Introduction to Soft Matter

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Lecture 39

Sinusoidal oscillations (Continued)

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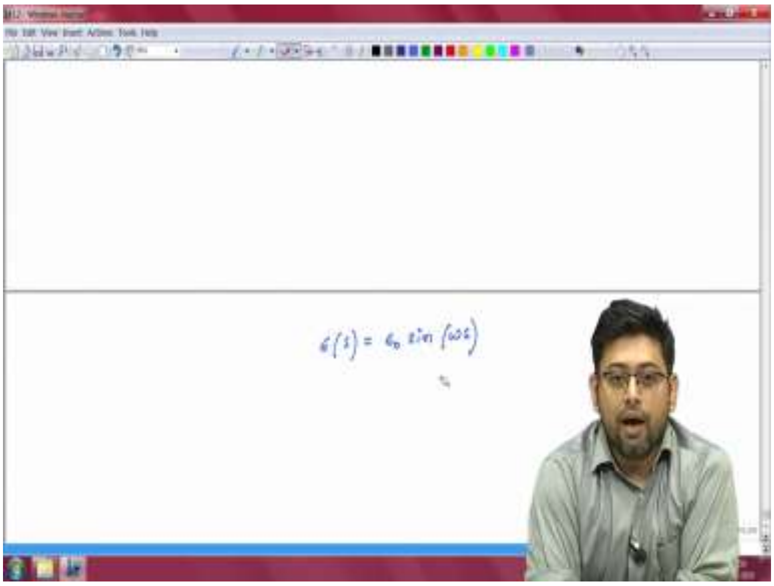

So, we have an imposed strain history, which is given by this sine function, right. And I am using s as the variable here, although we had to use s also in the Laplacian's, but let we should not get confused here because there is no Laplace transform being actually I used Laplacian. Laplacian is something else, so we restart, okay. I am going to restart the entire thing.

So, welcome back to one more lecture on introduction to soft matter. And this is the last lap of our class, where we were now going to discuss the imposition of sinusoidal stress or strain histories and the final the output that we get from a viscoelastic material. Now, we had seen in the last class that we started off with imposed strain history.

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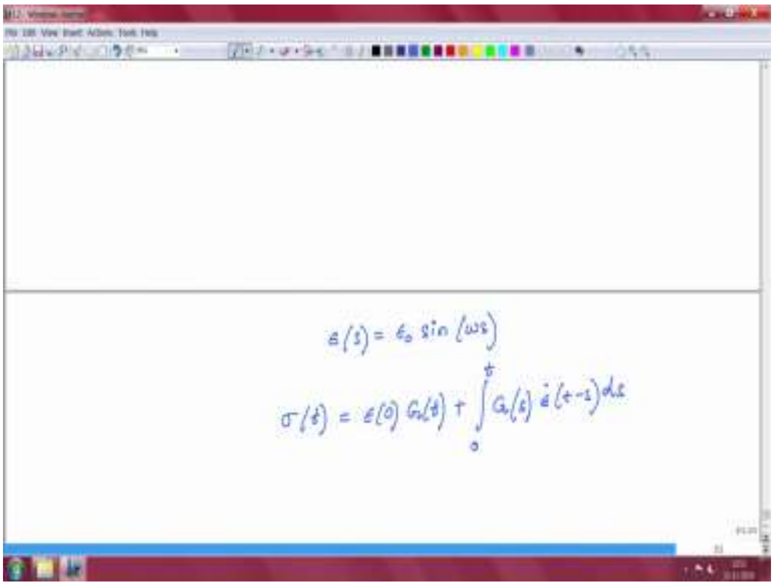

And we had said that the equation will now be of the form epsilon, which is now a function of time. Let us use the variable s for time here epsilon naught sine of omega s. Now, please do not get confused with this s here, I am using s here for time, we had discussion on Laplace transforms and s also appeared there, but that s is strictly the Laplace transform variable.

So, there is no real reason to get confused here, this is a different context altogether, we are working in the time domain not in the Laplace domain. And there is a reason why we are using s. We will see that this s is again a dummy variable. Finally, we will end up with t that we will use, okay.

So, this is the impose strain history. And now, we know that for a linear, so, we have assumed that epsilon naught is small enough that we can use the formulas that we have derived for a linear viscoelastic. So, for a linear viscoelastic, we know that the output sigma t, so, the stress at any function t is a function of its memory and we can write it as Gt plus integral 0 to t this is G of s epsilon dot t minus s ds. So, we have done this before.

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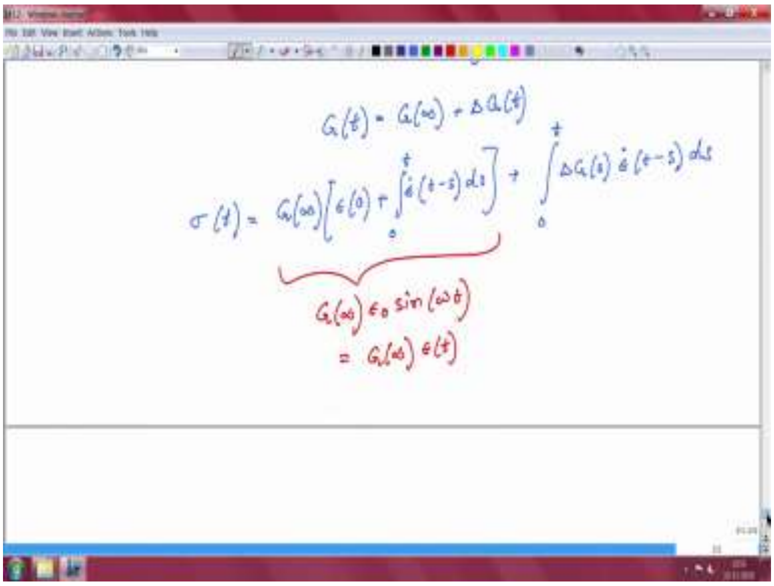

And where this finally led to was the equation that we assumed that the G of t can be written up as 2 parts. One is a time dependent function and another is a constant. And when we use this, we end up with sigma t equal to G infinity epsilon 0 plus integral of 0 to t epsilon dot t minus s ds plus the quantity integral 0 to t delta G of s epsilon dot t minus s ds. So, this is where we had to stop last time. Now, this particular, the first part of this of the right hand side, this simplifies.

Now carefully look at the, the terms inside the bracket, we have an epsilon 0, which for this case is 0. And we have an integral from 0 to t of epsilon dot, so the integral of epsilon dot is going to be the function epsilon itself. So, this is actually going to lead you to the function or basically, you can write it in 2 forms.

So, the first part actually simplifies for us. So, now, we have to use the expansion of the trigonometric identity for cosine and we have to expand the term inside

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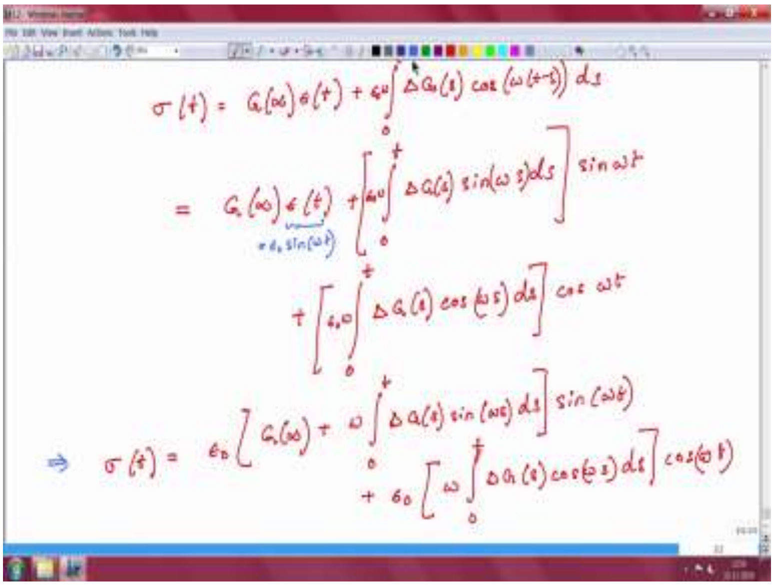

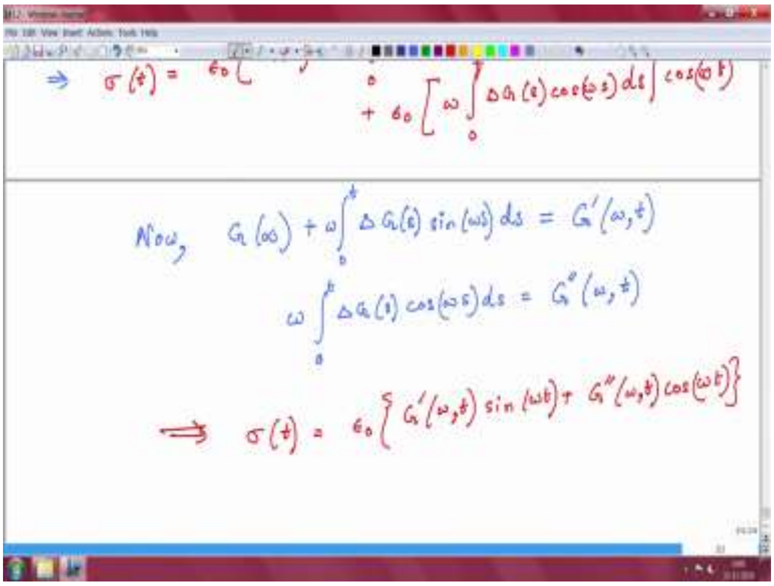

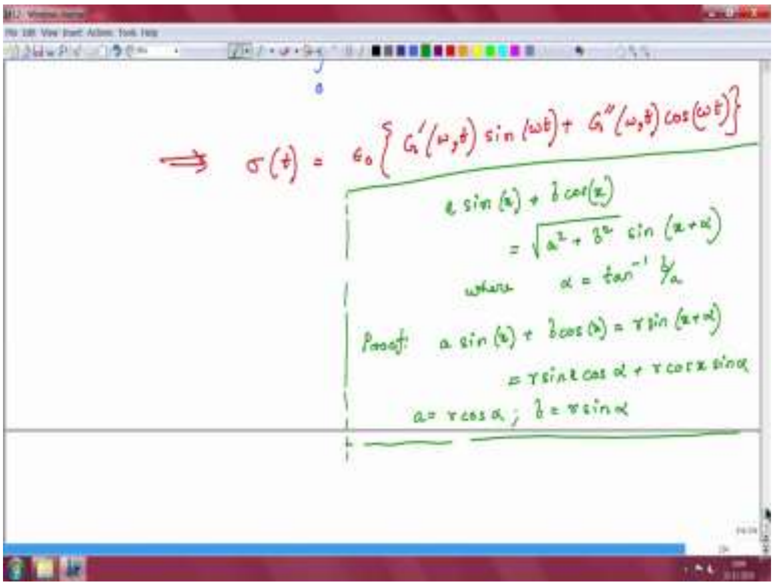

So, we have now sigma of t is equal to G infinity epsilon dot t plus integral, you have epsilon naught omega 0 to t cosine delta G of s cosine of omega t minus s ds, right. So, now, let us expand the cosine function. If we do that then we have, so we will (this) omega and we have delta G of s.

So let us bunch the, this is an integral in for ds. So we will just use the s term here, sine omega s ds. And this entire thing is now going to be multiplied with sine of omega t. Because t is not a variable for this integral, the sine omega t comes out.

Similarly, for the other, we will have delta Gs and this time we will have a cosine. We had the sine, so we have a cosine omega s ds and then this thing is multiplied with cosine of omega t. So, here we know that by the way epsilon t this term is equal to epsilon naught sine of omega t.

So, you end up having, so I can bunch this into 2 different quantities and the first quantity will be epsilon naught multiplied by G infinity plus omega integral 0 to t delta Gs sine omega s ds into sine omega t this a sine plus again you have this epsilon. So, you have an epsilon naught. This time you also have inside you have an omega and this is integral 0 to t delta Gs, you now have the cosine omega s ds into cosine omega t.

Now, please, this has become a big equation, but we are quickly very quickly I am going to simplify this. This entire quantity right here is an integral that is going to yield us some function of time, this is again something multiplied by sine omega t. And here on the other side, you have this quantity which is going to be some function of time again, but that is going to be multiplied by with a cosine wt.

So, I can say that this function now is equal to this function, I can say this is maybe some function, which is a function of omega and some other function of time. So, G dash is some function that has to be computed. Similarly, for the other quantity, we can say that, let us use these symbols, let us say that this quantity when evaluated leads to another function, we will call G double dash. And this is again a function of omega and time.

So, now, if we use these 2 terms and if we replace that huge term that we had this simplifies our term to epsilon naught multiplied by some G dash omega t that was multiplied by a sine omega t plus another G double dash omega t, that is now a cosine omega t, okay so this has led us to a very interesting point. We provided with a sine input.

And by the way, I think at this point you can probably appreciate why I used s instead of t in the beginning, because that s became a dummy variable and finally, everything has gotten transformed into a function of time t. Now, we had put in an input that was only a sine function. But in the output, which is a stress here, you end up getting both a sine and a cosine.

So, this is basically what it implies that you have a sinusoidal input which is generating another's, you can think of it as having being or being generating another sine function now as our output, but that which is phase shifted. So let us for a second here, what I am going to do is I am going to use a trigonometric identity, which you might not be familiar with, so I will just introduce it to you.

So, let us say you have a function a sine x plus b cosine of y. You can show that this is equal to root over of a square plus b square, this quantity can be written as root over of a square plus b square into sine of, okay this is not y this is an x (sorry) sine of x plus alpha where tan inverse alpha is equal to tan inverse of b by a.

If you want to derive this, so the proof I will just outline it, I will not proved the entire thing. The proof for this if you want to do it, then you say that you write this side as some r times sine of x plus alpha, where alpha and r are to be determined. So, if you now open this bracket, you will have r sine x cosine alpha. So, using the trigonometric identity for sine theta plus x, so, we are expanding the bracket term here. Here you have cos x sine alpha.

So, if you compare these two, you can see that a is equal to r cosine alpha and b is equal to r sine alpha, okay. So, we will close this here. So, from this we can see that this particular quantity probably can be written in a more simplified form.

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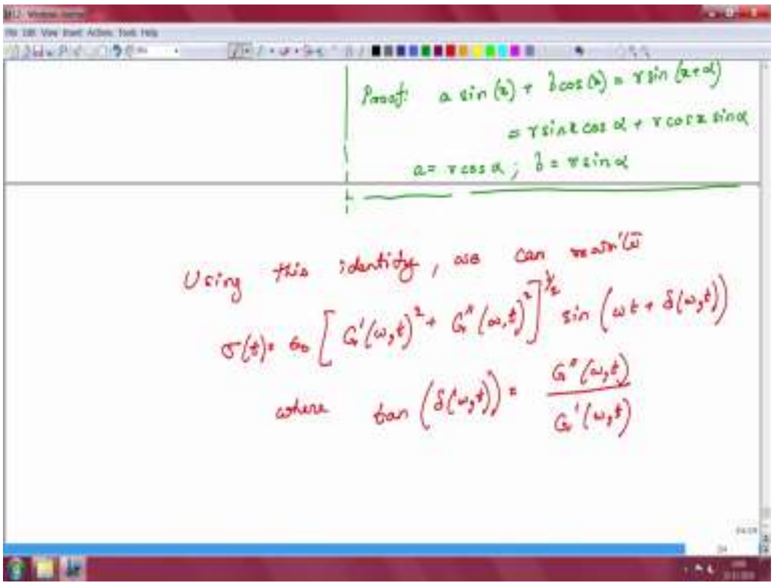

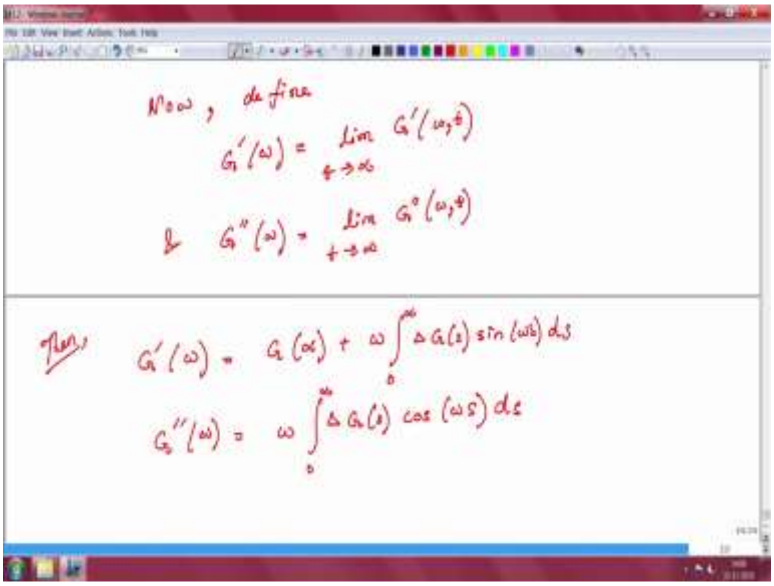

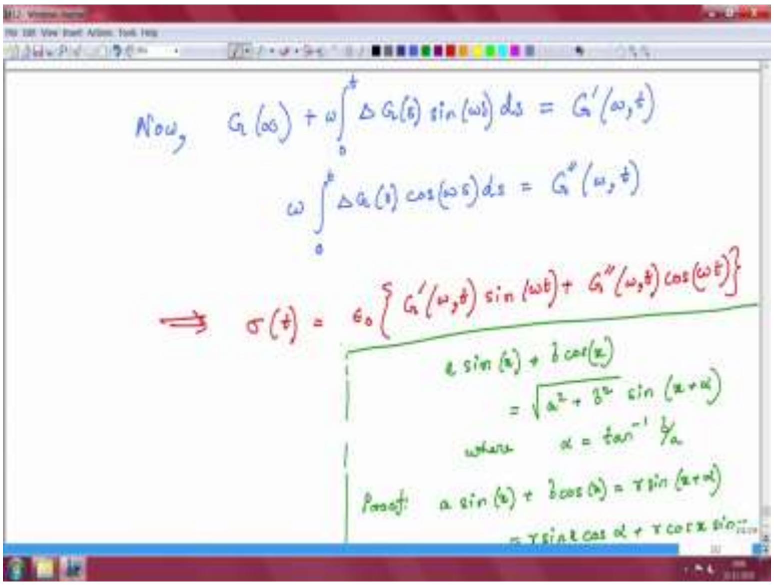

So, using this trigonometric identity, sorry just a second, we have, so using this identity we can rewrite sigma t now becomes epsilon naught multiplied by G dash omega comma t square plus G double dash omega comma t square the whole thing square root into sine of omega t plus some phase shift which is now a function of omega. Actually it can be function of omega comma t, right. But what is where tan of the phase shift is given as G double dash omega t is equal to the ratio of these 2 terms.

Now here, what we see is, so let us just quickly go back to the expression we have an output where obviously, the amplitude is also a function of time and the harmonic part is or the sinusoidal part is also a function of time. So, this amplitude part arises because you are also seeing the transient behavior.

So, if you want to see the long time scale steady state behavior, we can set the t is we can take the limit t tending to infinity. So, now define. So, now, we will define the limit terms, so, we will say that G of omega is equal to limit t tending to infinity G dash omega comma t and G double dash omega is equal to limit t tending to infinity, this is now G double dash omega comma t.

So, if you write it like this then if you put, so then or you can, so that once you define it like this, then you can write that then this term is basically in terms of the original integral or the original sum, it is now equal to this same quantity now, except that in the integral the definite integral, the limit t is replaced by t tending to infinity.

So, you have delta Gs and then we had sine of omega s ds and for the second term is now equal to some omega delta Gs cosine of omega s sorry, this is not a dot (just a s) omega s ds.

So, looking at long times steady behavior, steady behavior of the amplitude. We can say that sigma t is now going to be my epsilon naught into G dash omega square. Now this phase shift becomes also only a function of omega, right.

So, where the tangent of this is still given by the ratio of the G dash omega by G dash, G double dash omega by G dash omega. So, this becomes a very-very important result for us, this. So, this was derived using the idea that you had taken a sin function as an input. But in a more generic term you can also have a cosine input.

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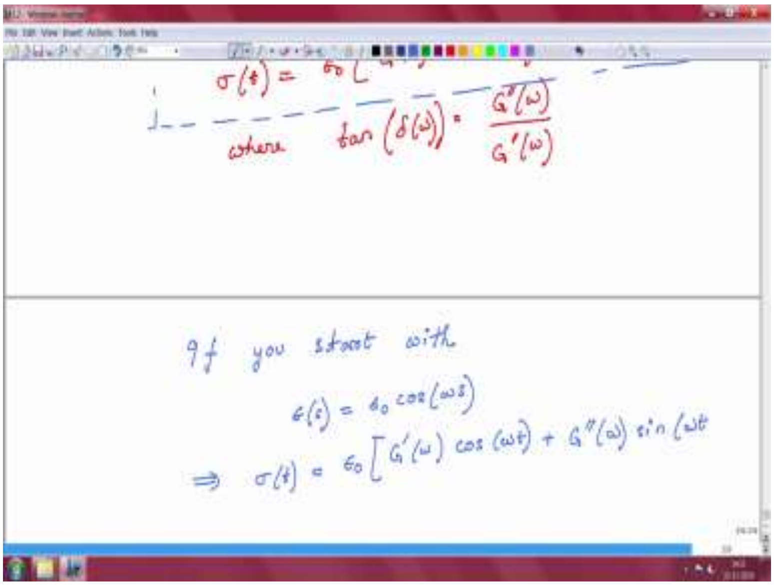

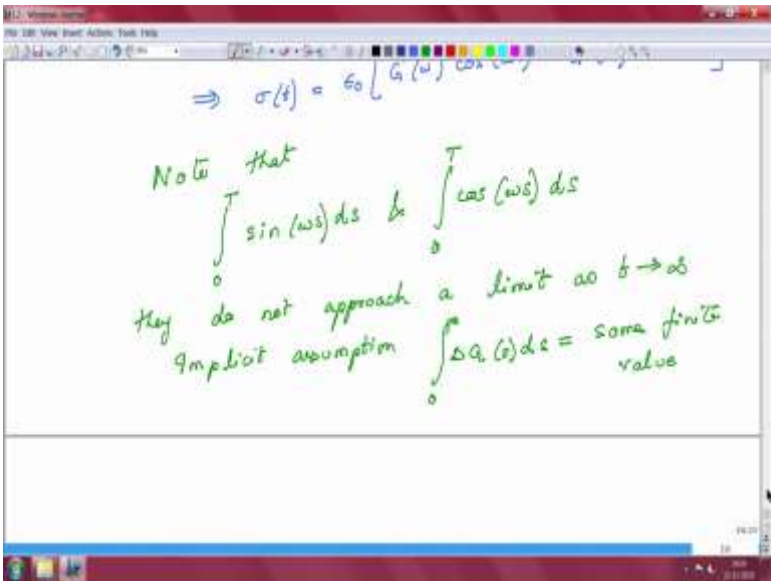

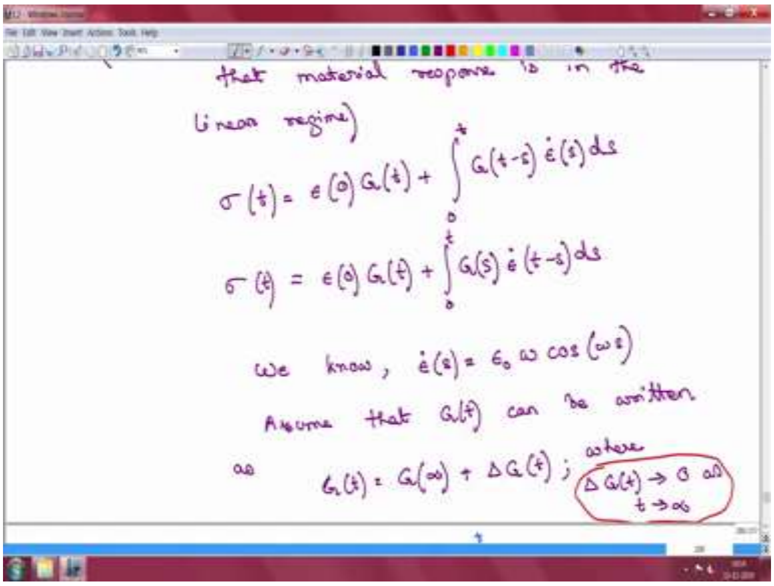

So, if you take a cosine input, if you assume, if you start with epsilon of s equal to some epsilon naught cosine of omega s, then you can repeat this entire process all over again as we just did, and then you will end up with a new equation. Actually the equation will not be really fully new, it will be seem very-very similar, okay.

And what you will find is if you do this calculation yourself, I am not going to re-do this entire calculation right now. But if you do this calculation, then what you will find is that this results this will lead to a situation where the stress as a function of time is given as epsilon naught.

You will again have a G dash omega t or omega and this time you will have a cos omega t plus G double dash of omega into a sine of omega t, sorry you will have a negative here. And this negative actually will result from the simple fact that you take the derivative of cosine you will also have a negative sign.

So, that will sort of flow down. So, now a couple of points that we, so once again we took some simplifications okay while we are deriving this one obviously in the very beginning was that epsilon naught is small, so, that we can assume linearity. The second was that the function Gt can be broken up into 2 parts, one is a constant part another was a time varying part.

And then it is important also to note, so note that integrals with cosine and sine are actually not very they do not converge properly. So, if you take time tending to infinity, you will not always get, they do not converge to a particular value. So, these integrals they do not approach a limit as t tends to infinity. But when you are multiplying it with another decaying function, it can give you it can reach a limit because here we do have there is with a cosine we have something that is getting multiplied.

Now, this function we have to assume that this is a decaying function because if this function does not decay then you cannot, the following results do not really follow. So, this, so another assumption is implicit assumption was that the integral of 0 to infinity delta Gs ds. This is some finite value, some finite value or basically that this integral converges.

And that is why we had required the delta G tends to 0 as t tending to infinity. So, if you remember in the very beginning we had done that, I will just quickly scroll back. And we, so here we had said if you recall the delta G is a decaying function of time. And the reason for that is so that these integrals can exist and we can write down the answer in that particular form.

So, if this is not true then the previous results will not hold true, why did we discuss this? What was our basic motivation?

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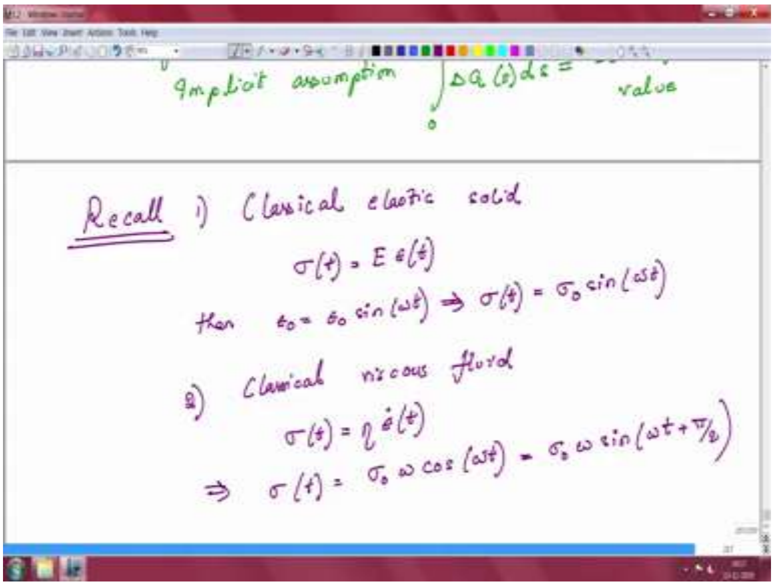

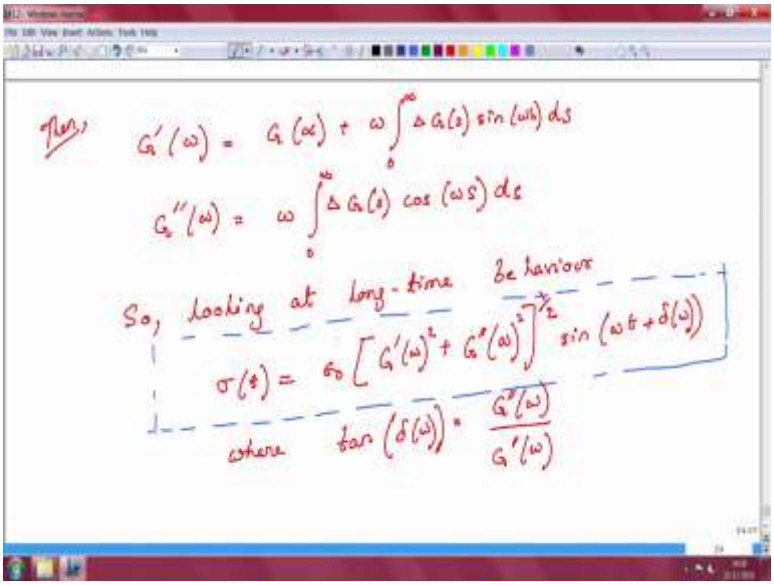

You want to recall that when we were first discussing, so, recall the case of the classical elastic solid, classical and this is this goes back to the very beginning of our lectures right, classical elastic solids.

We had said that the stress-strain relationship is given by this very-very straight forward relationship and this implies that when you have a strain, which is a time varying function, then your strain value is some sigma naught times of sine of omega t. So, for a pure or classical elastic solid, when you apply a sinusoidal strain history, the output is a stress, which is also a sine function and it is exactly in phase with the input.

But what happened when we did this for a classical viscous fluid, let us just quickly take a look classical viscous fluid. So, we had said that the stress-strain relationship in that case was given by a viscosity here, and this, this is what we are using. So, this implies that when you put in the same condition here you what you will get is a function here, which is some constant, we will again call it sigma naught into some omega into cosine of omega t.

We will take the derivative of epsilon t you will get a cosine, which is the same as writing it as some constant omega into sine of omega t plus Pi by 2. So, in the classical elastic solid, a sinusoidal input in terms of strain results in a sinusoidal output, where there is no phase difference between the input and output.

In the case of the classical viscous fluid, a sinusoidal input in terms of strain results in a stress where the stress and the strain again are sine functions, but there is of Pi by 2 phase difference or 90 degree phase difference, pi by 2 in radians, okay. But, when we are, when we did this calculation for a viscoelastic body, we got us, we came to a situation where the stress now is the stress is some constant some amplitude multiplied by another sine function, but this time there is a phase difference which is not going to be Pi by 2.

This phase difference obviously depends on the, this particular ratio of G double dash and G dash, but this is no longer Pi by 2. So, you end up with a different phase lag between the input and the output. So, this is a very-very important result. And this again goes back, we had discussed point that viscoelastic fluid, the phase there will be a phase difference. And now, we can see that mathematically.

At that time, we had just I had asked you to sort of believe me that if you have a sinusoidal input, then you will have an output which is phase lag, but now we can see from the mathematics how the phase lag behaves. And we have mathematical formulas governing the phase difference. So, we will stop here for this class, and in the next class, we will take up some of the other issues related to sinusoidal strain histories. So, we will stop here for today.