\ g$ of this second link is at a distance r_2 from here. So, what the acceleration of the $C\ g$ will be nothing but the acceleration of this plus some effect of this r_2.

So, that is what you get. So, you will get some term which is $r_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2$ and then of course we will have some $l_1\ s_2\ \dot{\theta}_1$ and $l_1\ c_2\ \dot{\theta}_1^2$ and so on. So, $\dot{\theta}_1^2$, this term is coming from here gs_{12} is coming from here, but then we have this centripetal term which is coming because the center of mass is at a distance r_2 from the origin of the second link.

So, we can find this acceleration of the C g of the second link and then we apply Euler's equation and Newton's law. Newton's law says that the force is nothing but m_2 times a_{c_2} which is what is written here. So, it is m_2 times this vector. So, you can see. The first term is $-l_1 c_2 \dot{\theta}_1^2$, here also it is $-\ l_1 c_2 \dot{\theta}_1^2$. So, this is just the x component. So, m_2 into the x component, m_2 into the y component and again 0.

An Euler equation tells me that this is $I_2(\ddot{\theta}_1 + \ddot{\theta}_2)$. So, these are the equations for $I\ =\ 2$, we have also found out what is for I = 1 which is the link 1.

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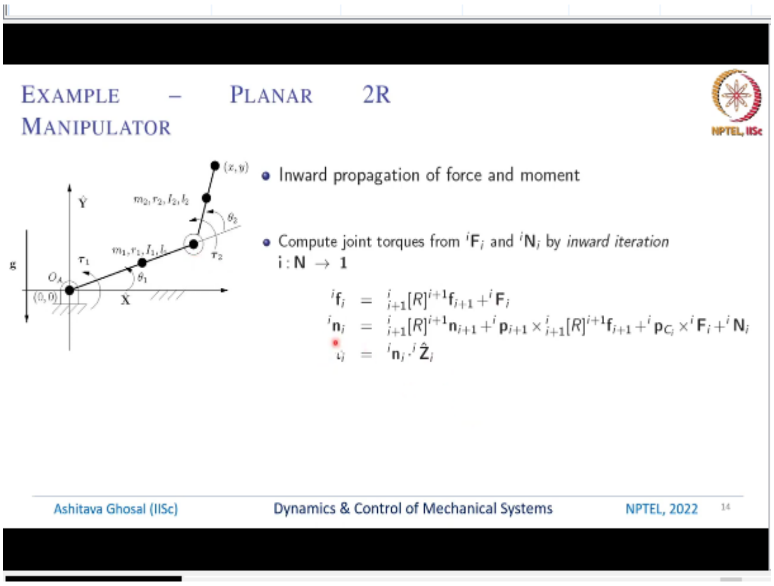

And then we go back and do this inward propagation of forces and moments. So, we want to compute the joint torques which is happening this τ_1 and τ_2. So, basically we have this force which is acting here which is F_i and then we have this external force in this case may be 0, there is no external force acting on this link, but if there were any we have to add that here.

And that gives me what is the force acting at the origin of the ith coordinate system, what is the moment which is acting at the origin of the ith coordinate system? And the torque in this case since they are rotary joints both of them we; will get the z component of the moment is resisted by this motor or this motor is giving the equivalent of the z component of the moment.

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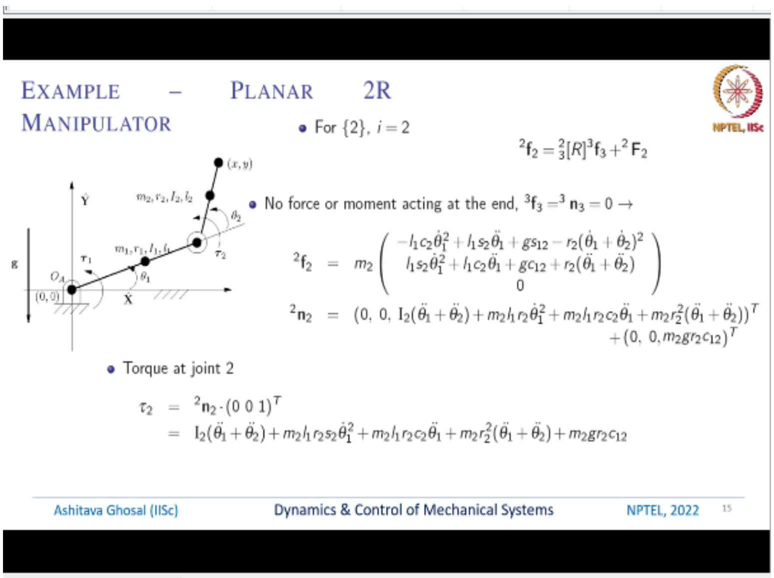

So, for $i = 2$, we have $2f_2 = 32[R]\ 3f_3 + \ 2f_2$. So, there are no forces which are acting on the third link. In this example if there are force and external force and moment which is acting on this link then we have to take this into account here. So, these two are 0 , $2f_2$ is nothing but this, $2f_2$ is nothing but F_2 which is mass times the acceleration and $2n_2$ is nothing but the I times alpha whatever you will get these terms.

So, $I_2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 r_2 \dot{\theta}_1^2$ and so on. So, this is like the alpha and the omega part. So, the torque at joint 2 is nothing but the z component of n_2. So, this is $2n_2$ dot product with 0 0 1 which is the z component and hence we will get this last 1, 2, 3, 4 terms. So, $\tau_2 = I_2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 r_2 s_2 \dot{\theta}_1^2 + m_2 l_1 r_2 c_2 \ddot{\theta}_1 + m_2 r_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 g r_2 c_{12}$. So, this is the equation of motion for of the two equations of motion which is τ_2 is given by some I into theta double dot plus some coriolis term and some gravity term.

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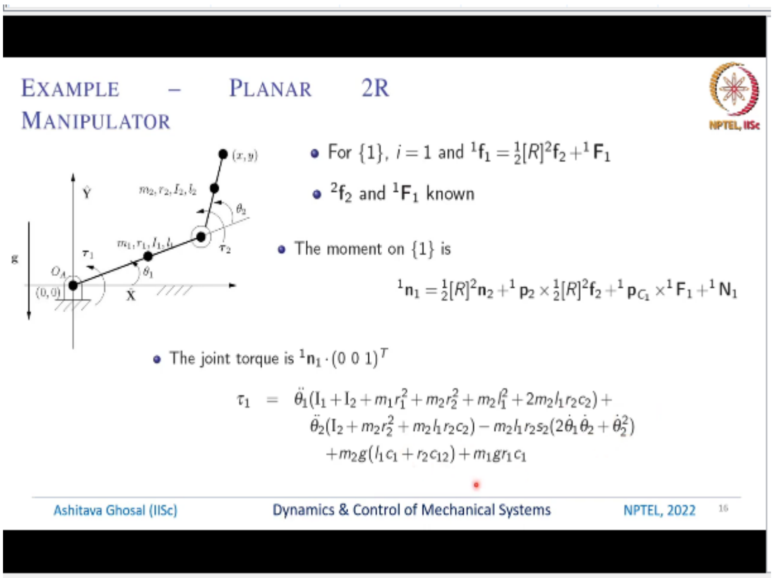

So, likewise for i = 1 we have $1f_1$ is nothing but $1F_1$ plus this $2f_2$. So, inward propagation. So, the force which is acting at the origin of the second link will be seen at the origin of the first link which is this and then $2f_2$ and $1f_1$ is known from the previous step. So, the moment which is acting at the origin of the first link is given by the x, this is from

Euler's equation, this is the moment of this force $1f_1$ and these are the force which are coming from the second link to the first link.

So, hence the joint torque is nothing but $1n_1$.(0 0 1). So, we know what is $2f_2$, we know what is $1F_1$, we know what is $1n_1$ from Euler's equation then we substitute everything here and then you take the z component of 1 and 1 and you will get τ_1 is given by this. So, this is the second equation for this planar 2R manipulator. So, the first equation said τ_2 will was something.

Now we have τ_1 is equal to $\ddot{\theta}_1$ into some terms here, $\ddot{\theta}_2$ into some terms here and then we have this $2\,\dot{\theta}_1\,\dot{\theta}_2 + \dot{\theta}_2^{\,2}$ and then some gravity term.

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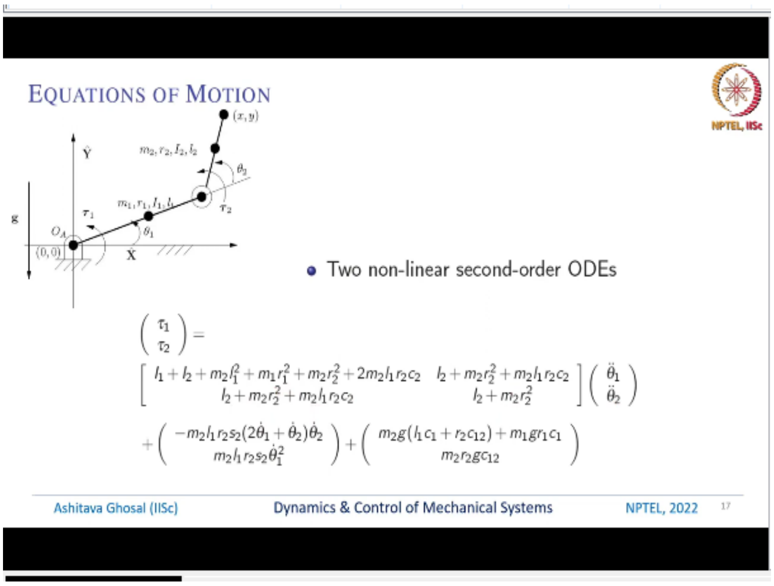

So, what do we have finally for this two link system? Two links connected serially by rotary joints, we have two non-linear second order ODEs and these second order ODEs are that we have τ_1 is something into $\ddot{\theta}_1$ plus something into $\ddot{\theta}_2$ plus some terms which are $\dot{\theta}_1\,\dot{\theta}_2 + \dot{\theta}_2^{\,2}$ and some term which is gravity which is $g(\,l_1 \cos\theta_1 + r_2 \cos\theta_{12}\,)$ and so on.

Likewise we have a term equation which is $\tau_2 = (I_2 + m_2 r_2^2 + m_2 l_1 r_2 c_2)\ddot{\theta}_1 + (I_2 + m_2 r_2^2)\ddot{\theta}_2$ plus a term which is $\dot{\theta}_1^2$ into something and one term which is gravity term. So, we have this two second order equations written in this vector and matrix form. So, τ_1 and τ_2 is a 2 by 1 vector. This is a 2 by 2 matrix multiplying $\ddot{\theta}_1$, $\ddot{\theta}_2$.

Then we have this vector 2 components one which is 2 $\dot{\theta}_1$ into $\dot{\theta}_2$ into something and then $\dot{\theta}_2^2$ and remember all these are multiplied by some sine θ_2 here is there, some cos θ_{12} here and this is cos θ_2. So, the elements of this matrix is are not constant. They are dependent on θ_1 and θ_2. So, this term here if this θ_2 is 30 degrees you will get something.

But if it is some other angle you will get some other terms here, I_1, I_2, m_2 l_1 r_1. These are all constant, but θ_2 is changing. So, if θ_2 is changing the elements of these 2 by 2 matrices are changing and which is why it is a non-linear second order ODE. So, the coefficients of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are not constant, some are constant. So, for example in the second equation this is$(I_2 + m_2 r_2^2)\ddot{\theta}_2$. So, this is constant, but this other term has cosine θ_2.

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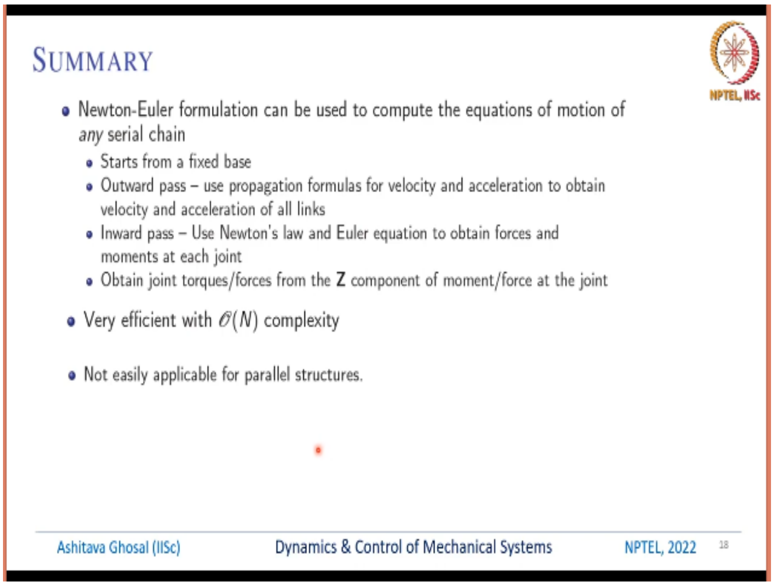

And the equations of motion that you have got are exactly same as what we had got earlier using hand calculations when we did not use these propagation formulas. So, in summary the Newton-Euler formulation can be used to compute the equations of motion of any serial chain this important word. So, if you give me a serial chain with 5 links yes, if you give me one with 100 links again I can find out the equations of motion.

Of course for a 100 link you will have huge 100 such second order ODEs. So, we starts from a fixed base, we do an outward pass, the outward pass consists of propagation formulas for velocity and acceleration to obtain velocity and acceleration of all links. And then there is an inward pass we use Newton's law and Euler's equation to obtain forces and moments at each joint and then we obtain the joint torques or forces from the z component of the moment or force at the joint.

So, in the examples which I showed you it was the z component of n_i, n_i was the moment but if it was a prismatic joint then we have to take the z component of F_i which is the force which is acting on the origin of the ith link. This is a very, very efficient algorithm, as I said it has O N complexity, it is linear in complexity. So, if the number of links becomes double the number of additions and multiplications and the complexity also scales by 2.

It does not scale by 2 square or even there are some algorithms which can be exponential complexity. So, the number of n is increased by twice, it will be some to the power of n cube or now it could be n square or e to the power n. So, that is not what is happening here. This scales linearly. The unfortunate part is it is not very easily applicable for parallel structures.

It is very, very efficient for serial chain but not really applicable for parallel structures and the basic problem with parallel structures which you have seen is that there are loops. So, I can start from one fixed end and I can come back to the fixed end again. So, hence we cannot go outwards till one last link and come back that is not possible.

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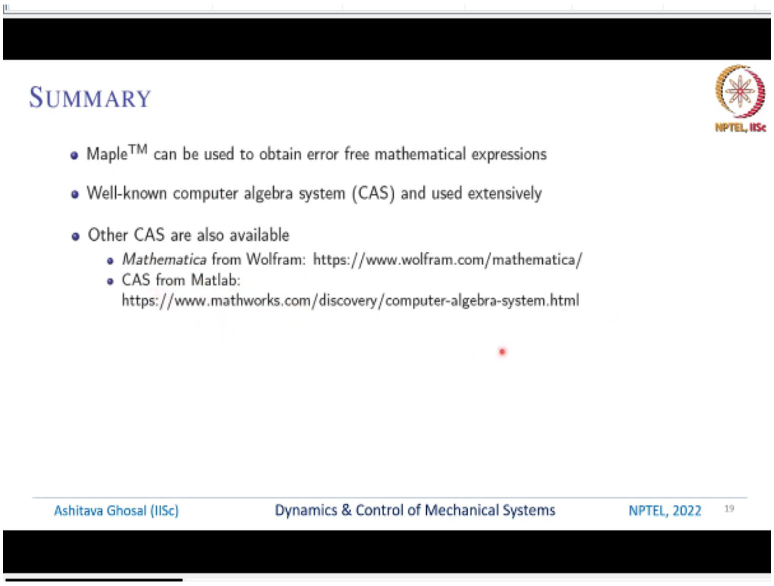

So, in summary maple can be used to obtain error free mathematical expressions. So, I can find out the elements of the mass matrix, the elements of the Coriolis term, the elements of the gravity term, the elements of the constraint matrix, the derivative of the constraint matrix and so on. Then we can form the equations of motion and solve the equations of motion in MATLAB or some other software which allows you to use numerical numbers or numerical tools like ODE solvers and so on which can solve this differential equations of motion.

So, what I have shown you is one of the computer algebra system which is well known. Maple it is used reasonably extensively, there are other computer algebra systems which are also available, there is another very well known CAS system which is called mathematica, it is available from this company called Wolfram and this if you are interested in mathematica please go to this website.

Wolfram.com slash mathematica. Matlab also gives you its own computer algebra system, this is slightly less well-known and less powerful than either maple or mathematica, but nevertheless you can use the computed algebra system from Matlab. For simple examples like the planar 2R or the angular velocity vector or even the four-bar mechanism we can obtain everything from the computer algebra system given by Matlab.

So, in this NPTEL course you have access to Matlab and all its tool boxes. So, it is a good idea, if you have time to try out the computer algebra system from Matlab, more details about the computer algebra system from Matlab is available in this website.