Introduction to Soft Matter
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Lecture 33
N Maxwell Model

Welcome back to one more lecture on introduction to soft matter. Last time we were discussing the solution to the general case the case of the nth, N models together. And what we saw was that, there is the constitutive relationship basically becomes a sort of a generalization of the previews case where we had polynomials on both sides and powers of D. And now you have basically the same thing where the power on that, power on each side is determined by the number of the element that we have.

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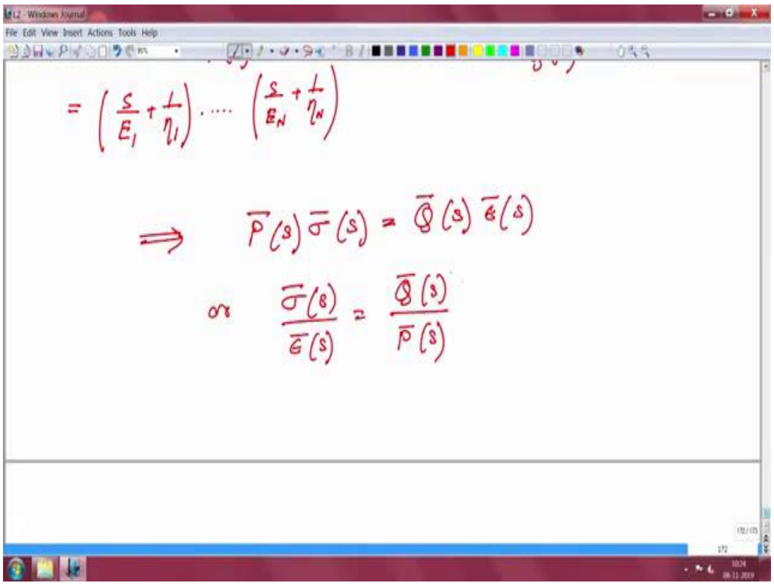

1) Stress - relaxation function

$$\epsilon(t) = \epsilon_0 H(t)$$

$$\Rightarrow \bar{\epsilon}(s) = \frac{\epsilon_0}{s}$$

$$\Rightarrow \bar{\sigma}(s) = \frac{\epsilon_0}{s} \cdot \frac{q_0 + \dots + q_N s^N}{\left(\frac{s}{E_1} + \frac{1}{\eta_1}\right) \cdots\cdots \left(\frac{s}{E_N} + \frac{1}{\eta_N}\right)}$$

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$$= \frac{G_0}{s} + \frac{G_1}{s/E_1 + 1/\eta_1} + \dots + \frac{G_N}{\frac{s}{E_N} + \frac{1}{\eta_N}}$$

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where $G_0, \dots G_n$ are constants that have to be determined.

So, we had come to the conclusion that when we take the Laplace of the polynomials, we end up with a simple expression like this if we write in terms of the polynomials. So, now let us go ahead and look at the different cases. So first thing we will do is look at the stress relaxation function. Look at the stress relaxation function where you are basically going impose a strain, a step strain which is again epsilon naught into H of t.

So the Laplace of this is already known and it is very simple, it is epsilon naught by S. So then the stress (relax) so in this case, so then once you apply this to the previous expression what you get is so applying this to that previous case you end up. So this implies that your Laplacian of the stress function looks something like this.

You will have epsilon naught by into S multiplied by this q naught, you have N terms here qN S to the power N and then on this side you have all the different, all the N terms multiplied together. So this is sometimes I will leave this in the factored form and it is easier for us to discuss, so here I am just leaving it in the factored form. So now what you are having is basically this is inner rational fraction form.

And the question is we cannot take the Laplace inverse of this, we have to express it in some simpler format. So you can show that for this rational fraction this is equivalent to some constants and now the fractions can be, you can just take the individual terms here. And you can go on constructing, you are all the terms till GN, and for the GN case you have a denominator that is S by EN plus 1 by eta 2 power N. So, we have N plus 1 constants, so where these are all new constants that we are introduced.

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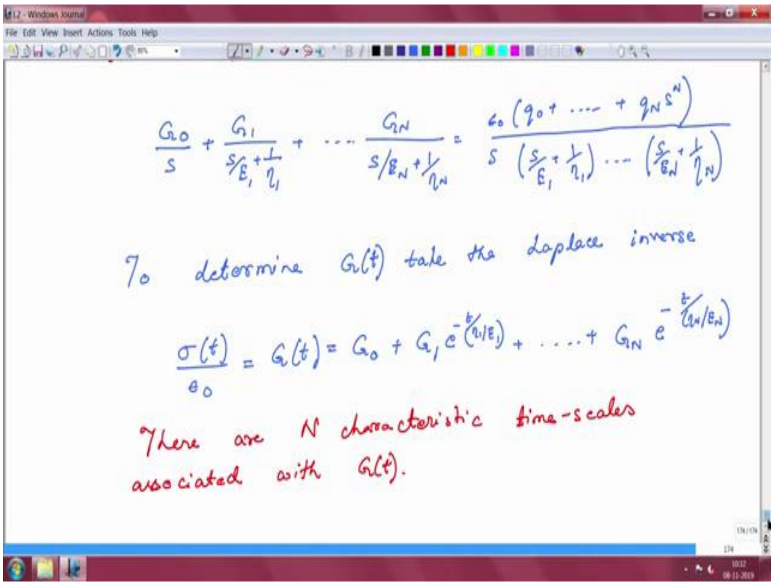

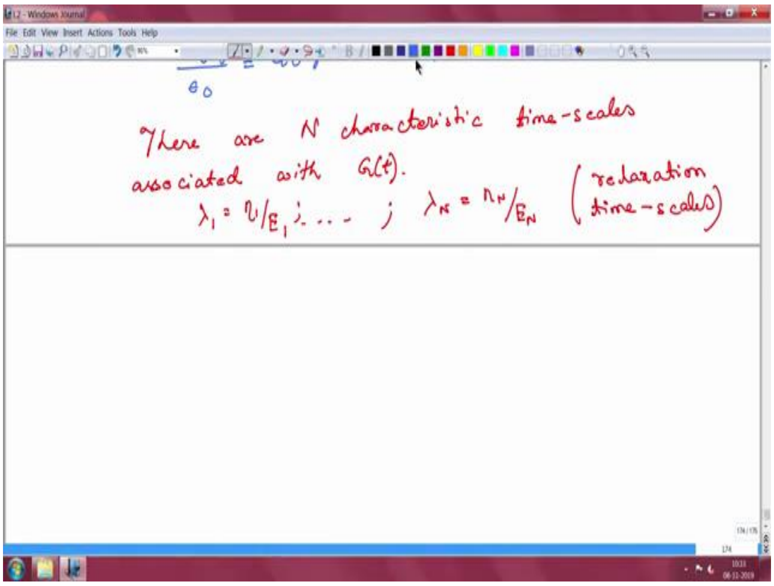

So, G naught till all the way till GN are constants that have to be determined. This video will need a another set of editing. So how can you determine G naught till GN. Well you can equate the two sides, so the two sides what we are claiming is that these are the same and this is the same as your epsilon naught.

So you can take this entire denominator here, bring it over here multiply this and you will end up having both sides will be two polynomials and here also you will have powers of S going from the from the 0 to N and you, on this side you again have a polynomial. So you will equate all the different, so you will equate all the different coefficients for the different powers and from that you will end up getting you will able to determine all this different constant.

So the different powers will give you N plus 1 different independent equations and you have N plus 1 unknown so you can solve for them. So in theory it is solvable, something it is going to be very-very cumbersome in terms of algebra here. And we are not going to do that, so what we are going to assume is that now that these constant can be uniquely determined.

So determine G of t which is what we after, take the Laplace inverse and when you do the Laplace inverse you see that which is the same as Gt becomes so the first one is G naught by S. So that is simply simply base G naught and in each of the other cases you have you will have some this G1 multiplied by an exponential function. And the exponential function would be such that it will be t by eta 1 by E1, and all the way till GN.

So there are, so this, so now here there are N characteristic time scales associated with Gt. And the time scales are such that you have lambda 1 equal to eta 1 by E1 and similarly you have lambda N, eta N by EN. These time scales are also called the stress relaxation time scales, sometimes often called the relaxation time scales. So what we have done now we can again redo for the Creep response. So we said so we did the stress relaxation function

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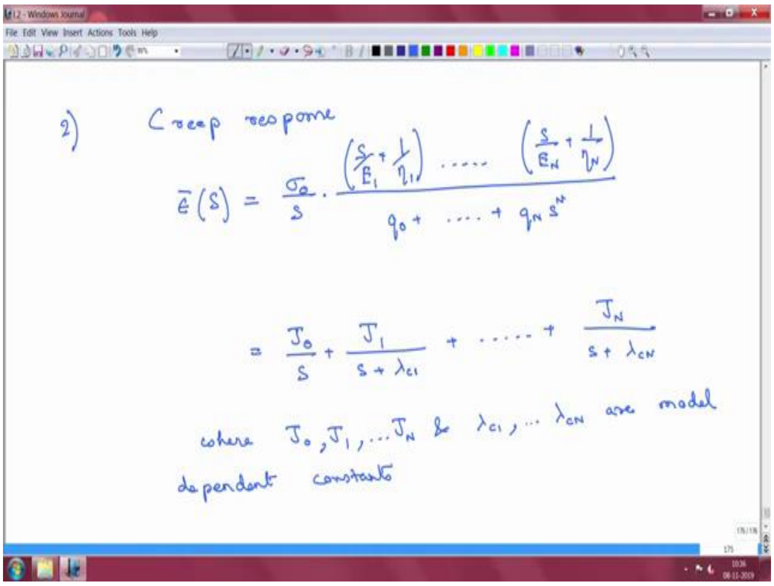

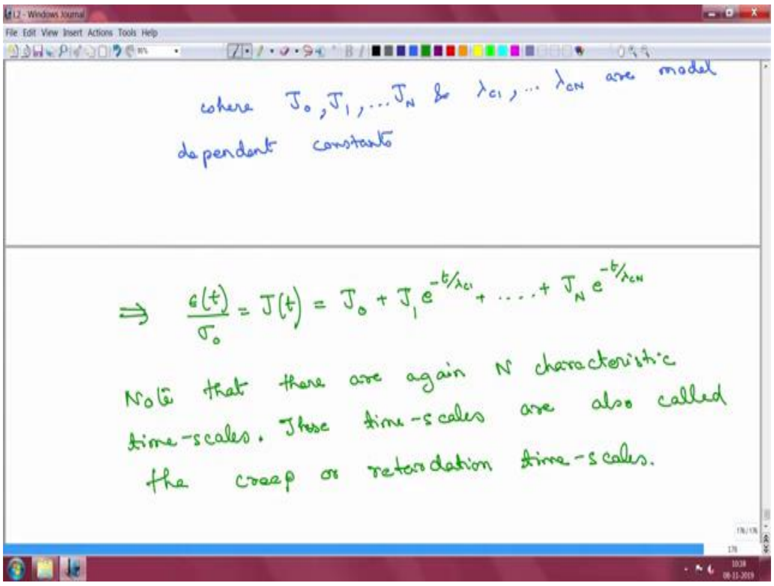

So, now we will do another one, which is the Creep compliance function or the Creep response issue that we have to look at, so let us look at Creep response. So in this case, you will just have the opposite so you are looking for epsilon at the Laplace of the strain and that is going to be some sigma naught by S and the numerator and the denominator will be flipped this time.

So you have on the numerator S by E1 plus 1 by eta 1 and all the way till all your N multipliers and on this side you have this q0 plus all the way till qN S N. So this entire thing again can be factored just like we have done it for the previous case this is again a rational fraction and the same rational in this rational fraction can be again be written as some J naught by S plus J1 by S plus some lambda and I am going to call this Creep 1 and the same way till JN S plus lambda.

I said that this denominator is not easily factorable but this is still a polynomial so you will have your different factor, different roots will be there. So if you determine all that then you now had to determine j naughts and all these different-different constants. And they are all because this previous equation is given to you, these are all again determinable. So where J naught, J1, JN and lambda C1 are model dependent constants.

So you have to now take the Laplace inverse of this to get the Creep response. So your epsilon t by sigma naught which is the same as the Creep compliance function is now going to be some J naught plus J1 e to the power minus t by lambda C1 plus and all the way up to JN e to the power minus t minus lambda CN.

So, note that, so now note that there are again, there are again N characteristic time scales in the system, there are N characteristic time scales in the system. And these time scales are not different from the ones that we had for the Creep, for the stress relaxation function. So these characteristics time scales there are also N, but they are different. So these time scales are also called, are also called the Creep or a relaxation, sorry or retardation time scales.

So the appropriate times scales actually depends on the particular function that you are looking at and if you are looking at stress relaxation function then you will have in a model with N parameters, you will have N characteristic time scales which are called relaxation time scales but if you are looking at the Creep function then you also again have N time scales but in this particular case they are called retardation times scales.

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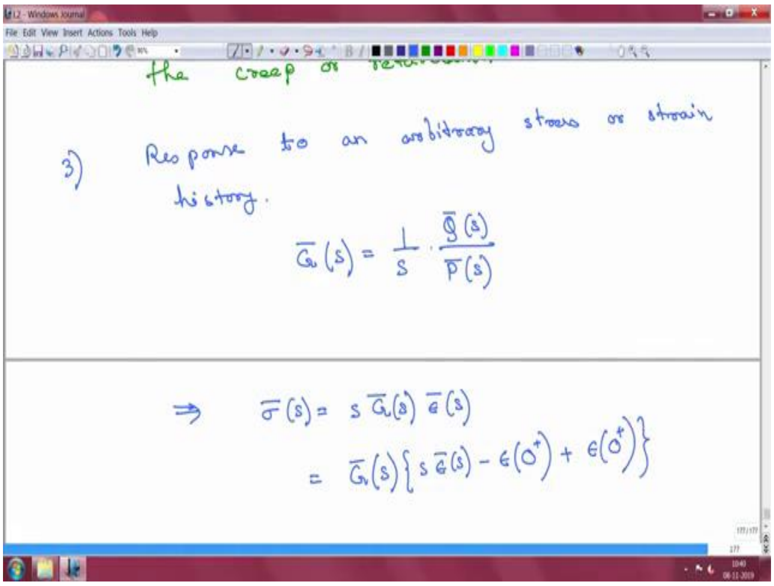

$$= \bar{G}(s)\{s\bar{\epsilon}(s) - \epsilon(0^+)\} + \bar{G}(s)\epsilon(0^+)$$

Take Laplace inverse

$$\Rightarrow \boxed{\sigma(t) = \epsilon(0^+)G(t) + \int_0^t G(t-s)\dot{\epsilon}(s)\,ds}$$

This 's' does not relate to Laplace transform

Taking Laplace inverse of both sides

$$\Rightarrow \boxed{\sigma(t) = \epsilon(0^+)G(t) + \int_0^t G(t-s)\dot{\epsilon}(s)\,ds}$$

Similarly work out the response to an arbitrary stress history

$$\boxed{\epsilon(t) = \sigma(0^+)J(t) + \int_0^t J(t-s)\dot{\sigma}(s)\,ds}$$

Creep Response

$$\sigma(t) = \sigma_0 H(t)$$

$$\Rightarrow \bar{\sigma}(s) = \frac{\sigma_0}{s}$$

Previously we saw,

$$p_0\sigma + p_1\dot{\sigma} = q_0\epsilon + q_1\dot{\epsilon}$$

$$\Rightarrow \bar{\sigma}(s)(p_0 + p_1 s) = \bar{\epsilon}(s)(q_0 + q_1 s)$$

Now, replace $\bar{\sigma}(s)$ with $\frac{\sigma_0}{s}$.

So, now that we have been able to do this particular two cases. Let us turn our attention to the case of the response to the arbitrary stress is to an arbitrary stress or strain history. So we saw that you can define a Laplacian which is 1 by S times of Q bar of S. And when we take the Laplace inverse of this we end up getting the stress relaxation function.

And we also know that this relationship holds, this particular relationship is also true right so if you combine this two these two equation so equation this one where which relates the ratio of the stress the Laplacian of stress and the strain to the different polynomials. Then you can write, then previous case you can write that sigma bar S is equal to S Times G bar of S epsilon bar S.

And just like what we had done previously. We want to take the Laplace inverse of this entire quantity on the left. So what you are going to do is you are going to club, you are going to this separate and you are going to club S and epsilon bar S and here you will add this initial condition subtract the initial condition and add it.

So you end up having, so will end up having these two expressions and then when you take the Laplace inverse, so take Laplace inverse and then you will end up with sigma t is equal to and then here you have. And remember with this it now we are going to introduced a dummy variable so I can just introduce S or I can introduce any other quantity with this S is going to be different from the Laplacian, so just please keep that in mind

I am keeping S because I just want to be consistent with what we have done till now. So just a note this S does not relate to Laplace transforms. So voila, what we have is this expression which we have seen before, we have derived this for the general case but we had seen this expression before also, when we had derived it for very special cases so if we go back up we will see that we have just trying to quickly find that expression.

So we had derived this particular expression for a special case and this special case was for this particular equation. So for this simple case of model we had seen that when you take the Laplace the relationship between an arbitrary given strain history and stress the current state of stress is given by this particular integral.

And it so happens that when you re-derive it for case of a more complicated system with N number of elements you end up getting back the same equation. So this shows that this particular equation is more fundamental in nature and it is not really related to the models and the other particular model that you have chosen and you can show this that this is generally applicable to the case linear viscoelasticity.

So, just like what you have for the stress given in arbitrary strain history you can re-derive the case of the strain for an arbitrary stress history when we flip the two and what you will find is that the expression just looks very-very similar to this one and that expression will now have J instead of the G that we have here. So that expression that exercise is left the derivation of that is left as an exercise for you so you can re-derive

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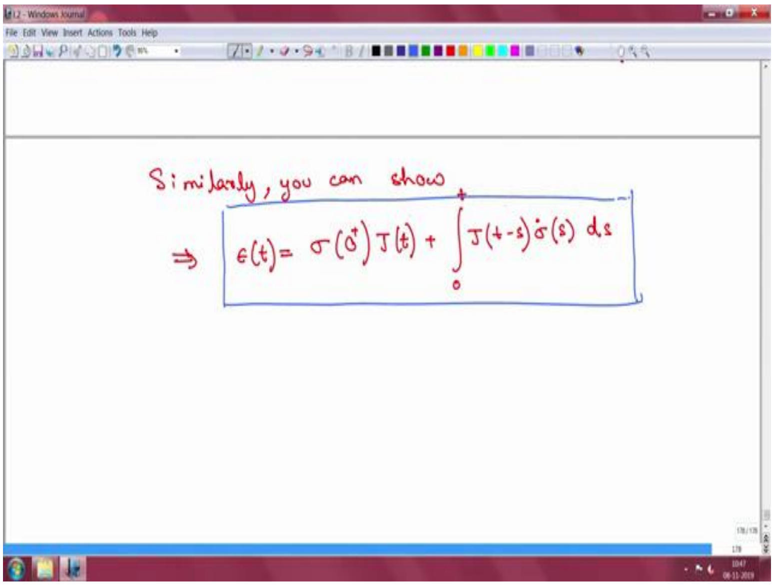

So similarly you can show, that you are epsilon t is equal to sigma 0 plus. So instead of J, G you will have the J the Creep compliance and here you will have a very similar convolutional integral and instead of J you will have, sorry J, instead of G you will have J, is becoming a bit of a tongue twister here.

So, this two expressions are going to be very important to us and they will be used quite a bit. And a simple note that in some of the books you will not find, you will find this expression in a slightly different format. And here if you look at the this integral on the left hand side, sorry right hand side you see that it goes from 0 to t, it is a definite integral and the lower value of time here that we have input is 0.

But you can also put minus infinity to the current time t, in which case this particular expression will simply be merged with the integral and you will just be left with the one convolutional integral. So often in some the books you will find this expression in slightly different format and just because it exists in a different format does not mean that the two equations are any different. In this particular case you just have to recall that you will always assume that there is a jump at t equal to 0. So this particular expression is valid for that kind of case.

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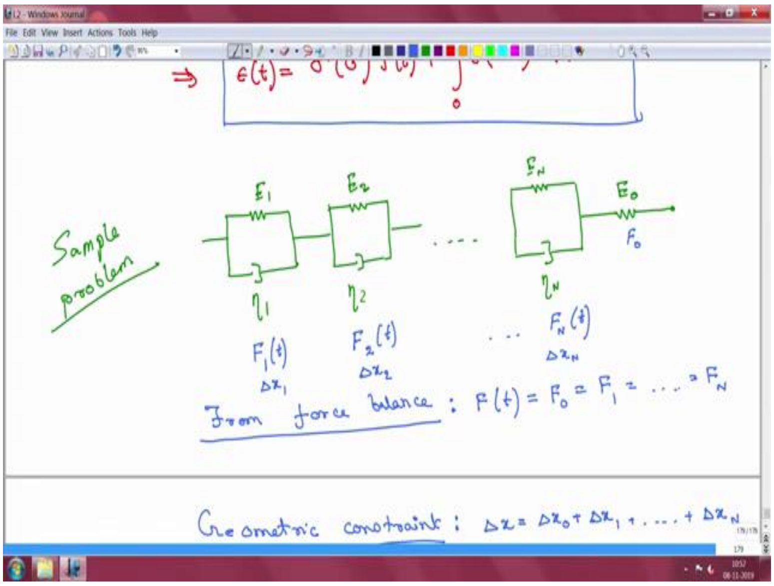

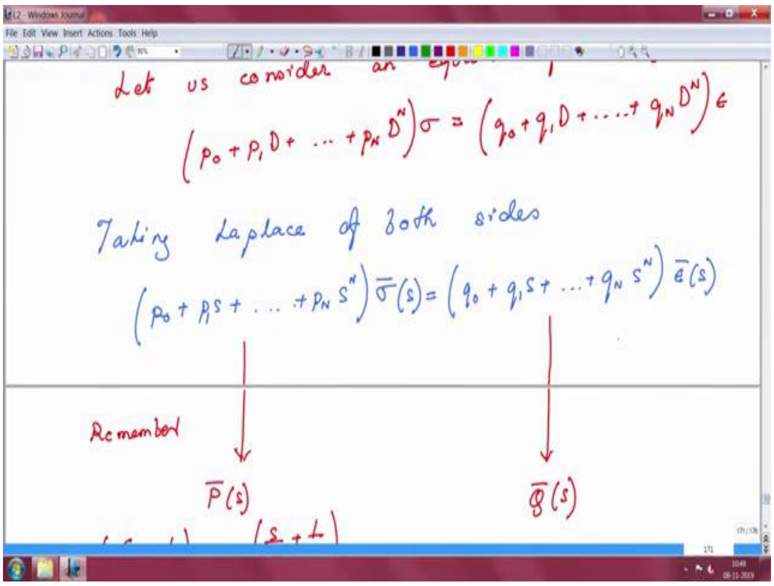

So the last may be I want to do one another sample problem where, so sample problem and we have seen that this formulation of the stress and the strain in terms of two polynomials seem like a very generic idea that would be valid for different kind of models that you want to apply. So it is only that the constant and the slide form will change given from one model to another.

So let us ask ourselves that if you have N Kelvin Voigt models and this goes all the way till N, so this eta 1, E1, so this is EN, this is eta N, and let say there is a last spring there. So you have N Kelvin Voigt models and you have one spring. So can we re-derive force deformation relationship and see whether or not it falls in the same category as what we have done for the N Maxwell.

So the only what I am trying to do is we had shown that this particular form. This form we have gotten form the N Maxwell models in parallel. So what happens if we put a lot of Kelvin Voigt models, Kelvin Voigt Mayer body's and one spring in series like this? So, what are, so there would be, so each of these different bodies will have when you apply a certain amount force it will share a fractional part of the force.

So let us just say that each of them is experiencing force F1 another one is experience F2 and all the way till FN. And this is this another force here. So from force balance, we can say that the net force has to be the same in all the cases. Because these are all massless, so these individual forces are no longer, not a fractional part it is actually the full of that.

So this is just F naught sorry this is all are equal and all the way. And what is now going to be geometric constrain on this system. The geometry constraint well be such that all of this will be experiencing different deformations and the total deformation delta x is now going to be a sum of all the different deformations in all the different units.

So now on the next, so to solve it further, what we have to do is we have determined the force balance of the geometry constraint that is the quite straight forward and then we have to write the individual force deformation relationships and then try to see how we can combine them into one equation. So I leave this here and I would like you to try and finished this problem as part of yourself self-work and we will finished will begin with this particular problem in the next class. So we will stop here today.