Dynamics and Control of Mechanical Systems
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Lecture-12
Equation of Motion

Welcome to this NPTEL lectures on dynamics and control of mechanical systems. In the last week we had looked at the properties of a body in terms of mass, it is distribution of mass also the inertia matrix. And we had looked at the linear and angular momentum and the various types of external forces which act on a rigid body or a multi body systems. In this lecture we will look at the equations of motion of a rigid body.

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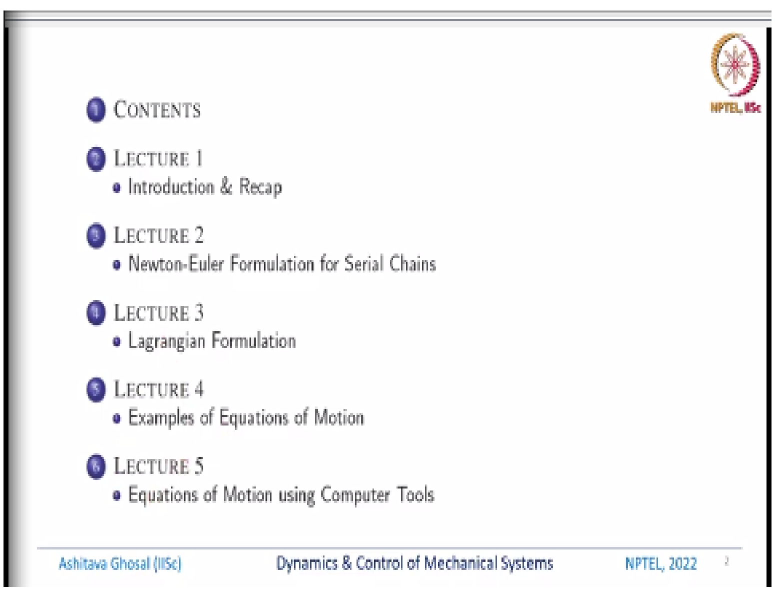

On the topic of equations of motion for a rigid body and multi-body systems, there will be 5 lectures these will be spread over the next 2 weeks this week and the next week. In the first lecture we will look at introduction and a recap of the contents which we have done before. In the second lecture we will look at the Newton-Euler formulation for serial chains, this is a very well known approach which is used extensively in many robots and other mechanical systems.

In the third lecture we will look at what is called as the Lagrangian formulation and this is also a very well known approach to derive equations of motion for rigid bodies and multi-body systems. In lecture 4 we look at examples of equations of motion which we have derived and I will show you the steps required to derive the equations of motion. And finally in lecture 5 we will look at how these equations of motion could be obtained using computer tools.

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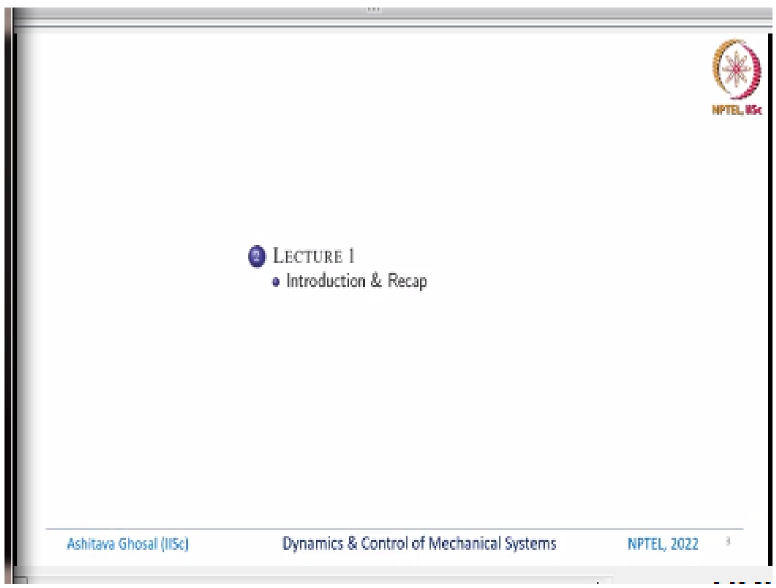

Lecture 1, introduction and recap.

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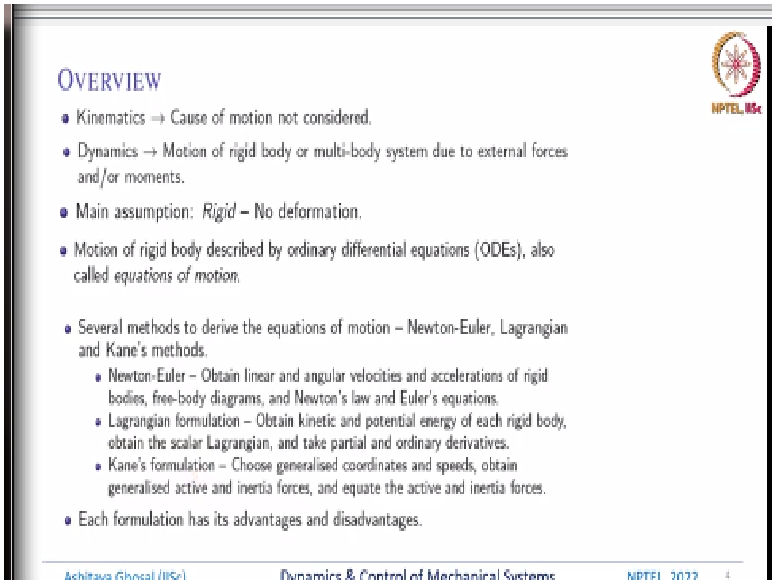

In this lecture we will quickly overview this whole area. So, first important thing is till now in kinematics the cause of the motion was not considered. In dynamics the motion of a rigid body or a multi-body system due to external forces and not movements are considered, so that is the topic of dynamics. The main assumptions in this whole set of lectures and in this course in fact is that the rigid bodies are used, there is no deformation in the body or in the link or whatever is making up the mechanical system.

The motion of the rigid body described by ordinary differential equations, also called equations of motion. So, if you have a rigid body we will naturally get ordinary differential equations, if you have deformations in the rigid body we do not get ordinary differential equations of motion, we will have partial differential equations. So, we will not worry about partial differential equations, we will only limit our scope to rigid bodies which naturally give only ordinary differential equations.

There are several methods to derive the equation of motion of a rigid body or a multi-body system these are the 3 main ones, one is called as a Newton-Euler, one is called Lagrangian and one is called Kane's method. In the Newton-Euler approach we obtain the linear and angular velocities and acceleration of rigid bodies. We draw free body diagrams and then we apply Newton's law and Euler equations to each of the free body and then we do lot of simplification to arrive at the equations of motion.

In the Lagrangian formulation we obtain the kinetic and potential energy of each rigid body. We obtain a scalar quantity called the Lagrangian of the system and then we take partial and ordinary derivatives, so this is the energy based approach. Whereas in the Newton-Euler we have to obtain velocities and acceleration, in the Lagrangian formulation we can get away with only obtaining up to velocities.

In the Kane's formula we choose what are called as generalized coordinates and speeds, we obtain generalized active and inertia forces and equate the active and inertia forces to obtain the equations of motion. We will not be looking at this in this course, anybody who is interested can search Kane and then there is lot of material available on how to use Kane's formulation to obtain equations of motion.

Each formulation has it is own advantages and disadvantages, so we will look at Newton-Euler and Lagrangian formulation their advantages and disadvantages. But as I said we will not study Kane's formulation in this course.

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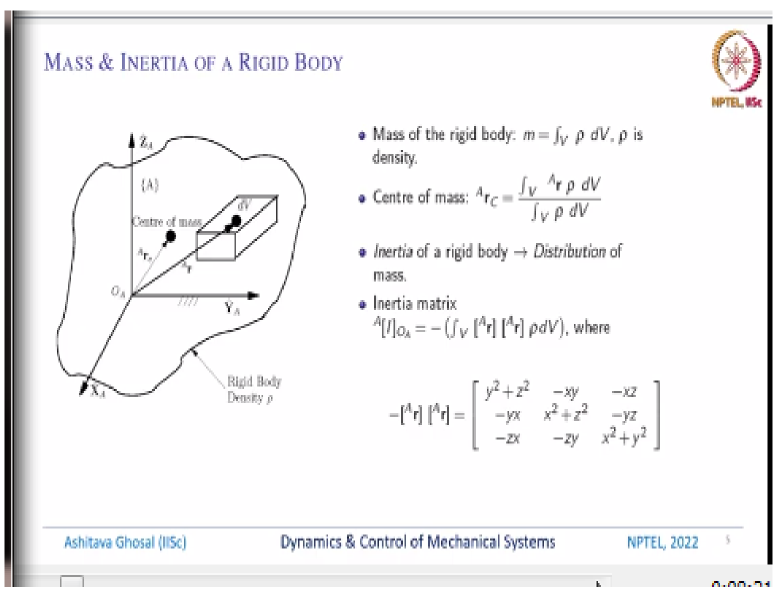

So, let us continue, this is to recap we have something called as the mass and inertia of a rigid body, so this figure shows a rigid body. So, there are rigid body and we have associated coordinate system $\hat{X}_A$ $\hat{Y}_A$ $\hat{Z}_A$ with an origin O_A. And we look at a volume element dV and let us say the volume element is located by this vector Ar . So, we can find the center of mass of a rigid body by performing this operation which is the location of the centre of mass is given by the

$$Ar_C = \frac{\int_V Ar \ \rho dV}{\int_V \rho dV}.$$

So, the $\int_V \rho dV$ is the total mass of the rigid body and this vector locates the centre of mass as shown here in this figure. We also have something called the inertia of a rigid body; this is basically the distribution of mass about this 3 axis X, Y and Z. And we had looked at this in the last week, so we have something called as an inertia matrix and this is nothing but the $\left(\int_V [A\,r][\,A\,r]\rho dV\right)$.

And ${}^A r\, {}^A r$ remember is some kind of a vector X, Y, Z but written in a skew symmetric form of a matrix. So, and this inertia matrix is with respect to a coordinate system in this case A and it is about a point, so we need to mention the inertia matrix of a rigid body is with respect to which coordinate system and about which point. So, in this it is shown that it is with respect to the A coordinate system and it is about this origin of the A coordinate system O_A.

So, as I said this skew symmetric matrix ${}^A r\, {}^A r$ if you multiply - ${}^A r\, {}^A r$ into ${}^A r\, {}^A r$, so it is square of this skew symmetric matrix you will get terms like $y^2 + z^2, x^2 + z^2, x^2 + z^2$ along the diagonals. And the off diagonal terms will be -x y, -y x, so it is still a symmetric matrix.

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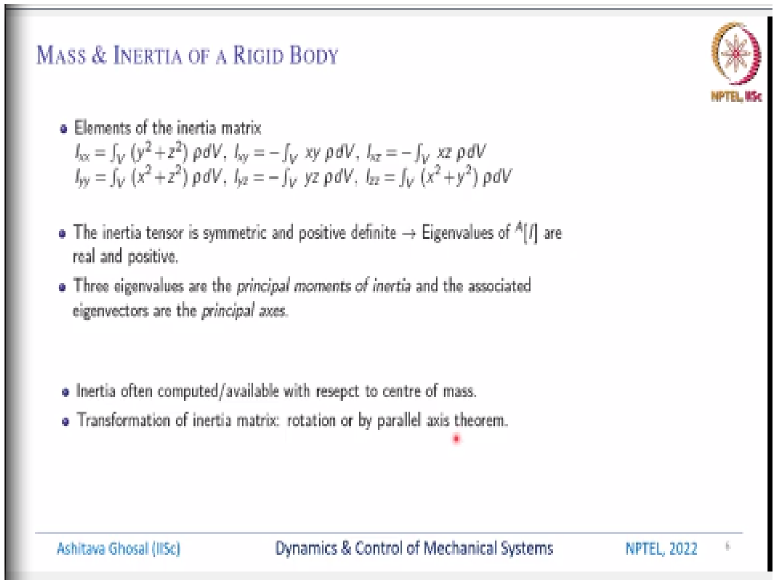

And based on this skew symmetric matrix $A\,r$ and the square of the skew symmetric matrix and the volume integral we can define elements of the inertia matrix which are I_{xx} which $\int_V y^2 + z^2 \rho dV$. I_{xy} is $-\int_V xy\rho dV$, $I_{xz} =-\int_V xz\rho dV$. So, you can see that the subscript x and z and here also we have x and z.

So, likewise $I_{yz} =-\int_V yz\rho dV$ the diagonal terms I_{xx}, I_{yy} and I_{zz} they are square, so it is $y^2 + z^2$ here. In I_{yy} we have $x^2 + z^2$ and in I_{zz} we have $x^2 + z^2$, so this is the 3 by 3 matrix. The inertia matrix is symmetric and positive definite and as a result the Eigenvalues of this matrix ${}^A[I]$ are real and positive. So, we have 3 Eigen values which are also called the principal moments of inertia and the associate eigenvectors are the principal axis for the rigid body.

Inertia is often computed or available with respect to the center of mass and this is available in many textbooks and some mechanics textbooks and at the end of the appendix. We can obtain the inertia matrix with respect to some other point and about some other coordinate system by using the transformation of the inertia matrix. There are 2 of them which are possible one is rotation or by parallel axis theorem.

Parallel axis theorem is translation from the Cg to another point and rotation is at the Cg if we have 2 coordinate systems A and B we can relate the inertia matrix in the A coordinate system and in the B coordinate system by multiplying by the rotation matrix $BA[R]$.

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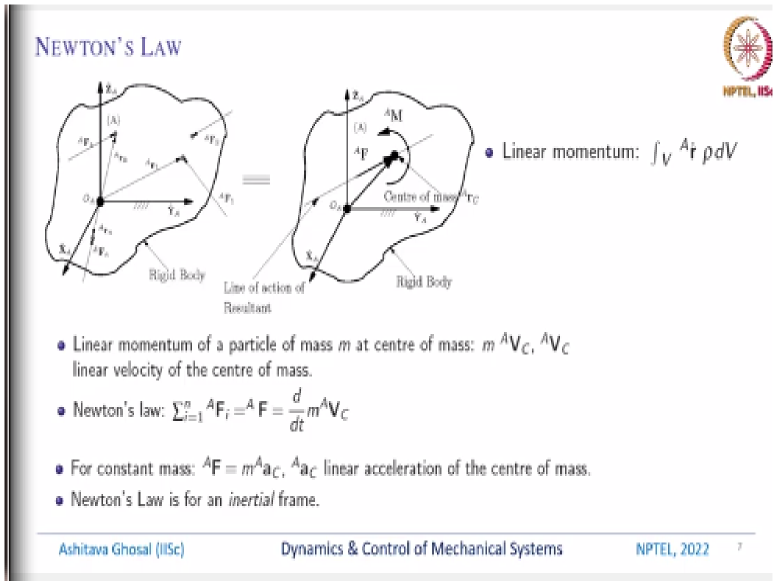

So, let us continue. So, a very important notion or use or to obtain the equations of motion is Newton's law. So, let us briefly recapitulate what is Newton's law. So, basically, we have a rigid body and let us say there are these external forces AF_1, AF_2 and so on which are acting in this rigid body at these different points. So AF_1 is acting at Ar_1, AF_2 is acting at some other point and so on. And what are these AF_1, AF_2? They are the external forces.

Last week we had looked at different kinds of external forces, say gravity, spring, damping and so on. So, if you have a rigid body subjected to these n external forces which are acting at n points on the rigid body we can obtain a equivalent description of all these forces on this rigid body. As that there is a resultant force AF and there is a resultant moment which is acting about the Cg or the center of mass.

So, these 2 pictures are equivalent instead of having n forces we can have a single force and a single moment about the center of mass. The linear momentum of this rigid body can be obtained as volume integral of $\int_V {}^A\dot{r}\ \rho dV$. What is ${}^A\dot{r}$? it is the velocity of the center of mass or velocity of some other point, most of the time we will use center of mass. The linear momentum of a particle of mass m at the center of mass is given by m into velocity of the center of mass.

So, velocity of the center of mass with respect to the A is denoted by ${}^AV_C$ and it is the linear velocity of the centre of mass in the A coordinate system and this C is the center of mass. Newton's law states that the $\sum_{j=1}^{n} {}^AF_i$ which is the same as this ${}^AF$ is nothing but $\frac{d\left(m\ {}^AV_C\right)}{dt}$. So, remember m into ${}^AV_C$ is the linear momentum. So, Newton's law says that this resultant force is equal to mass times a time derivative of the linear momentum.

For constant mass we can take this m outside this derivative and what we have is the force is equal to m times acceleration, this is very well known standard expression of Newton's law. That the force which is acting on a rigid body will result in an acceleration which is denoted by F equals m ${}^Aa_C$. And what is this ${}^Aa_C$? It is the linear acceleration of the center of mass. One important thing to remember is Newton's law this expression is valid for an inertial frame.

Meaning, that this A coordinates system must be an inertial frame and if you go back to your undergraduate or even high school physics. There is no such thing as a really as an inertial frame. So, inertial frame is basically something which is in a simplistic way which is not rotating. So, what is an inertial frame? A good approximation of an inertial frame is the lab or the surface of the earth.

But we know that the earth is rotating, it is spinning about it is axis, so it is not strictly an inertial frame. So, if you think that the center of the earth is an inertial frame that also is not true because the earth is going around the sun and so on. So, although there is strictly no such thing as a inertial frame but Newton's law is valid only for an inertial frame. And we have reasonably good approximation of inertial frame when we say that the lab or the floor of the lab is an inertial frame.

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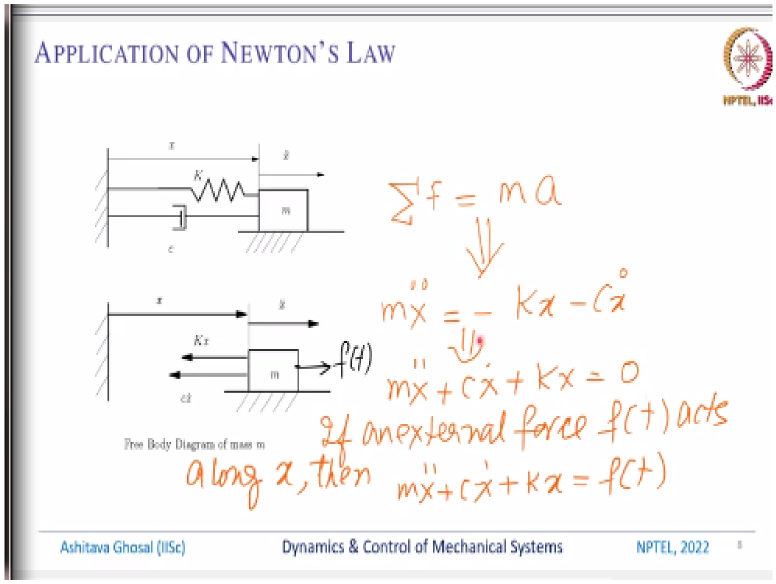

So, let us apply this Newton's law and we will take a very, very simple example. So, what we have is a mass which is connected to a fixed ground by means of a spring K and also a damping element which damping is given by this c. Remember, one way to measure the force exerted by a damping element is something like $c\,\dot{x}$. So, the mass is at a distance x and it is accelerating to the right by $\ddot{x}$.

So, Newton's law says that the sum of all these forces is equal to m times a. So, what is a? it is $m\ddot{x}$ and what are the forces which are acting on this mass? So, what you can see is that the K x which is going in the opposite direction to x dot and there is a $c\,\dot{x}$. So, what we have is $m\ddot{x} = -\,kx\,-\,c\dot{x}$. And this we can rewrite it as $m\ddot{x} + kx\,+\,c\dot{x}$. = 0.

It is also possible that there is somebody pulling this mass, so maybe there is some string attached and you are pulling it or maybe there is an external force of some kind which is acting on this mass and let us denote this external force by $f(t)$, it could be time varying. So, then the equation of motion of this mass is $m\ddot{x} + kx\,+\,c\dot{x}$. = $f(t)$, this is a very, very standard equation which is used in vibration and everyone must have seen this.

And so basically what we are doing is, we are applying Newton's law which is sum of all forces is equal to ma and we get this equation.

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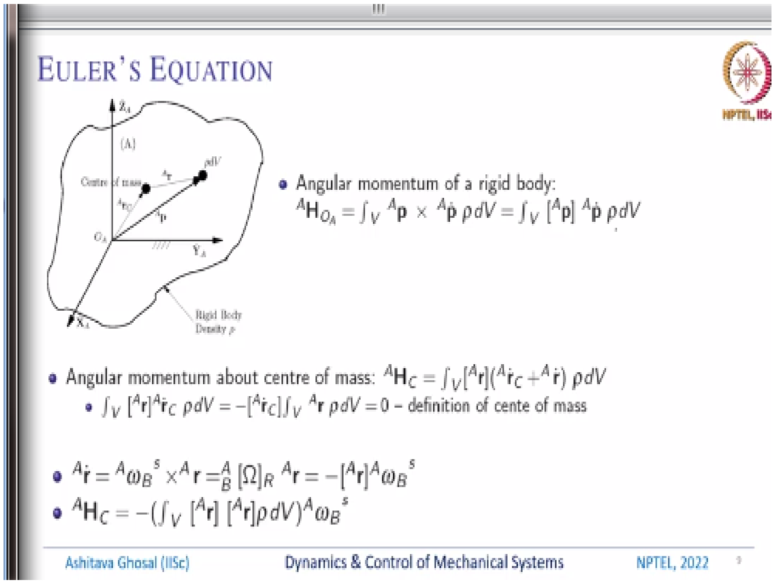

We can also look at what is something called as an Euler's equation. To study, look at Euler's equation we again go back to this picture of a rigid body, we have a $\int_V \rho dV$ which is located at a point Ap. The center of mass of this rigid body is located at this point and is located by this vector Ar_C. And from the center of mass to this volume element it is Ar and all of these are with respect to a reference or a fixed coordinate system $\hat{X}_A$ $\hat{Y}_A$ $\hat{Z}_A$ with origin O_A.

So, the angular momentum of this rigid body is defined as this $\int_V Ap \times \dot{Ap}\, \rho\, dV$, so what is $\dot{Ap}$? That is the velocity of this volume element. And then we take the cross product of this velocity vector, so the velocity vector will be in some direction with this Ap, so it is like $r \times mV$. So, this is like m this is like V, so this is $r \times mV$ and again we integrate over this volume element.

And just like other vectors we need to define this angular momentum with respect to which point in this case it is with respect to O_A and in which coordinate system? In this case it is with respect to the A coordinate system. So, now as you can see there is a cross product involved and we can write this cross product as a $[\,Ap]$ into the rest of it which is $A\dot{p}\ \rho\ dV$.

So, instead of writing $r\times\ mV$ we can write a skew symmetric matrix into mV as this form, this is perfectly reasonable, it helps later on we will see when we write it in this form. So, the angular momentum about the center of mass instead of about O_A can be written as AH_C, again A means reference coordinate system A, and C means about the center of mass is again we can write this vector as $\int_V\ [\,A\,r](\,A\dot{r}_C\ +\ A\dot{r}\,)\rho dV$.

Because $A\dot{p}$ can be written as the derivative of $A\dot{r}_C\ +\ A\dot{r}$. So, this $A\dot{p}$ is written in this form and then if you integrate you can show that the $\int_V\ [\,A\,r]\ A\dot{r}_C$ is given by $-\ A\dot{r}_C$ because it is a cross product of 2 vectors and you can change the order. So, like A cross B we can write as -B cross A, so we can take out this $A\dot{r}_C$ with a minus sign and then we are left with the $\int_V\ \ Ar\ \ \rho dV$.

So, now this $Ar\ \rho dV\ =\ 0$, because that is the definition of center of mass. So, as you can see using this nice way of using a cross product as a skew symmetric matrix we can show that this part of the term $[\,A\,r]\ \ A\dot{r}_C$ is 0. So, this $[\,A\,r]\ \ A\dot{r}_C$ is 0, so we are left with only one more term. So, what is $A\dot{r}$ which is this part here? This $A\dot{r}$ is nothing but $A\omega_B^s\times\ A\,r$ and of course we have to be very careful when we are taking cross product.

And also which omega we are interested in, in this case we are interested in the space fixed omega which is $[\dot{R}]\ \ [R]^T$. So, $A\dot{r}$ is nothing but $A\omega_B^s\times\ A\,r$ and which is remember the space fixed angular velocity vector can be written in terms of a skew symmetric matrix $BA[\Omega]_R\ \ A\,r$.

So, again instead of writing $A\omega_B^s \times\ A\,r$ we can write $-\ [A\,r]\ A\omega_B^s$ because if you change the order of the cross product we have to put a minus sign.

And hence we can write AH_C as minus this $A\,r$ is here, so we are rewriting $A\dot{r}$ as $-\ A\,r$ into $A\omega_B^s$. And then $[A\,r][\ A\,r]\rho dV$ and we can take out $-\left(\int_V [A\,r][\ A\,r]\rho dV\right)$ outside. So, what do we have? That the angular momentum about the center of mass of this rigid body is a volume integral which contains the square of this 2 skew symmetric matrix basically it is like r square, where r is a vector from the center of mass to a volume element and then into angular velocity.

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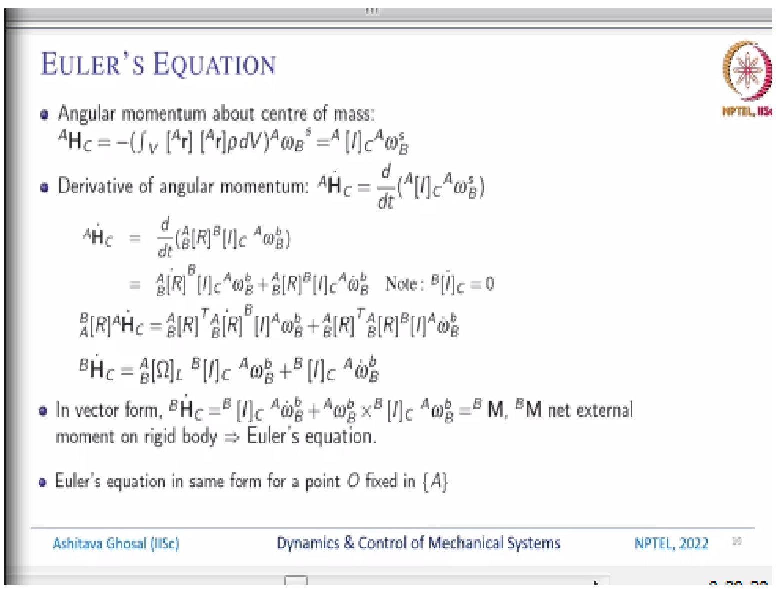

So, what do we do with this angular velocity? We can rewrite this angular momentum and angular velocity as AH_C just from this previous slide can be written in terms of the inertia matrix. So, if you go back and see the definition of the inertia matrix this $-\left(\int_V [A\,r][\ A\,r]\rho dV\right)$ is the inertia matrix about the center of mass in the reference coordinate system A into $A\omega_B^s$.

So, the angular momentum is $A\left[I\right]_c \; A\omega_B^s$ which is what we have seen in undergraduate. So, if you take the derivative of this angular momentum which is AH_C which is derivative of both of these and then you can use the chain rule. So, we can see that this derivative $A\left[I\right]_c \; A\omega_B^s$ this can be written in 2 ways. One is we can go back to this $A\omega_B^s$ and we can pre multiply by a rotation matrix and $A\omega_B^b$ in the body fixed coordinate system.

And then we also have $A\left[I\right]_c$ that can be converted to the body fixed as some $BA[R]$ $B\left[I\right]_c \; BA[R]^T$. So, those of you who have if you go back and see the notes you can see that the inertia matrix in the B coordinate system and in the A coordinate system are related by $BA[R] \; B[I]BA[R]^T$. And that $BA[R]^T$ into this will give you the body fixed angular velocity vector.

So, whatever these 2 terms in the bracket can be written as some rotation matrix into inertia matrix in the body coordinate system and body fixed angular velocity vector. Why are we doing this? Because this inertia matrix in the body coordinate system does not change with time, it is a number, it is a bunch of numbers only, it is constant. So, when you take the derivative of this we will get some $BA[\dot{R}]$ into this $B[I] \;\; A\omega_B^b \; + \; BA[R] \;\; B\left[I\right]_c \; A\dot{\omega}_B^b$.

Because the derivative of this term will be 0 into $A\dot{\omega}_B^b$, derivative of the angular velocity, so this is like angular acceleration. The third term in the chain rule is derivative of $B[I]_C$ this one will be 0, so not $B[\dot{I}]_C$ will be 0. So, what do we have here that the rate of change of angular momentum is given by this expression. So, let us pre multiply both left and right side by $AB[R]$, so again we are basically converting it into another coordinate system.

So, if you pre multiply $A\dot{H}_C$ with $AB[R]$, we will get some $BA[R]^T$, why $BA[R]^T$? Because B A R is same as $BA[R]^T$. This is the inverse of $BA[R]$ which is same as the transpose. And hence we have $BA[R]^T\ B\dot{A}[R]\ B[I]\ A\omega_B^b + BA[R]^T\ BA[R]\ B[I]\ A\dot{\omega}_B^b$, so this is like angular acceleration into some matrix.

See, note that these 2 will become identity, so this is like $[R]^T[R]$. How about this? This is $[R]^T\ [\dot{R}]$. So, this is like again angular velocity but the body fixed angular velocity. If you go back and see your notes, the space fixed angular velocity is $[\dot{R}]\ [R]^T$, whereas the body fixed angular velocity is $[R]^T\ [\dot{R}]$. So, hence the rate of change of the angular momentum in the B coordinate system can be written in terms of a skew symmetric matrix which is $BA[\Omega]_L$.

So, this is the left multiplication or we will see later it is the body fixed angular velocity vector into some inertia matrix into angular velocity plus this inertia matrix in the body coordinate system into $A\dot{\omega}_B^b$ again the rate of change of the body fixed angular velocity vector. So, what is this? This is like some kind of a skew symmetric matrix, this part is like a vector, it is like some matrix into a vector which will give you a vector and this is like a skew symmetric matrix.

So, this is like some vector cross this, so this is the cross product of these 2 this part and this part together and this is some matrix into angular acceleration. So, hence in vector form this above expression can be written as the rate of change of the angular velocity vector about angular momentum about the centre of mass in the B coordinate system is something like $B[I]_c\ A\dot{\omega}_B^b$, this is from this part and this part is $A\omega_B^b \times\ B[I]_c\ A\omega_B^b$.

So, this is very well known for Euler's equation. And what is the Euler's equation states that the rate of change of the angular momentum is equal to the net external moment acting on the rigid body. So, this part from here which is something like I alpha plus omega cross I omega is equal to the external moments. And all of these are written in the B coordinate system that is the important part.

The Euler's equation is also valid instead of about the center of mass we form the equation for a point O which is fixed in A. So, we do not need to write the Euler equation about for the center of mass but we can also write about the any other point which is fixed in A, remember A is the reference coordinate system and we will use this fact very shortly.

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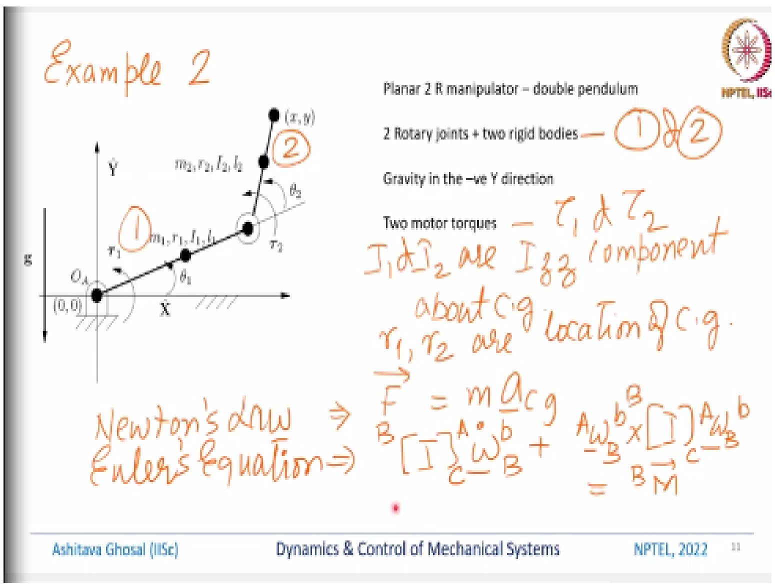

So, let us look at an example. So, this is an example of a 2 degree of freedom system. So, basically it is a plane R, 2 R manipulator, it is very well known in robotics. It is also a double pendulum basically I have drawn it this way but if I had drawn it vertically hanging down it would be a double pendulum. So, there are 2 rotary joints, this one joint here, another other joint here and there are 2 rigid bodies, this is one rigid body, this is another rigid body.

So, between the fixed ground and this first link or the first rigid body is one rotary joint and between the first link and the second link there is another rotary joint. The gravity is acting downwards, this is the origin of the fixed coordinate system $\hat{X}$, $\hat{Y}$, O_A, (0, 0). And we have for each link the following parameters. So, we can have a mass of this link, so m_1, the location of the CG of the link is given by r_1.

And in this example we will assume that it is along the link, it need not be on the link but for simplicity and simple calculations we say this is r_1. We also have the z component of this inertia which is I_1. Remember, this is a planar motion, so I_{xx} and I_{yy} they will not appear, I will show you that also. So, I_1 and I_2 are I_{zz} component about the CG and r_1 and r_2 are the location of the CG from this O_A and as well as from this fixed point here, not fixed point the second origin of the second link which is here.

So, and this is the end of this link which is (x, y). The first link is rotating by θ_1 with respect to the x axis; the theta 2 is with respect to the reference coordinate system which is along this line, so this is a relative joint rotation. Likewise we have a torque which is acting on this first link. So, maybe there is a motor here if it were a robot, there would be a motor here which would be rotating this link and it would be giving a torque τ_1, similarly if there is a motor here it will give you a torque τ_2.

So, we want to find the equation of motion for this very, very simple planar_2 R manipulator or a double pendulum. So, what do we do? We will apply Newton's law which is force equals mass times acceleration and Euler's equation which is that $B\left[I\right]_c \; A\dot{\omega}_B^b + A\omega_B^b \times \; B\left[I\right]_c \; A\omega_B^b$ is the net external moment. And remember we are doing some matrixes and vectors, so they better be all in the right coordinate system.

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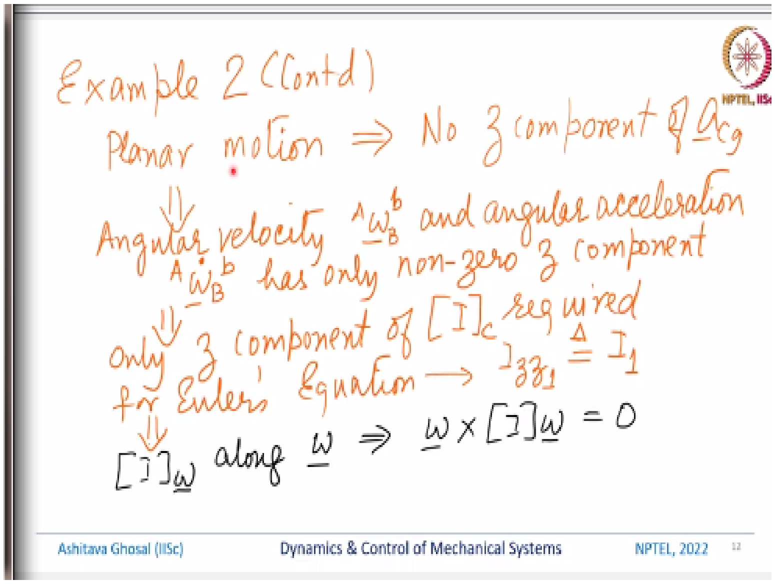

So, let us continue, remember as I said it is a planar motion, so there is no z component of the acceleration, so it is only x and y component. Likewise the angular velocity $A\omega_B^b$ and the angular acceleration time derivative $A\dot{\omega}_B^b$ has only non-zero z component. It is coming out of the page the motor axis and that is what is giving the $\dot{\theta}_1$ and $\ddot{\theta}_1$. So, hence if you have only $A\omega_B^b$ and $A\dot{\omega}_B^b$ with only z non-zero z component then only the z component of the inertia matrix is required.

Because I_{xx}, I_{yy} and the terms which multiply omega x, omega y they will go to 0 because omega x, omega y is 0. So, we are only interested in the I_{zz} into the omega z part. So, for hence for Euler equation we need only the z component which is I_1 for link 1 and I_1 for link 2. Moreover this vector I ω will be along ω because this is a body fixed inertia matrix with only the z component, this omega is also only the z component, so it will be like theta or_1 dot.

So, I_1 into $\dot{\theta}_1$ will be along omega only, because you are multiplying the angular velocity which is along the z axis into some scalar which is I_1. So, hence $\omega \times [I]\ \omega$ will be 0. So, in the

Euler's equation we had $[I]\dot{\omega} + \omega \times [I]\omega$ equals the net external moment, so $\omega \times [I]\omega$ term will be 0.

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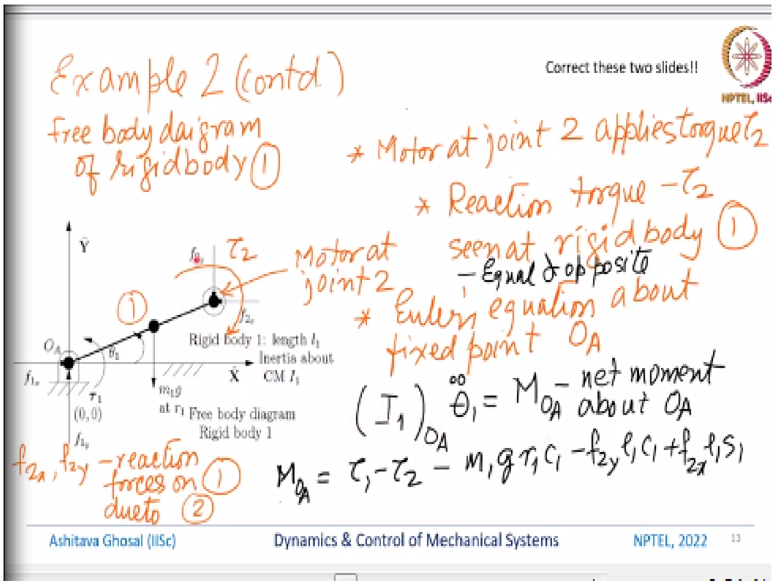

So, let us continue in the Newton-Euler formulation or when we are trying to use Newton's law and Euler's equation we have to draw what is called as a free body diagram. And what is a free body diagram? We just take a look at each link at a time, so this is link 1. So, when you draw the free body diagram you are basically breaking the link or removing this joint. So, if you look at this joint what is happening is there will be some reaction forces from the ground, so there is an f_{1x} and an f_{1y}.

Likewise when you break the second link away from the first link or remove the second link from the first link we have to say that there is a reaction force or a force coming from the second link on to the first link. So, this is f_{2x} and f_{2y}, so these directions are chosen randomly, you could have chosen in some other way. So, f_{2y} is this way and f_{2x} is this way, it could have been the other way round but does not matter it will get automatically adjusted.

Because we will see for the second link when we draw the free body diagram we will have to also assign a force f_{2y} and an f_{2x} equal and opposite in the opposite direction, this is as far as forces is concerned. If you have a motor at this joint then the motor is applying a torque to the second link, so there will be a reaction torque equal and opposite in the opposite direction. So, if you go back and see the previous slide there is a anti-clockwise torque which is acting on the second link.

Hence there is a clockwise stock τ_2 which is acting on the first link. So, this is equal and opposite to the torque which is acting on the second link. Now we can apply Euler's equation about the fixed point O_A, remember Euler's equation can be applied either about the CG or about a fixed point. So, in this case it is smarter to apply about O_A because if you apply the moments or Euler's equation about this fixed point the effect of f_{1x} and f_{1y} will not show up because there is no moment about these forces.

We have to worry about what is f_{2x} and f_{2y}. So, what does Euler's equation about fixed point O_A tells me? That $I_1\ \ddot{\theta}_1$ about O_A, so now we have to find the inertia about O_A is equal to the external moment which is acting about O_A and what this M_{OA}? So, this is τ_1 which is acting there is a τ_2 which is in the opposite direction, so it is τ_1 - τ_2 and then we have this moment due to this force which is acting at the CG.

So, this is $m_1\ g$ gravity is acting this way, so hence we have $-m_1\ g\ r_1\ c_1$. So, remember this is the moment term this way and this is -$f_{2y}\ l_1\ c_1$ because f_{2y} is also acting this way and this distance is l_1, this is cos θ_1, so this is the moment term. So, we have these 2 terms minus because they are clockwise, whereas this τ_1 is anti-clockwise, so we are assuming anti-clockwise is positive like anywhere else.

And then we have this f_{2x} into this moment term and what is this moment term? This l_1 sin θ_1, the vertical about distance is sine θ_1. So, the external moment is τ_1 - $\tau_2 - m_1 g\, r_1\, c_1$-$f_{2y}\, l_1\, c_1$+ -$f_{2x}\, l_1\, s_1$, where f_{2x} and f_{2y} are reaction forces on link 1 due to link 2.

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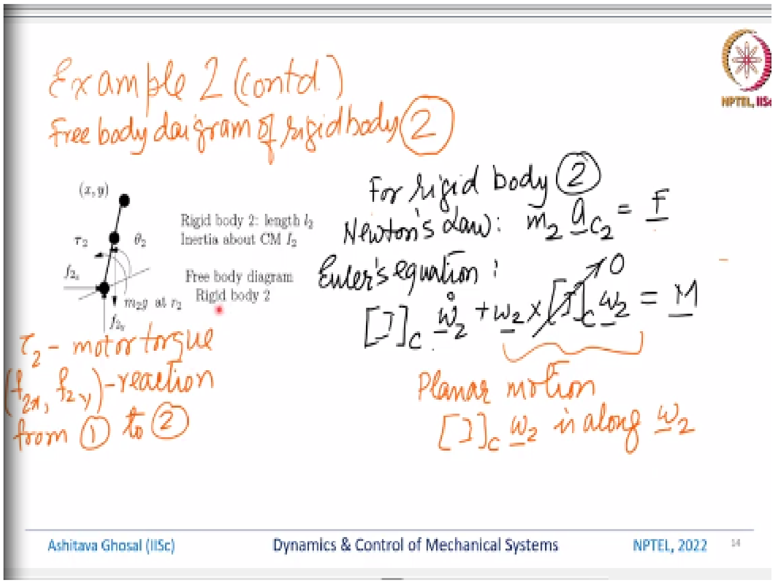

Next we draw the free body diagram of the second link. So, what do we have here? We have f_{2x}, remember f_{2x} was acting in the opposite direction for link 1, so here it is acting in the x direction and f_{2y} is in the y direction. The gravity is a force which is m_2 g which is acting at the CG. So, for rigid body 2 we can apply Newton's law which is m_2 into a_{c2} = F, the net external force.

And we can also apply the Euler's equation above the Cg. Now in this case we do not have any fixed point but we can apply Euler's equation about the Cg. Which is that I about the CG into alpha which is $\dot{\omega}_2 + \omega_2 \times [I]\, \omega_2$ = M. Now this is 0 because why? Because ω_2 is coming out of the page $[I]_c \omega_2$ and ω_2 they are in the same direction.

So, something like 2 vectors cross product of 2 vectors which are in the same direction is 0,. So, this is a planar motion, so $[I]_c\ \omega_2$ is along ω_2. So, we can use Newton's equation, we can use Euler's equation. So, why did we not use Newton's equation earlier because we were taking the

Euler's equation about the fixed point and there is no motion of that fixed point that does not have any acceleration?

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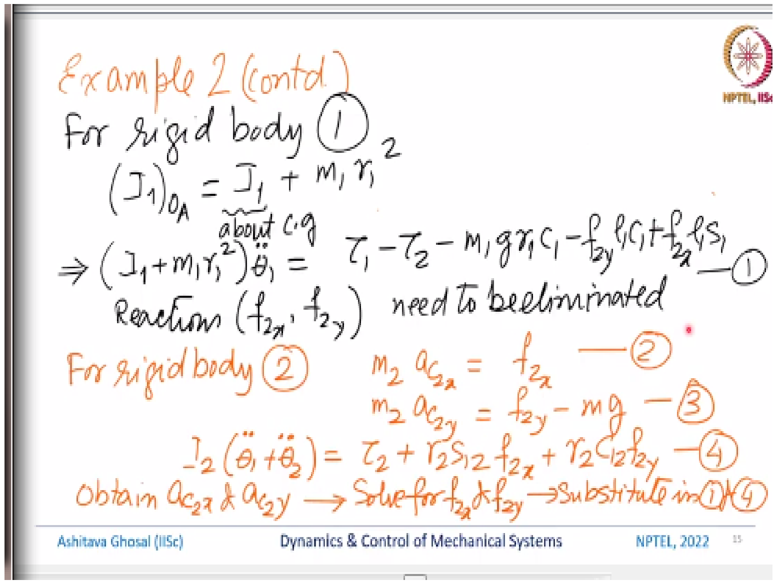

So, for rigid body 1 the inertia matrix about O_A is nothing but I_1 which is the inertia matrix about the CG + $m_1 r_1^2$, so this is parallel axis theorem. And then we can rewrite that equation the Euler's equation as($I_1 + m_1 r_1^2$) $\ddot{\theta}_1$ is right hand side remains same $\tau_1 - \tau_2 - m_1 g r_1 c_1 - f_{2y} l_1 c_1 + f_{2x} l_1 s_1$. So, these reactions f_{2x} and f_{2y} we do not know, so we need to somehow get rid of them, they need to be eliminated and that can be done by considering rigid body 2.

So, we know that from Newton's law $m_2\, a_{c2x}$, so which is nothing but the x component of the acceleration vector is f_{2x} which is the x component of the force which is acting on the second link. Similarly $m_2 a_{c2y} = f_{2y} - mg$, remember there is a force in the negative y direction which is the weight of the second link. And we also have the Euler's equation for the second link which is I_2 into $\ddot{\theta}_1 + \ddot{\theta}_2$.

Remember, θ_2 is relative to θ_1, so the angular velocity and the acceleration will be $\dot{\theta}_1 + \dot{\theta}_2$ and $\ddot{\theta}_1 + \ddot{\theta}_2$, respectively. That will be equal to τ_2 which is the torque acting and the moment arms which are $r_2 s_{12} f_{2x} + r_2 c_{12} f_{2y}$. So, from these 2 equations 2 and 3 if I know what is a_{c2x} and a_{c2y}, so the x and y component of the acceleration of the center of mass of the second link, I can find out what is f_{2x} and f_{2y}.

Then I can substitute f_{2x} and f_{2y} in this equation and also I can substitute τ_2 in this equation and then we can simplify and get rid of f_{2x} and f_{2y}. So, we can eliminate f_{2x} and f_{2y} if I know what the x and y component of the acceleration of the center of mass of the second link is.

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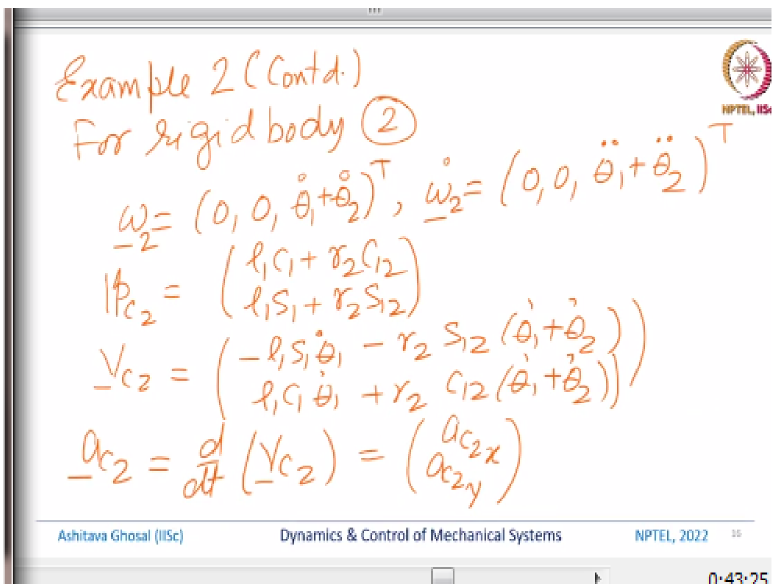

So, let us continue. So, for the rigid body 2 we know the angular velocity is nothing but (0, 0, $\dot{\theta}_1$ + $\dot{\theta}_2$), this is obvious, however angular acceleration it is 0, 0, $\ddot{\theta}_1 + \ddot{\theta}_2$ both are vectors along the z axis. The position vector of the second link the CG of the second link is $l_1 c_1 + r_2 c_{12}$ and the y component is $l_1 s_1 + r_2 s_{12}$ which is nothing but simple vector.

So, you go from the origin O_A to the end of the first link and then again you go from there by r_2 to the CG of the second link. The velocity of this center of mass of the second link is nothing but the time derivative of this, so we use chain rule. So, derivative of l_1 c 1 is $- l_1 s_1 \dot{\theta}_1$ and the derivative of this is $-r_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2)$. So, the y component will have $l_1 c_1 \dot{\theta}_1$ and $-r_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2)$. How about acceleration?

You can take the derivative of this velocity of the center of mass and these are the x and y component of the acceleration of the center of mass, so this is shown next.

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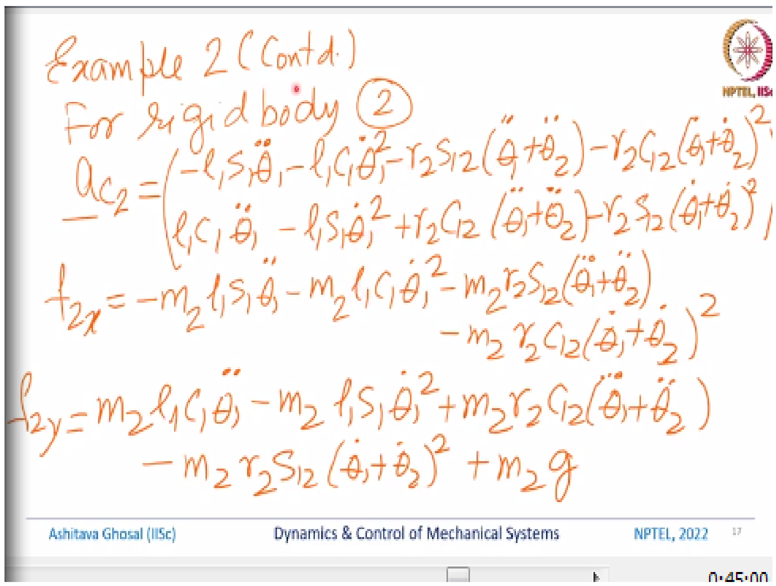

So, let us continue, so for rigid body 2 we find that the acceleration of the center of mass is the derivative of the velocity vector. So, the velocity vector are 2 terms, so now when you take the derivative and use chain rule you will have 4 terms. So, the x component for example is

$$-I_1 s_1 \ddot{\theta}_1 - I_1 c_1 \dot{\theta}_1^2 - r_2 s_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - r_2 c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2.$$

So, s_1 means $sin\theta_1$, c_1 means $cos\,\theta_1$, s_{12} means $sin(\theta_1 + \theta_2)$and c_{12} means $cos(\theta_1 + \theta_2)$. The y component will have $-I_1 c_1 \ddot{\theta}_1 - I_1 s_1 \dot{\theta}_1^2 - r_2 c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - r_2 s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2$. So, once we have the acceleration of the center of mass of the second link f_{2x} is nothing but m_2 into the x component.

So, we have $m_2\, l_1\, s_1\, \ddot{\theta}_1\ -\ m_2\, l_1\, c_1\, \dot{\theta}_1^2$ and so on. How about f_{2y}? That is m_2 into the y component of the acceleration + $m_2\, g$ because there is a weight also acting. So, we can find expressions for f_{2x} and f_{2y}, once we know what the acceleration of the center of mass of the second link is.

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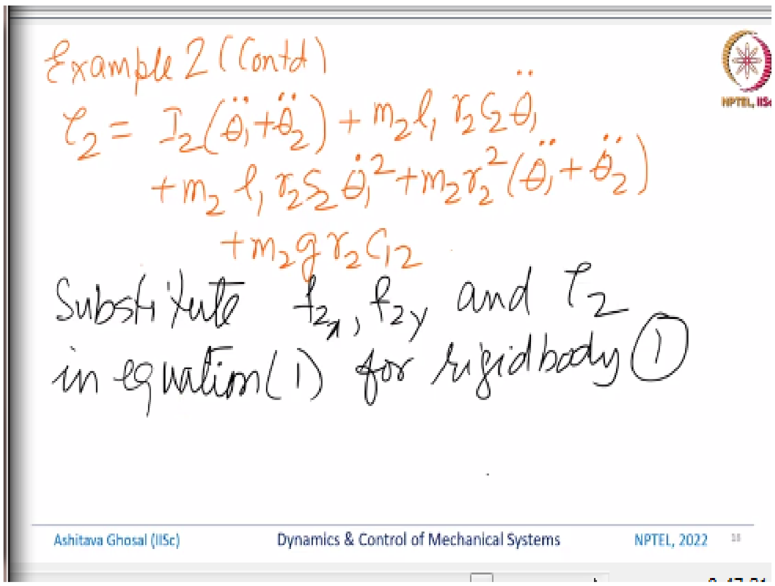

So, next from the Euler's equation we can show that τ_2 is nothing but $I_2\,(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2\, l_1\, r_2\, c_2\, \ddot{\theta}_1 +\ m_2\, l_1\, r_2\, s_2\, \dot{\theta}_1^2 + m_2\, r_2^2\, (\ddot{\theta}_1 + \ddot{\theta}_2)$. And we also have this gravity term which is $m_2\, g\, r_2\, c_{12}$. So, what is the next step? We substitute f_{2x} and f_{2y} which we obtained in the previous slide and τ_2 which is here in equation 1 for the rigid body 1.

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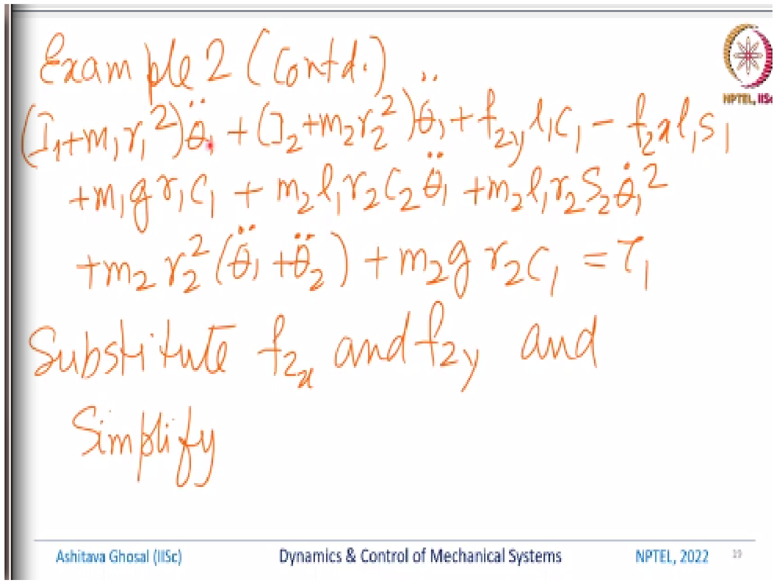

So, if you substitute, so what we can show that you will get some term which is $(I_1 + m_1 r_1^2)\ddot{\theta}_1$ + I_2 + $m_2 r_2^2$ $)\ddot{\theta}_1$ and then we have this previous terms f_{2y} and f_{2x} and so on. So, we will substitute f_{2x} and f_{2y} and simplify, so f_{2x} is a long term remember it is m_2 a_{c2x} and f_{2y} is m_2 a_{c2y} + mg. So, this becomes very long that is why I have not yet fully substituted but you can do it, you can substitute and then simplify.

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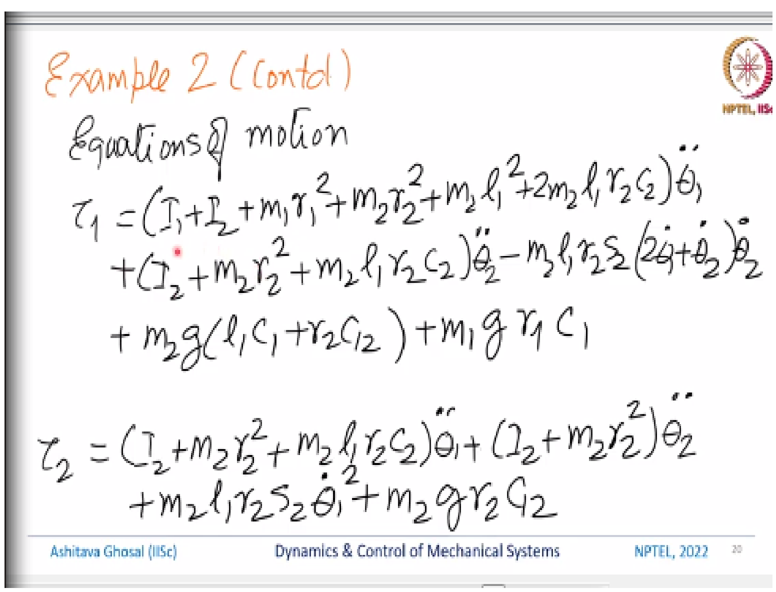

So, once we substitute f_{2x} and f_{2y} which were nothing but $m_2\ a_{c2x}$ and f_{2y} was $m_2\ (a_{c2y} + mg)$. And then simplify everything together we will get these 2 equations of motion, let us go over it term by term. So, τ_1 is the torque which is acting on link 1 and it is given in terms of $(I_1 + I_2 + m_1 r_1^2 + m_2 r_2^2 + m_2 l_1^2 + 2m_2 l_1 r_2 c_2)\ \ddot{\theta}_1$.

So, what is I_1? I_1 is the z component of the moment of inertia of link 1 about the CG. I_2 is the z component of the moment of inertia of link 2 about the CG, so this is I_1 is I_{zz1}, $I_2\ is\ I_{zz2}$. $m_1 r_1^2$ is nothing but $I_1 + m_1 r_1^2$ is like parallel axis theorem. So, we find the moment of inertia about the fixed point O_A likewise for I_2 and $m_2\ r_2$ square. And this $m_2 l_1^2$ is because the second link is at a distance l_1 interesting term is this one.

So, this is $2m_2\ l_1\ r_2\ c_2$, so this is something like an equivalent moment of inertia as seen by joint 1. So, this is $\ddot{\theta}_1$ is like $\dot{\omega}\ or\ \alpha$. But there is a term in the inertia which is multiplying $\dot{\theta}_1$ which is the function of the second rotation angle theta 2, c_2 is θ_2. Likewise, we can have another term which is multiplying $\ddot{\theta}_2$ which is $I_2 + m_2 r_2^2 + m_2 l_1 r_2 c_2$.

So, again we can see one term which is a function of θ_2 then we have this Coriolis and centripetal type of term which is $m_2 l_1 r_2 s_2 (2\dot{\theta}_1 + \dot{\theta}_2)\ \dot{\theta}_2$. So, this is like $\dot{\theta}_2^2$ and this is like 2 $\dot{\theta}_1 \dot{\theta}_2$. So, $\dot{\theta}_2$ is like centripetal and 2 $\dot{\theta}_1 \dot{\theta}_2$ is like coriolis and then we have this effect of gravity. So, the gravity comes in 2 terms which is $m_2\ (g\ l_1\ c_1 + r_2\ c_{12}) + m_1 g r_1 c_1$.

The second equation of motion is τ_2 which is the torque given by second motor on the second link. So, that is given by $(I_2 + m_2 r_2^2 + m_2 l_1 r_2 c_2)\ \ddot{\theta}_1 + (I_2 + m_2 r_2^2)\ \ddot{\theta}_2 + m_2 l_1 r_2 s_2 \dot{\theta}_1^2 + m_2 g r_2 c_{12}$, c_{12}

stands for $cos(\theta_1 + \theta_2)$. So, as you can see there are many, many terms and we have to be very careful while doing this substitution and simplification.

At the end of this module I will show you one way of getting error free equations of motion using computer tools. But for the moment let us assume that this has been done by hand carefully and we get these 2 equations of motion which are second order because second derivatives $\ddot{\theta}_1$, $\ddot{\theta}_2$ is there and there are 2 of these.

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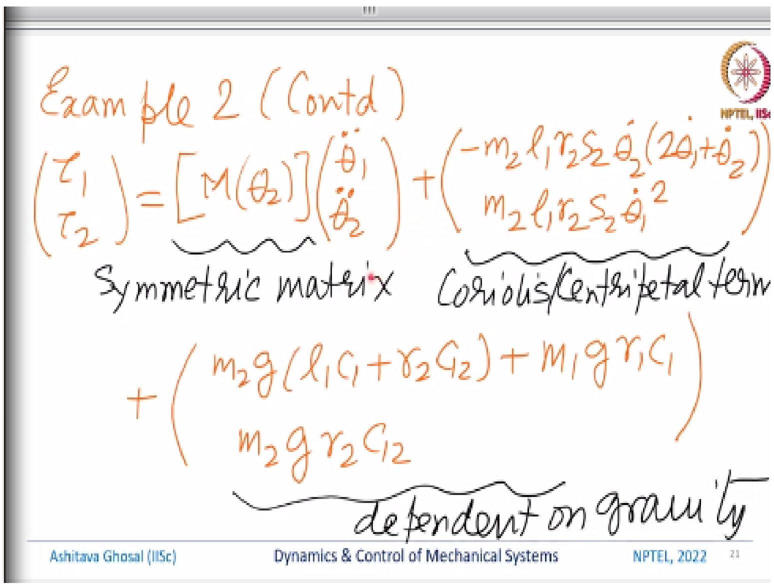

So, let us continue. So, we can reorganize those 2 equations of motion as in this form. So, we have τ_1 and τ_2 which are the external torques given by these 2 motors. One matrix which I am going to call as M of theta 2, note it is only a function of θ_2. Because remember in the terms multiplying $\ddot{\theta}_1$ and $\ddot{\theta}_2$ some $cos\,\theta_2$ was there, θ_1 was not there.

And then we have this vector which is $-m_2 l_1 r_2 s_2 \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2)$ and the y component or the second component is $m_2 l_1 r_2 s_2 \dot{\theta}_1^2$. And similarly we have a gravity term which is $m_2 g(l_1 c_1 + r_2 c_{12}) + m_1 g r_1 c_1$ and for the second equation it is $m_2 g r_2 c_{12}$. So, it turns out and you can see carefully that this matrix which is also called the mass matrix is always symmetric which means m_{12} is same as m_{21}; this is a 2 by 2 matrix, so this is a 2 by 1 vector.

So, hence this one is a 2 by 1 vector. So, we have this as a 2 by 2 matrix and then this is a symmetric matrix, so these 2 terms are the coriolis and the centripetal term. Because $\dot{\theta}_2^{\,2}$ is sort of like the centripetal term and $\dot{\theta}_1$, $\dot{\theta}_2$ is like the coriolis term, so this is a coriolis term $\dot{\theta}_1^{\,2}$. And these 2 terms here the $m_2 g$ and $m_1 g$ you can see that they do not contain theta dots or theta double dots.

They are only functions of θ_1 and θ_2 which is the configuration of this 2 degree of freedom system this planar 2R manipulator.

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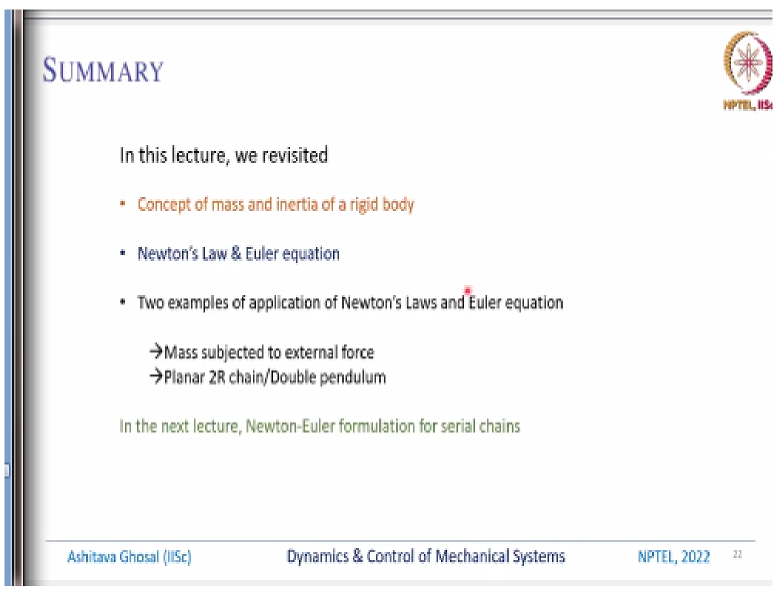

In this lecture we revisited the concept of mass and inertia of a rigid body. I introduced the Newton's law and Euler's equation of motion for a rigid body. I showed you 2 examples of application of Newton's law and Euler's equation. The first example was that of a mass subjected to a force acting along the x direction and we saw what happens to the motion of the mass. The second example was little bit more complicated it was a planar 2R chain or a double pendulum.

And I showed you how using basic mechanics and the concept of free body diagram we could derive the equations of motion. In the next lecture an algorithmic form of the Newton-Euler's equation called the Newton-Euler formulation for serial chains will be discussed.