Sound and Structural Vibration
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Lecture – 5
Classical Problem Continued

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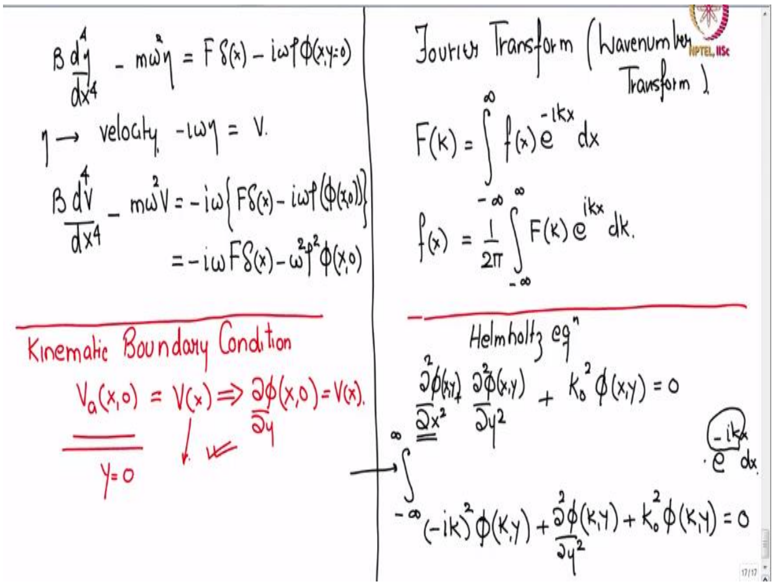

Welcome to this next lecture on sound and structural vibration. Last class we left off at the kinematic boundary condition. So, that is given by this equation which equates the acoustic particle velocity at the surface to the velocity of the plate. Now, we need a Fourier transform definition, we are going to do this in the Fourier domain. When it is used in the spatial form, not temporal form, it is also called a wave number transform.

So, I will give you the definition we are going to use. The forward transform

$$\mathrm{F(k)} = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx,$$

this is the forward transform and the inverse

$$\mathrm{f(x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} \mathrm{dk},$$

this is the pair that we are going to use.

So, now if you recall the Helmholtz equation, the time removed wave equation Helmholtz equation was

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} + {k_0}^2 \phi(x,y) = 0.$$

This is the Helmholtz equation for the acoustic half space. And so now, we are going to apply the wave number transform which is an integral from $-\infty$ to ∞ and we are going to have e^{-ikx} dx.

So, it works on the x and when you have a derivative you should know that this $-$ik will fall out twice. So, what happens is the result is

$$(-\mathrm{ik})^2 \phi(\mathrm{k},\mathrm{y}) + \frac{\partial^2 \phi(\mathrm{k},\mathrm{y})}{\partial \mathrm{y}^2} + \mathrm{k}_0^2 \phi(\mathrm{k},\mathrm{y}) = 0,$$

that is the resulting equation after wave number transform.

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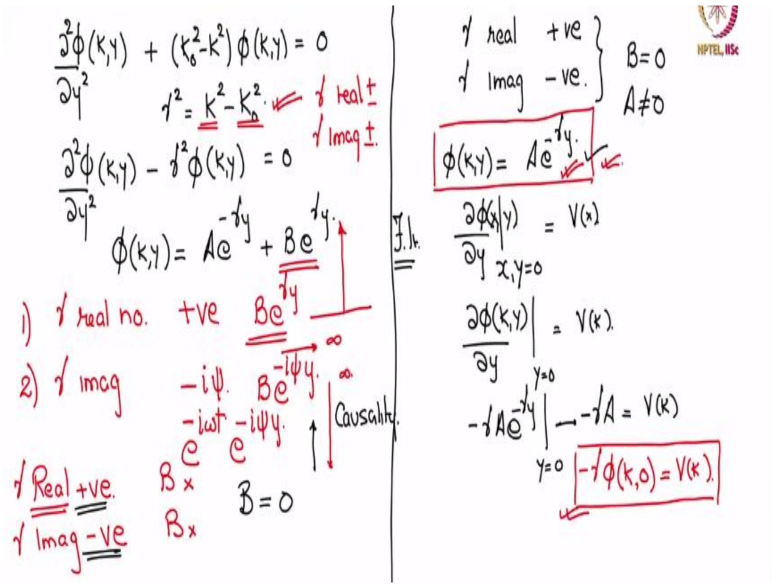

So if we write it what do we get? We get

$$\frac{\partial^2 \phi(k,y)}{\partial y^2} + \left({k_0}^2 - k^2\right)\phi(k,y) = 0.$$

So, I am going to define a variable γ^2 which is actually $k^2 - {k_0}^2$ so that it becomes

$$\frac{\partial^2 \phi(k,y)}{\partial y^2} - \gamma^2 \phi(k,y) = 0.$$

This is somewhat like the spring mass system except with a minus over here, so what is the solution to this?

Solution to this is $\phi(k,y) = Ae^{-\gamma y} + Be^{\gamma y}$. Now because γ^2 has this nature of a difference at times k can be bigger than k_0, at times k can be less than k_0 so in which case γ could be a real number positive or negative, gamma could be an imaginary number positive or negative. Now, these have implications.

If γ happens to be a real number and we choose the positive value, then look at the B term, $Be^{\gamma y}$ and y is going off to infinity from plate and therefore this term will go to infinity as y tends to infinity if γ is a positive real number which cannot be allowed. We cannot allow infinite pressures or infinite displacements, infinite pressures in this case at infinity that is one point.

The other is let us say γ is now imaginary, yeah we choose the negative quantity. Then what happens is that again the B term has how does it look like? It looks like $e^{-i\psi y}$. Now the temporal description was $e^{-i\omega t}$ and the spatial description is $e^{-i\psi y}$, so what this would imply is that there is an incoming wave from infinity in the y direction.

So, if we decide to choose when the real quantity when γ is real, if we want to choose the real positive value then B is not allowed. If we decide to choose the γ imaginary quantity but negative then also B is not allowed, both these are counter intuitive. This incoming wave from infinity violates causality. There is no source at infinity. The source is our plate.

In the entire universe there is only one source which is our vibrating plate and so all the waves should move from the plate towards infinity, so there cannot be a returning wave that is the idea of causality. So, in these two cases having B violates it, so we say that B we set to 0. That means when γ is real we will choose the positive value, when γ happens to be imaginary we will choose the negative imaginary value and with this convention B will have to be set to 0, A will not be 0.

So, my $\phi(k, y)$ is given by A $e^{-\gamma y}$. Now, the kinematic boundary condition if you recall was

$$\left.\frac{\partial \phi(x, y)}{\partial y}\right|_{x,y=0} = v(x).$$

So, now if we take the Fourier transform first or wave number transform first, then I get

$$\left.\frac{\partial \phi(k, y)}{\partial y}\right|_{y=0} = V(k).$$

So, now $\frac{\partial\phi(k,y)}{\partial y}$ gives me $-\gamma \mathrm{A} e^{-\gamma y}$.

When y is set equal to 0, I get

$$-\gamma \mathrm{A} e^{-\gamma y}|_{y=0} \rightarrow -\gamma \mathrm{A} = \mathrm{V(k)},$$

and A I write as we look at it here $\phi(k,0)$. So, I get

$$-\gamma\phi(k,0) = V(k).$$

So, one after the other this is what we derived. Now, we have to get back to our plate equation and substitute this, the transform plate equation we substitute this, so what do we get?

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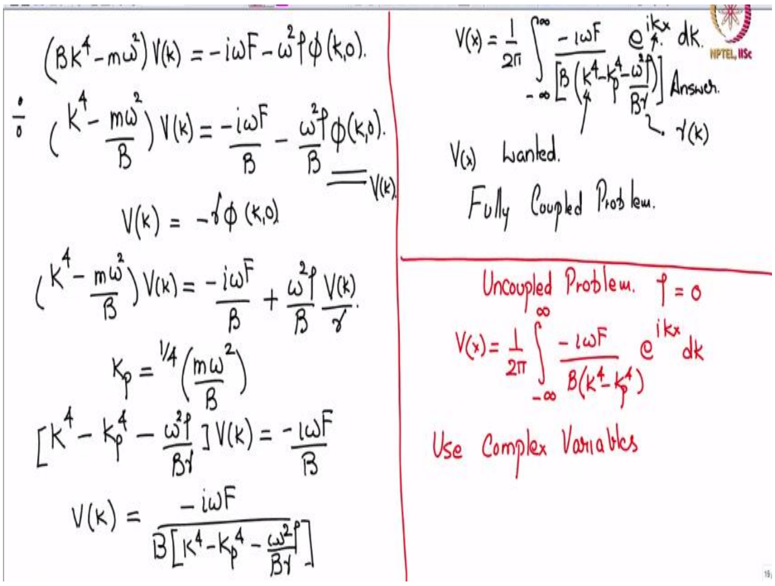

So, let us just recall what it was. The transformed plate equation was

$$(Bk^4 - m\omega^2)V(k) = -i\omega F - \omega^2\rho\phi(k,0).$$

Now, let us divide both sides by B. So I get

$$\left(k^4 - \frac{m\omega^2}{B}\right)V(k) = \frac{-i\omega F}{B} - \frac{\omega^2\rho\phi(k,0)}{B}.$$

So, this also can be put in terms of $V(k)$ if you recall. This was equal to or rather $V(k)$, if you recall $V(k)$ was equal to $-\gamma\phi(k,0)$.

So, this also can be written in terms of $V(k)$. So if we do that, let us do that so

$$\left(k^4 - \frac{m\omega^2}{B}\right)V(k) = \frac{-i\omega F}{B} + \frac{\omega^2\rho}{B}\frac{V(k)}{\gamma}.$$

So, now we will combine and before that I have given you the definition right in front that k_p the free plate wave number in vacuum at a given frequency ω was $\left(\frac{m\omega^2}{B}\right)^{1/4}$.

Therefore, I get first of all

$$\left[k^4 - {k_p}^4 - \frac{\omega^2 \rho}{B\gamma}\right] V(k) = \frac{-i\omega F}{B}.$$

In other words

$$V(k) = \frac{-i\omega F}{B\left[k^4 - {k_p}^4 - \frac{\omega^2 \rho}{B\gamma}\right]}.$$

Now, this has to be inverse Fourier transformed.

We want velocity in space, so we have to inverse Fourier transform this,

$$v(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i\omega F}{B\left[k^4 - {k_p}^4 - \frac{\omega^2 \rho}{B\gamma}\right]} e^{ikx}\, dk.$$

γ has k in it mind you, γ is a function of k. So, there is k in the denominator here, there is k in the denominator here, there is k in the numerator here and we are going to do this integral.

So, this will give me $v(x)$ which is what is wanted the velocity of the plate under fluid loading under forcing that is what is wanted. So, we are going to do that. So, this is the fully coupled problem, but before going fully all the way we will look at the uncoupled problem, what the uncoupled problem simply means is that the fluid density is 0. So that means we are in vacuum and then what happens?

The velocity I need is

$$v(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i\omega F}{B\left(k^4 - {k_p}^4\right)} e^{ikx}\, dk.$$

So, this is the inversion we have to do for the invacuo problem. Now, here we are going to invoke the complex variables theorems, we are going to use complex variables to compute this integral.

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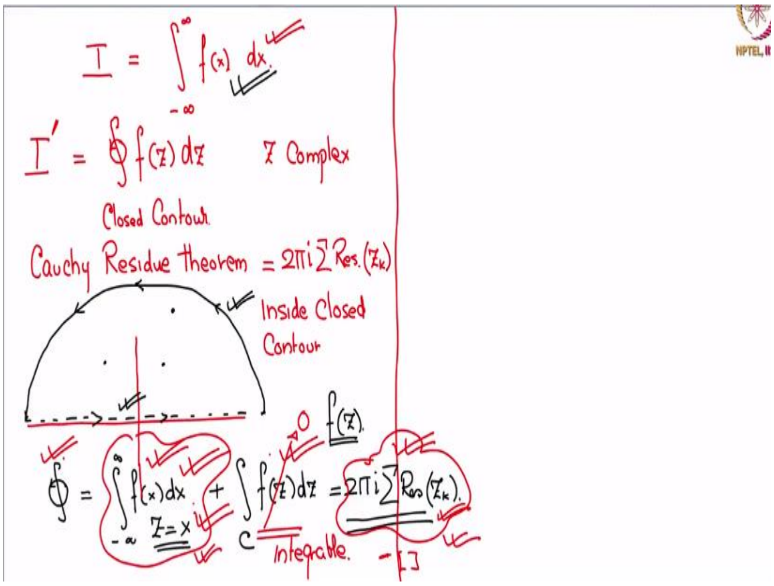

So, what that means? That means is if we are going to do some integral let us say I,

$$I = \int_{-\infty}^{\infty} f(x)\ dx.$$

What we are going to do in order to compute this integral is we are going to take an I star or I', let us say I' and compute the integral over a closed contour of this function or some related function,

$$I' = \oint f(z)\, dz.$$

Let us say for now dz where now z has become a complex variable, variable in the complex plane, complex variable. Now this integral from Cauchy Residue theorem is equal to $2\pi i \sum Res.(z_k)$. For those who have forgotten, I would advise you to refresh this use of complex variables, there are some very good books and you may have already seen it.

So what is the advantage? Advantage is this, I am just taking some example, I need an integral that is going from $-\infty$ to $+\infty$ I need this integral. Instead of that, I am going to do an integral on a closed contour, typically we take it in the counterclockwise direction, these are all rules of complex variables. So, the importance is that when I choose the closed contour the portion I want to integrate must be part of the contour.

So, I have this portion which I wanted to do

$$\oint = \int_{-\infty}^{\infty} f(x)\ dx,$$

z becomes x on the real axis that was integral I wanted to do. So, my contour includes the portion I want to do and then there is this additional contour that has come up. This additional contour which has come up $f(z)$ dz on the contour.

$$\oint = \int_{-\infty}^{\infty} f(x)\ dx\ + \int_{C} f(z)\, dz\ =\ 2\pi i \sum Res\, f(z).$$

It means that singularities of this z function at z function inside the contour where the isolated singularities where $f(z)$ goes to infinity. So, now, this side can be computed, the right hand side can be computed, the residues can be computed and this additional contour integral which has come up should be integrable, that means we should be able to find it, find its value integrable. I will tell you ahead of time usually it goes to 0.

So, what has happened? I need this integral, instead I have chosen to integrate over a contour but that contour includes this portion and some additional portion and is equal to the residues which are known. So, if this can be computed or sent to 0, let us say it goes to 0, then the value of the integral I want is equal to $2\pi i$ sum time sum of the residues So, without touching the integral I have found the value.

Without actually attempting the integral I have found the value. If this integrate to some nonzero value, some finite nonzero value, then I have residue minus that value as the integral value. So, that is the advantage, we integrate without touching the original integral we want. The time is up, I will close this lecture here. We will begin from here in the next class. Thank you.