Sound and Structure Vibration
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Module No # 05
Lecture No # 22
Asymptotic Method Continued and Maple Demo

Good morning and welcome to this next lecture on Sound and Structural Vibration.

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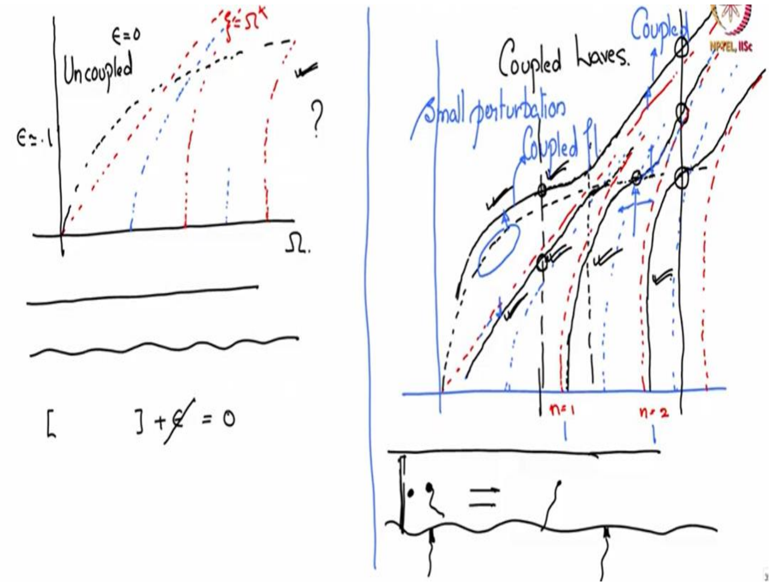

We ended up with this picture in the last class. A schematic of the coupled wave where I showed you the light dotted lines as the uncoupled solutions and the dark black lines as the coupled solutions and the nomenclature, I was going to use. Now it is time to start finding the mathematical solution from the dispersion equation.

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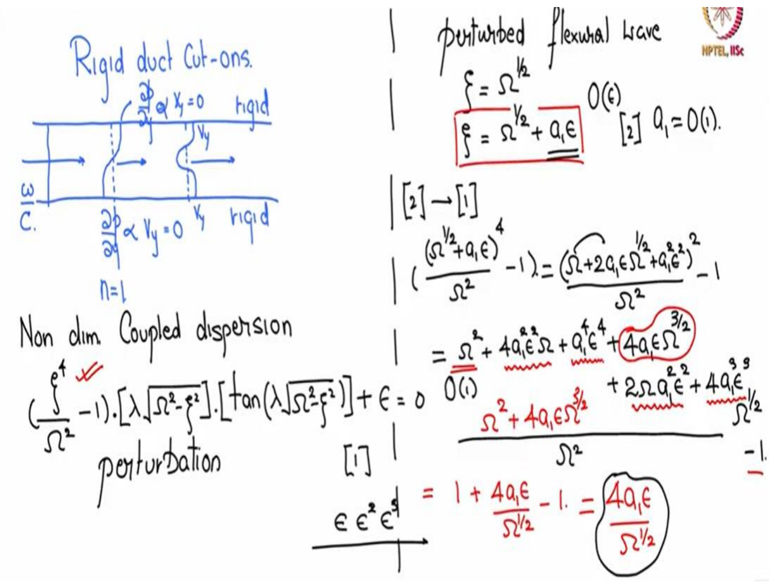

But before we do that, let me just show you what I mean by the rigid duct cut-ons**.** So, we have this duct 2D duct and now top and bottom are rigid, lower plate and upper plate are rigid**.** So, what is the plane wave the plane wave looks like this is a plane wave it is plane on the wave front parameters do not change and it propagates this is $\frac{\omega}{c}$**.** Now what is the next cut-on n equal to 1 cut-on it looks like this in terms of pressure it looks like this.

This is a pressure wave front here you have $\frac{\partial p}{\partial y}$ which is particle velocity in the y direction or proportional to the particle velocity in y direction is 0 here also $\frac{\partial p}{\partial y}$ is 0 is proportional to particle velocity equal to 0. So, this is the $n = 1$ cut-on it looks like this. Next $n = 2$ cut-on it looks like this; it is here it goes like this, and it comes back**.** So here again $\frac{\partial p}{\partial y}$ or v_y is 0.

Here again $\frac{\partial p}{\partial y}$ or v_y is 0 and it has a higher deformation profile. And so forth so these are the cut-ons and they will propagate if they satisfy the omega condition they will propagate. Now as I said let us start deriving the solutions. So let me have the non-dimensional Coupled dispersion equation is given by $\left(\frac{\xi^4}{\Omega^2} - 1\right)\left[\lambda\sqrt{\Omega^2 - \xi^2}\right]\left[\tan\left(\lambda\sqrt{\Omega^2 - \xi^2}\right)\right] + \epsilon = 0.$

So, we have to solve this**.** So, we are going to use the asymptotic method, or the perturbation method and we have seen it before in the classical problem what we are going to do is that every solution is perturbed by a small quantity and how do we give the small quantity. So first let us say I am going to look at the perturbed or coupled both are equal now flexural wave perturbed flexural wave.

What was the unperturbed flexural wave or uncoupled flexural wave was given by this($\xi = \Omega^{\frac{1}{2}}$)? So, the coupled value is going to be $\xi = \Omega^{\frac{1}{2}} + a_1\epsilon$ it is like a Taylor series so this is a small quantity then order ϵ quantity that is the language of asymptotes it is an order ϵ quantity this total term is an order ϵ quantity a_1 is an order 1 quantity**.**

Now if we substitute this into this equation, so let us give this will call this 1 and we will call this 2 so if I substitute 2 into 1 what do we get we get $\frac{(\Omega^{1/2}+a_1\epsilon)^4}{\Omega^2} - 1$. That is the first term so what to do with this let us look at this we get I will do a square first. So, I will get

$$\left(\frac{\left(\Omega^{1/2} + a_1\epsilon\right)^4}{\Omega^2} - 1\right) = \frac{\left(\Omega + 2a_1\epsilon\Omega^{1/2} + {a_1}^2\epsilon^2\right)^2}{\Omega^2} - 1\,,$$

$$= \frac{\Omega^2 + 4{a_1}^2\epsilon^2\Omega + {a_1}^4\epsilon^4 + 4a_1\epsilon\Omega^{3/2} + 2\Omega{a_1}^2\epsilon^2 + 4{a_1}^3\epsilon^3\Omega^{1/2}}{\Omega^2} - 1\,.$$

Now I said ϵ is a small quantity if ϵ is a small quantity our understanding is that ϵ^2 is a much smaller quantity, and ϵ^3 is a further smaller quantity**,** the powers will reduce in magnitude. So, let us look at the ϵ order so this is the order one term that means no ϵ in it, this is order one term let us take red this is the order one term and the ones this with only one ϵ are over here this is the only term with one ϵ**.**

The one is with ϵ^2 this is the term with ϵ^2 it is a smaller quantity; this is the term with ϵ^2 also the same quantity. This is the only term with cubic in ϵ that is a further small quantity, and this is the only term with fourth power the last quantity. So, we will ignore with neglect the ones which are higher than ϵ so we will keep Ω^2 + $4a_1\epsilon\Omega^{3/2}$ and divided by Ω^2.

$$\left(\frac{\left(\Omega^{1/2} + a_1\epsilon\right)^4}{\Omega^2} - 1\right) = \frac{\Omega^2 \ + \ 4a_1\epsilon\Omega^{3/2}}{\Omega^2} - 1,$$

$$= 1 + \frac{4a_1\epsilon}{\Omega^{1/2}} - 1 = \frac{4a_1\epsilon}{\Omega^{1/2}}\,.$$

So, this first term is what we have found using this substitution we have all the other terms also to take care**.**

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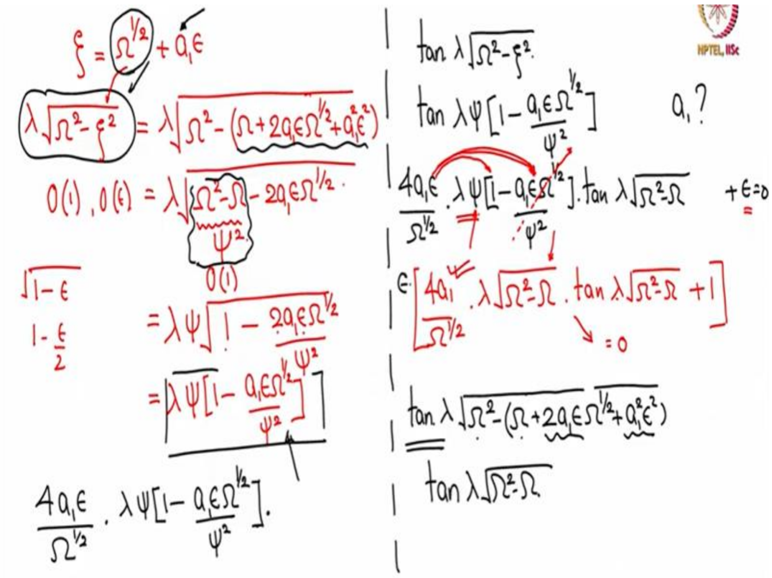

So, what to do let us see so I have the second term here, let me write $\xi = \Omega^{1/2} + a_1\epsilon$ and the term is $\lambda\sqrt{\Omega^2 - \xi^2}$**.** So, this ξ gets substituted in here in this form so if we do that, I get

$$\lambda\sqrt{\Omega^2 - \xi^2} = \lambda\sqrt{\Omega^2 - (\Omega + 2a_1\epsilon\Omega^{1/2} + {a_1}^2\epsilon^2)}\,.$$

So, again there is order 1 quantities and there is order ϵ quantities so let us keep till order ϵ quantities. So, what do we get

$$\lambda\sqrt{\Omega^2 - \xi^2} = \lambda\sqrt{\Omega^2 - \Omega - 2a_1\epsilon\Omega^{1/2}}\,.$$

Now because this is a first time or this is quite new, I will do it here so suppose this is a I will call this some ψ and ψ is an order one quantity. let me call this ψ^2 is an order one quantity.

So, what I will do I will take this λ take ψ^2 outside as ψ that means I am dividing inside by ψ^2. So, I get

$$\lambda\sqrt{\Omega^2 - \xi^2} = \lambda\psi\sqrt{1 - \frac{2a_1\epsilon\Omega^{1/2}}{\psi^2}}\,.$$

So, this is a small quantity that is our assumption because of ϵ this is smaller than this so if I do a binomial expansion of this I get

$$\lambda\sqrt{\Omega^2 - \xi^2} = \lambda\psi\left[1 - \frac{a_1\epsilon\Omega^{1/2}}{\psi^2}\right].$$

So, I get a half in front which cancel with two this is what happens right square root of $1 - \epsilon$ is $1 - \frac{\epsilon}{2}$ that is what I have done. So, this term here with this substitution has ended up like this so if I want what do I have let us see just one second here, so this was the first term I just want to remember $\frac{4a_1\epsilon}{\Omega^{1/2}}$.

So, I have $\frac{4a_1\epsilon}{\Omega^{1/2}}$ from the first and from the second, I get some $\lambda\psi\left[1-\frac{a_1\epsilon\Omega^{1/2}}{\psi^2}\right]$, we should remember what ψ is, this thing we call ψ^2 so we should not forget. There is one more tan term so one more tan term so what happens there we have one more tan term we have which is $\tan\left(\lambda\sqrt{\Omega^2-\xi^2}\right)$.

Now we have done $\lambda\sqrt{\Omega^2-\xi^2}$ so we should not have to do anything here so it will look like $\tan\lambda\psi\left[1-\frac{a_1\epsilon\Omega^{1/2}}{\psi^2}\right]$. So, in totality now how does it look like

$$\frac{4a_1\epsilon}{\Omega^{1/2}}\,\lambda\psi\left[1-\frac{a_1\epsilon\Omega^{1/2}}{\psi^2}\right]\tan\lambda\psi\left[1-\frac{a_1\epsilon\Omega^{1/2}}{\psi^2}\right]+\epsilon=0.$$

What is the whole idea we should not forget in the mess idea is to figure out a_1 which gives me the correction on the flexural wave number which was $\Omega^{1/2}$ we should not forget that? So out of this my question is what is a_1? Now I will pull out first the coefficient of ϵ in this if I do that so, this is a total tan term with ϵ inside it we cannot touch it anyway.

But here I have ϵ multiplied with 1 and $\lambda\psi$ which is which will be ϵ order quantity then here I have ϵ multiplied by ϵ and other things, so this is an ϵ^2 quantity so this can be neglected. ϵ^2 quantity I am going to neglect so what am I going to get here as a coefficient of ϵ, and including this ϵ I will get

$$\epsilon\left[\frac{4a_1}{\Omega^{1/2}}.\lambda\sqrt{\Omega^2-\Omega}\ \tan\lambda\sqrt{[\ ]}\right].$$

So let us do this let us do one more let us just do one thing here I have a tan then I have a $\lambda\sqrt{\Omega^2-(\Omega+2a_1\epsilon\Omega^{1/2}+a_1{}^2\epsilon^2)}$ Now because it is inside the tan I cannot separate it out algebraically like this. So, I will take the dominant terms, so ϵ is a small term compared to Ω or Ω^2, ϵ^2 is a further small term.

So, I will just take it as $\tan\lambda\sqrt{\Omega^2-\Omega}$. So, I apologize I will take this out here and in black write it as $\tan\lambda\sqrt{\Omega^2-\Omega}$. And so here $\tan\lambda\sqrt{\Omega^2-\Omega}$ which is what happened here that is the same thing that happened here. I just showed you how it is second ϵ comes and which is neglected because I get ϵ^2, over here.

$$\epsilon\left[\frac{4a_1}{\Omega^{1/2}}.\lambda\sqrt{\Omega^2-\Omega}\ \tan\lambda\sqrt{\Omega^2-\Omega}+1\right].$$

So, the coefficient of ϵ now must be set to zero so this thing must be set to zero we set it to 0. I will find my a_1**.** So let us see if I can $\frac{4a_1}{\Omega^{1/2}}$.

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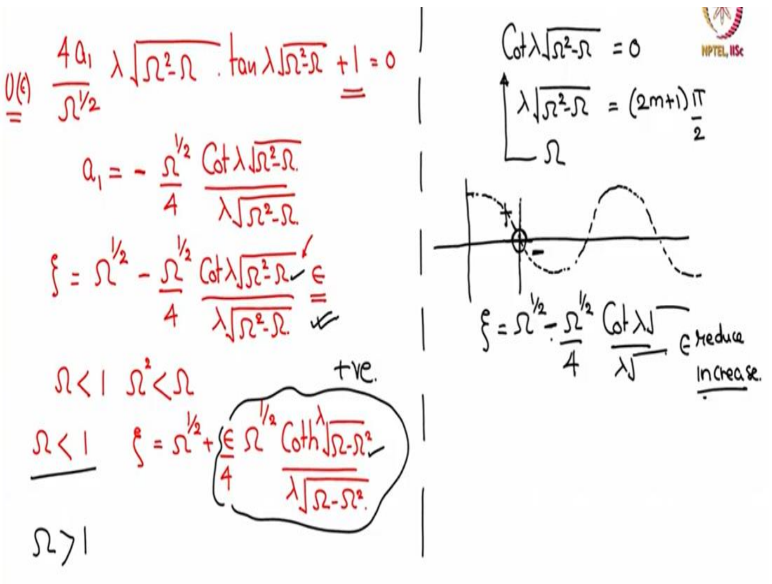

Let us see $\frac{4a_1}{\Omega^{1/2}}$, so I have

$$\frac{4a_1}{\Omega^{1/2}}.\lambda\sqrt{\Omega^2-\Omega}\ \tan\lambda\sqrt{\Omega^2-\Omega}+1=0.$$

So, we have balanced the equation at order ϵ , order ϵ we have balance that equation**.** So, then what happens my a_1 can be found out a_1 now is equal to taking the whole thing to the other side I get

$$a_1=\frac{-\Omega^{1/2}}{4}\frac{\cot\lambda\sqrt{\Omega^2-\Omega}}{\lambda\sqrt{\Omega^2-\Omega}}.$$

So, what is the total solution perturbed solution now

$$\xi=\Omega^{1/2}-\frac{\Omega^{\frac{1}{2}}}{4}\frac{\cot\lambda\sqrt{\Omega^2-\Omega}}{\lambda\sqrt{\Omega^2-\Omega}}\epsilon\,,$$

cot is an oscillatory function because by sign. Now if Ω is less than 1 then between Ω^2 and Ω , $\Omega^2<\Omega$ so we have to get into cot hyperbolic**.**

So, for Ω less than 1 take my answer ξ looks like

$$\xi = \Omega^{1/2} + \frac{\epsilon}{4}\,\Omega^{\frac{1}{2}}\,\frac{\coth\lambda\sqrt{\Omega^2-\Omega}}{\lambda\sqrt{\Omega^2-\Omega}}.$$

The sign is switched please note this sign is switched and that is below Ω less than 1.

So, $\frac{\epsilon}{4}\,\Omega^{\frac{1}{2}}\,\frac{\coth\lambda\sqrt{\Omega^2-\Omega}}{\lambda\sqrt{\Omega^2-\Omega}}$ thing is a positive number this happens to be a positive number it is a positive number. Whereas, for Ω greater than 1 we still have this term over here and it changes sign about a certain frequency. So let us see that frequency so let us say so I have $\cot\lambda\sqrt{\Omega^2-\Omega}$**,** which let us say is 0 at some point that means cos is 0 at some point what is cos over sin the cos is 0 at some point.

Which means $\lambda\sqrt{\Omega^2-\Omega}$ is $(2m+1)\frac{\pi}{2}$**.** So now so this thing at that Ω value at that Ω value there is an Ω value where this term is 0 right where this term is 0**.** So now that means let us look at how cos looks like, cos starts off here and then it goes like this**.** Now what happens is that let us look at this over here below?

The below that particular frequency here you are correction you are positive you are about below this frequency where cos goes to 0 you are positive here above your negative over here**.** So, if you look at ξ equation $\Omega^{1/2} - \frac{\Omega^{\frac{1}{2}}}{4}\frac{\cot\lambda\sqrt{\Omega^2-\Omega}}{\lambda\sqrt{\Omega^2-\Omega}}\,\epsilon$ what happens is that below that frequency this correction is positive so that means this negative will reduce**.**

And above that frequency your negative means that negative and this negative will increase. For other locations the sign of cos in the numerator and the sign of the sign in the denominator both will matter because it is a cot which is cos over sin**.** So let me quickly plot what I what we have over here in this picture, let me plot in red or let me go back.

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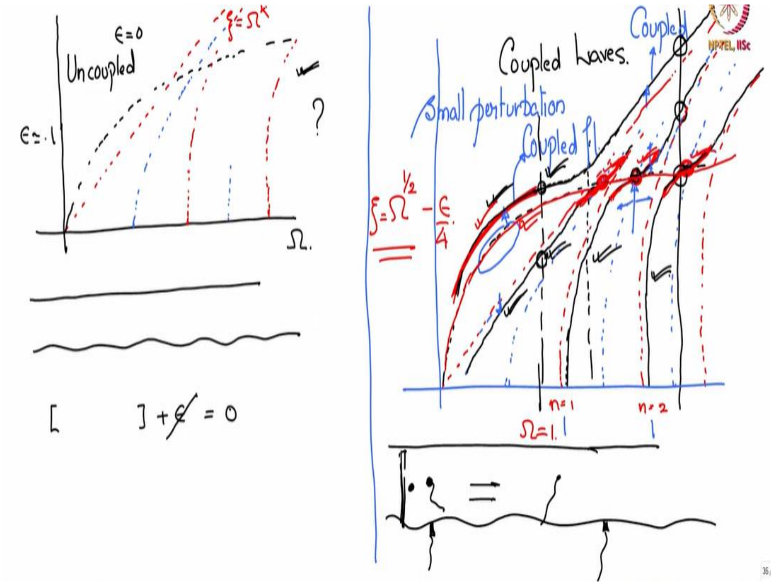

So, we are looking at perturbations to the flexural wave. Let me use red this time so this is the coincidence frequency, below coincidence frequency the correction was positive that means to this flexural wave. I will have an increment so this will be the coupled flexural wave below coincidence frequency. Above coincidence frequency there are frequencies where cot goes to 0 those are these frequencies**.**

I will talk more about them these are the frequencies so below you will have a reduced value above you have an incremented value. So, this portion is the coupled flexural wave this portion this little portion is the club coupled fractural wave. So, the equation we have derived you know the solution we have derived we did ξ is equal to $\Omega^{\frac{1}{2}}$ and some correction**.**

$\frac{\epsilon}{4}$ etc, that is represented pictorially by this red portion I have these short red portions**.** So that means what this original flexural wave which I now completely show in red localized regions as modified to this, this is the coupled flexure**.** In next class we will find all the other solutions the coupled plane wave, the coupled rigid duct cut-on n_1 and n_2 etc, we will do that in the next class couple plane wave here that in the next class thanks.