Introduction to Soft Matter
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Lecture 40
Sinusoidal oscillations (Continued)

So, welcome back to one more lecture on introduction to Soft Matter. This is as we said, this is towards the end of our lecture series and the last chapter that we are discussing as part of this course, is the issue of sinusoidal inputs and the output, the consequent output from a viscoelastic material.

And last class we saw that when you input a sinusoidal strain history, you get an output which is also a sine function, but which has now a phase lag and a phase difference there exists a phase difference between the sine between the input and the output, right.

And all that where you are using regular trigonometric and there is also another way similar but very useful format in which we can put the same equations, which is a complex variable representation. So, we will just do that today.

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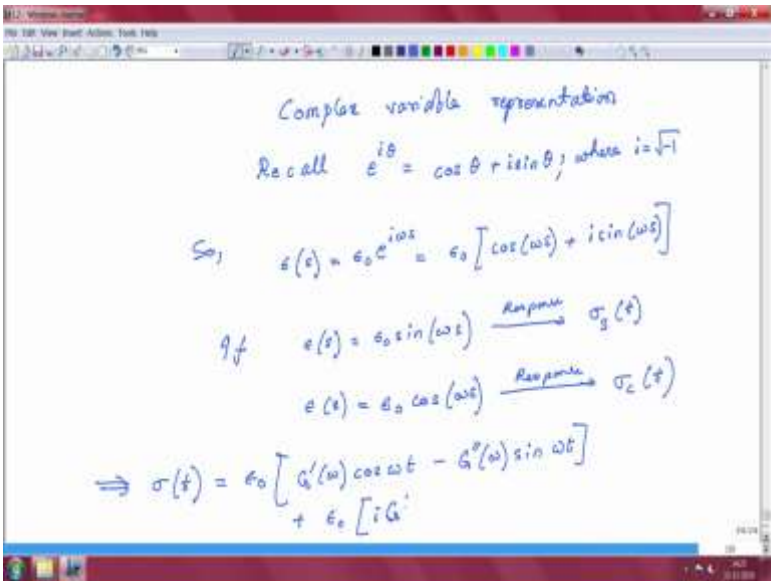

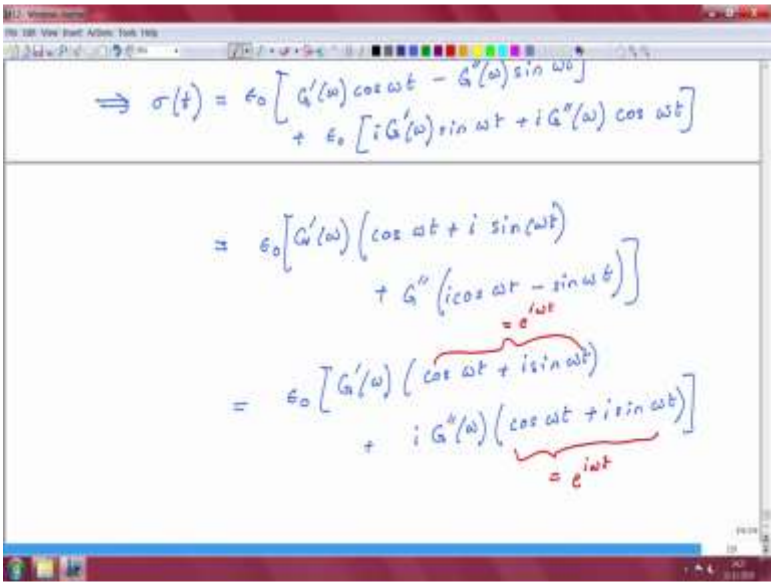

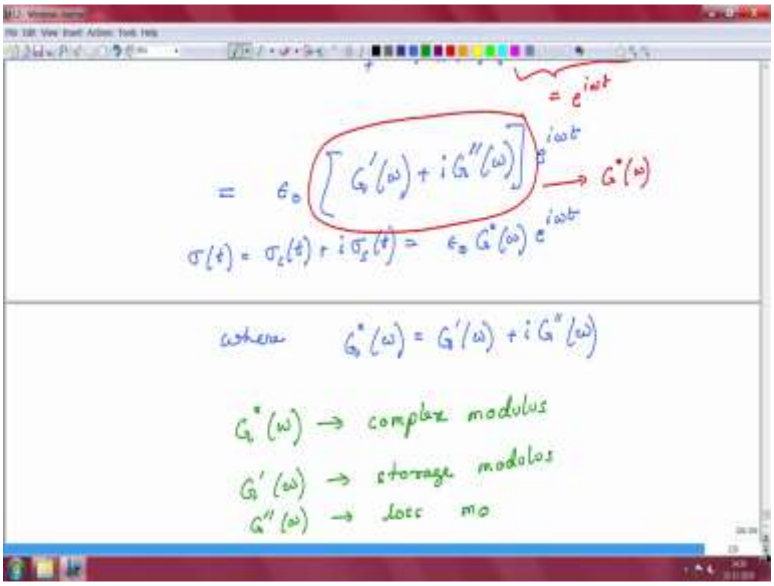

So, we want to look at the complex variable representation. So, recall e to the power i theta is equal to cosine theta plus i times of sine theta where i is root over of minus 1. So, we can represent our input in terms of a complex variable using this. So, how will we do that?

We will use, so now, our input is epsilon s is epsilon naught, instead of writing a sine function, we will just write the complex variable representation which is e to the power i times of omega s which is epsilon naught, you have cosine omega s plus i times of sine omega s and s is again a real number between 0 and infinity. Now, if so, we had discussed what the, if the inputs are cosine and a sine function, their responses we discussed in the last class.

So, let us just write down, if you have epsilon s as epsilon naught sine of omega s then it leads to a response and the response now is some functions of time, which we did last time.

And if you have the response as a cosine function, sorry this is, let us write s for sine function and let us write c for cosine that will help us distinguish.

Then the response is some sigma c of t, this is an s, s for sine c for cosine. So, now if we use the result that we just derived in the previous class then we can say that what is the expression sigma t that is going to be equal to epsilon naught. So, now we just replace for the real part, we know that in the real part we have a cosine input.

So, we will write down the output for a cosine function cosine omega t minus G double dash which is a function of omega into a sine omega t plus for the imaginary input part of the input, we have epsilon naught into i times G dash g dash f omega into sine of omega t plus i times of G double dash omega cosine of omega t.

So, what we want to do now is let us get all the imaginate the sine terms together sorry G dash terms together. So, what we should do now is we should get all the similar terms together, so we will say epsilon naught into G dash of omega say put in a bracket. So, if we combine the terms here, then we have plus i G dash sorry the G dash is outside already (just a second) into sine of omega t.

Similarly, you have the G double dash, which is now going to give me cosine i times of cosine omega t minus sine of omega t. So, what I am going to do is I am going to try and represent this itself as a complex variable. So, see what we are trying to do? Our input was a complex function, we are trying to reduce this also to a complex function. A complex function that is easy to understand this is already in a complex format.

But something that might be easier to, so we have the first term which we will leave it as it is. And then the second term, I am going to bring the i out. So, I will just put i and this is G dash f omega. This is now going to be cosine of omega t and this is going to be plus i times sine omega t. But what is now you see can see that both of these two terms have the same time varying part, this is equal to e to the power i omega t and similarly, this part is equal to e to the power i omega t.

So, I can simplify my case further by writing epsilon naught, this is now G dash of omega plus i times of G double dash omega e to the power i omega t. So, I had in the beginning written sigma t is equal to sigma c plus i times of sigma st. This is just going back to the output from the cosine part and this is just the output from the sine part and but this is now equivalent to some epsilon naught.

And now, I can rewrite this particular expression. So, this is now another complex number and now I can give it another name, I can call it G star omega. So, this is equal to c star omega e to the power i omega t where G star omega equal to G dash i times of G double dash and this is a very important number, this is called the complex modulus and G dash of omega is called the storage modules and the G double dash omega is also called the loss modulus

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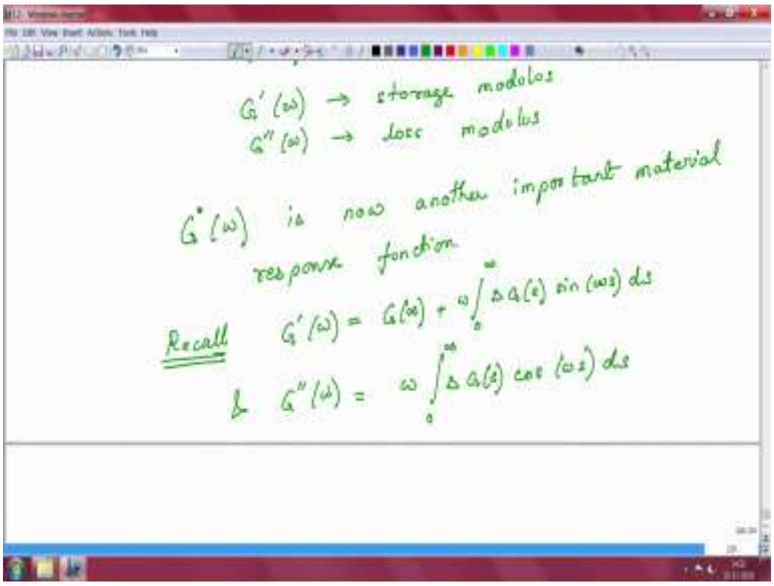

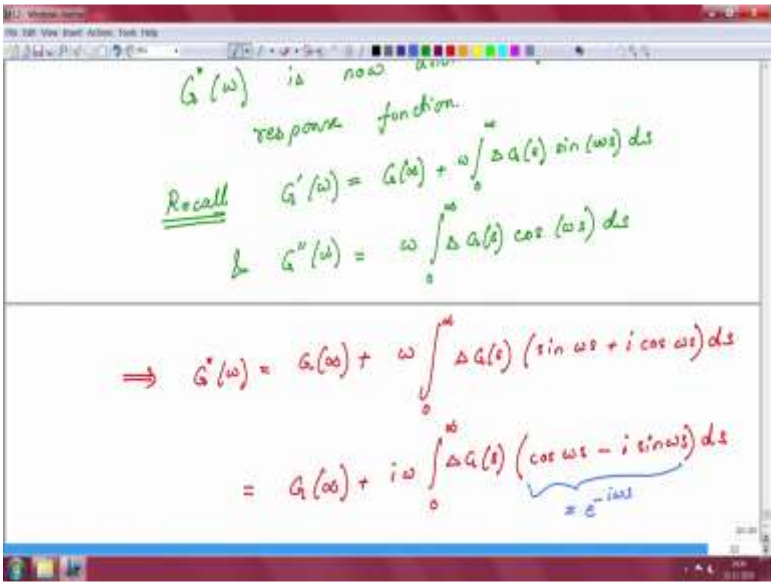

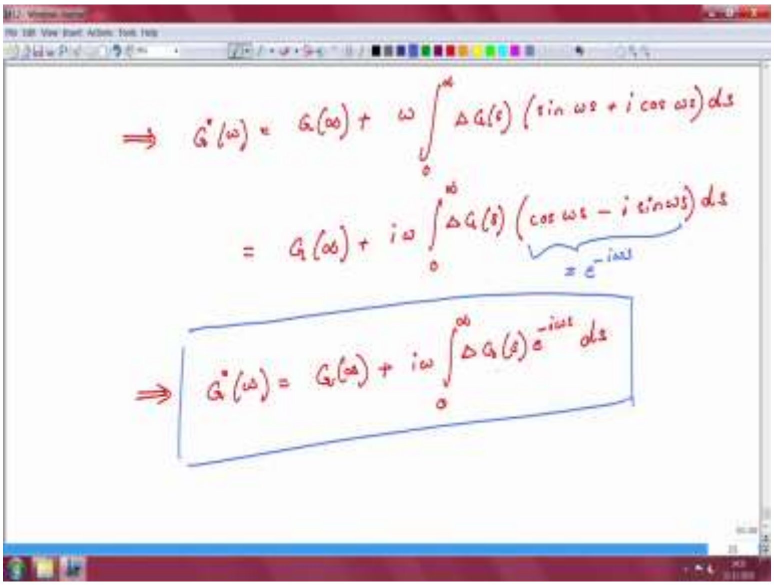

This complex modulus is now another important material response function which means that G dash omega can be used to characterize the behavior of a material. Earlier we had when we did not have sinusoidal inputs when we just had a step stress or a step strain we saw that there were two important functions that were material responses and that was the stress relaxation function Gt and the creep compliance J, right J of t.

And now, we see that when we apply a sinusoidal strain or a sinusoidal input, when we put it in, when we express the sinusoidal input as a complex number, we can get, we can characterize the output in terms of another complex number, where you now encounter this new quantity which we are going to call the complex modulus, right.

Now, just in terms of the previous variables, so recall we had written before the what G dash omega was, so this was an integral and the integral had a G infinity, so there was a constant term and then there is omega 0 to infinity delta Gs omega s ds and G double dash omega was omega integral 0 to infinity delta Gs. We have seen this before.

So, if we have to write, so now if we use these expressions, this implies that my complex modulus is nothing but G infinity plus, now, I can put add these 2 integrals. So, I just say omega, omega is common to both the integrals, both the integrals go from 0 to infinity and you have the function delta Gs that is common to both and this is going to be sine omega s plus i times of cosine omega s ds.

Or if you want, you can, so the inside is, if you want to represent it in i form e to the power i form, then what you can do is you can take an i omega out let me just write it. So, we said G infinity, I put an i outside of the integral and then I just rewrite the inside part, which now becomes cosine omega s minus i times of sine omega s ds. And this part, if you recognize this is going to be equal to e to the power minus i omega s.

So, this implies the G star of omega equal to G infinity plus i omega delta Gs e to the power minus i omega s ds. So, see this is giving an important equation right here, because it is tying your complex modulus with the response that you have determined before right.

Gt is a response that you can determine separately and if you can rewrite it as G infinity a constant term plus a time varying function, then you can this is the method by which you can compute the complex modulus analytically from that. So, obviously here we imply that the functional forms of G infinity and delta Gs, this must be known to us.

Now, we will go on with this writing this entire thing in this in the form of e to the power I omega.

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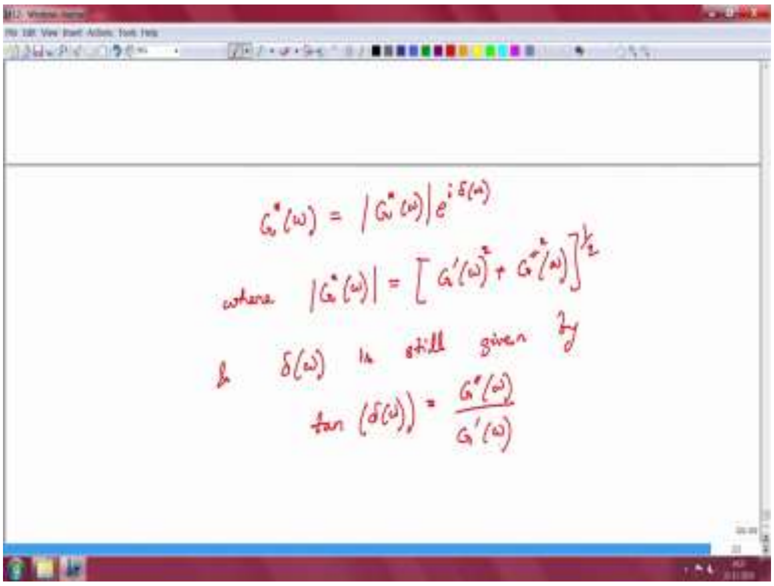

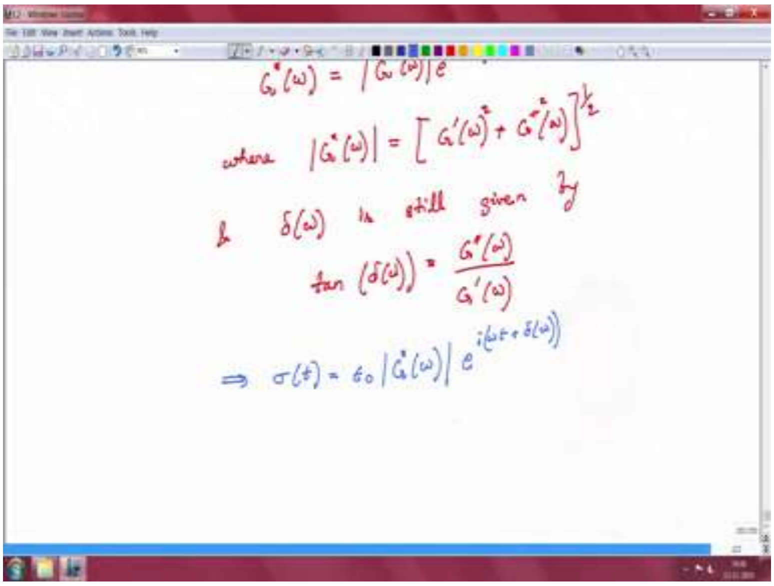

So, we can make some more simplifications, some more important equations we can do. So, this G omega term, this is now obviously a complex number. So, this complex number can again be rewritten as the amplitude of this complex number multiplied by e to the power of something.

So, we can say that, this is let this represent the amplitude and then we have e to the power i some phase difference where this amplitude how will you compute for a complex number? We know from complex mathematics, that this is simply this term and this is still given by tan of delta omega and that is just a ratio of G double dash omega by G dash omega.

So, these are the different forms that you will probably encounter in literature at different points of time. If you go through different books or different materials, then these are usually the same idea is put in very different forms and these are some of the forms that you are very-very likely to encounter.

So, now this tells us that your stress is going to be, so this was, so, what is going to be the output stress? This is simply epsilon naught g omega t, where we can just rewrite this as and you already had e to the power i omega t. So, you have e to the power i omega t plus, so this is your output in this particular form.

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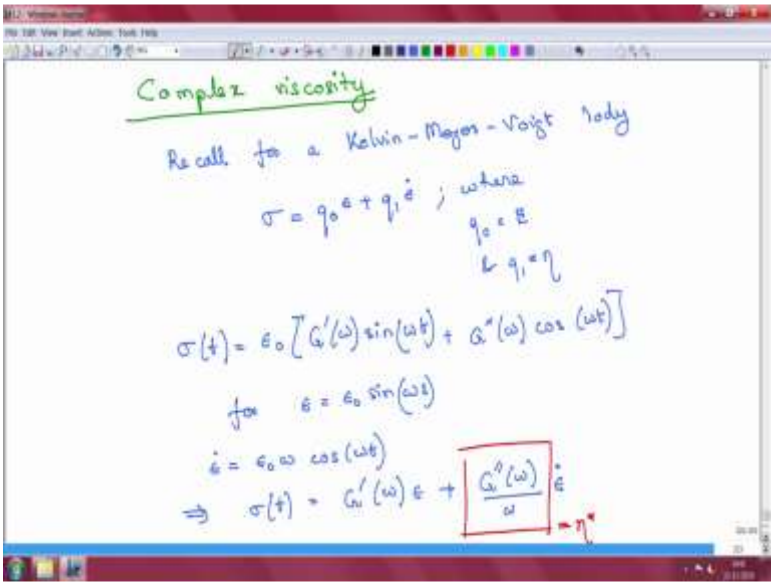

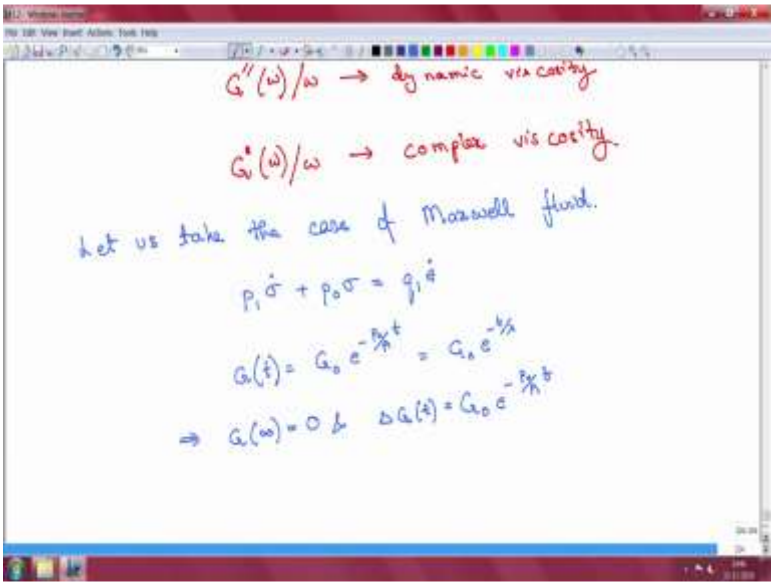

Another term that you are going to you are very likely to find in literature is the term called complex viscosity and this is very much related to the equations that we just derived. So, let us use the idea of the Kelvin voigt model.

So, in the Kelvin voigt model for a Kelvin Meyer voigt body your equation was sigma equal to some a term which is only has epsilon and there was another term which was going to have epsilon dot right, where Q naught was e and this other term Q 1 was viscosity. So, given the complex form, what can we do about the stress?

So, let us say, so in this particular case your stress is now going to be epsilon naught G dash omega sine, I am just rewriting the previous equation cosine of for epsilon equal to epsilon naught. So we have already seen that. Now epsilon dot is equal to epsilon dot omega cosine of omega t. So, this implies that my stress is, so this first part is epsilon and the second part.

So, I recast that equation. This is the generic response, right. And I have recast this for the case of the Kelvin Meyer voigt body equation. So, this is the governing equation for this and using the general response, I have been able to rewrite it in this particular form because epsilon naught sine omega t is epsilon t and this other one is related to epsilon dot.

So, this variable here this is often called a dynamic viscosity in this case. So, this term is called and when you do not take (())(21:24) as G double dash when you take the complex entire complex number when you take it as G dash or G star omega by omega, this term is called a complex viscosity.

So, we use this as a motivation. So, today what we have done is we have seen a number of different formulations or quantities that you are very-very likely to encounter when you are reading literature on viscoelasticity. And one of the important terms that we today discuss was G star which is a complex modulus and we saw that this broken up into 2 other moduli, which is called a storage modulus and the loss moduli.

Now, the 2 names are obviously given because one of them represents the storage or the storage of energy in the system. The other one represents the dissipation of energy. And this idea of storage and dissipation goes back to our previous discussion on the classical elastic bodies, the classical elastic solid and the classical viscous fluid and from there on, we have motivated ourselves and seen that these 2 that one of those terms is a represents a storage and the other one represents a dissipation, right.

Okay so, having derived all this, there is one example that we are going to start and I will encourage you to try and finish it in the next class and we will, that sample problem we will finish next time but I am going to start the problem today. So, let us take the case of Maxwell fluid, the governing equation for this we already know is some P 1, we are recasting it now that we know that both the sides are basically polynomials in the operator D. So, we are just rewriting it as in that particular form.

So, P 1 has a certain value that you can find out going back to the notes and so, with the first derivative I write P 1, the term which does not have a derivative at P naught. And on the other side, we have the derivative of epsilon. So, I write Q 1.

And for this you had found out that the function Gt had a particular form, it was actually a G naught e to the power minus in this particular case it was P naught by P 1 t or even rewrite it, you could rewrite this as G naught e to the power of minus t by lambda where lambda was the characteristic time scale associated with this also called the relaxation time scale.

So, for the case of the Maxwell fluid, we have, so we are trying to go back and see re-evaluate some of this moduli in terms of for the Maxwell case. So, you know that G infinity in this case is 0. So, and that means that the time dependent term is simply this one.

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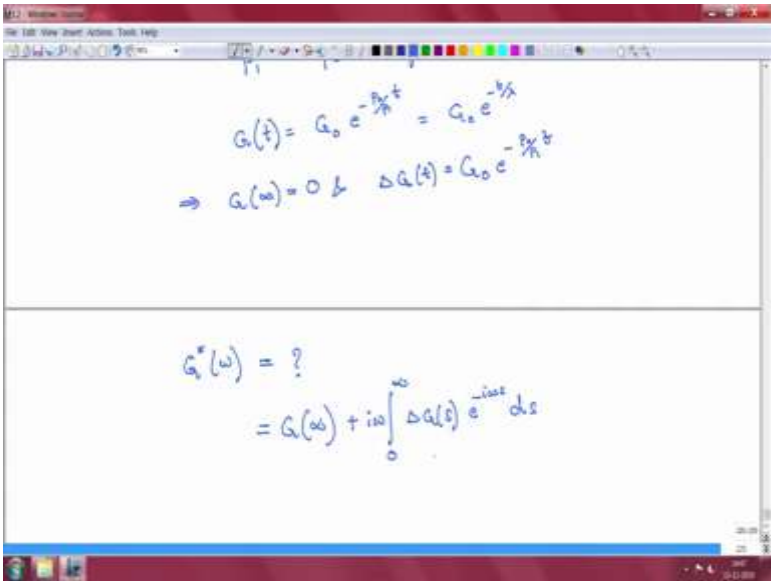

So, what I am asking you to find is, for example, what is the value of the complex modulus? And we know already the equation. So, I just want you to compute the integral. We have, we know that the general solution for this is given as G infinity i omega then there is an integral here that goes from 0 to infinity, you have delta Gs e to the power minus i omega s ds.

So, you want to replace this, the first time is actually going to cancel. It is just becoming going to become 0 and then you replace this term here and you try to see what result you finally get. So, today, this is the second last lecture. And what we are trying to do right now, this is what we are finishing up the chapter on sinusoidal response to sinusoidal inputs.

And what we found is, we found ourselves, we familiarize ourselves with some other very important variables that come again and again in viscoelastic literature, which has a complex moduli, loss modulus, storage modulus and also complex viscosity. And towards the end, we as this is the case of the Maxwell fluid and what we are trying to do is to find the functional form for the complex modulus case.

So, I left I leave you to here, this is almost done. So, for the next class, I would encourage you to go ahead and solve this and we are going to discuss the result. And we are also going to wrap up in the last lecture. So, today we will stop here.