Introduction to Soft Matter
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Lecture No. 31
Two Maxwell Model

So, welcome, everybody to another lecture on introduction to soft matter. Last time we left off while were discussing Jeffrey's model, we are not completed our discussion. So today we will start off where we left off and we will finish up that discussion. So, what we had done is we had written down the two expressions for the forces and then we had to simplify that. So just let us write down that one more time. So, we have the force for the Kelvin Voigt body, the Kelvin Voigt Meyer body, and then the force for the spring, sorry the dashpot we have to write those two separately.

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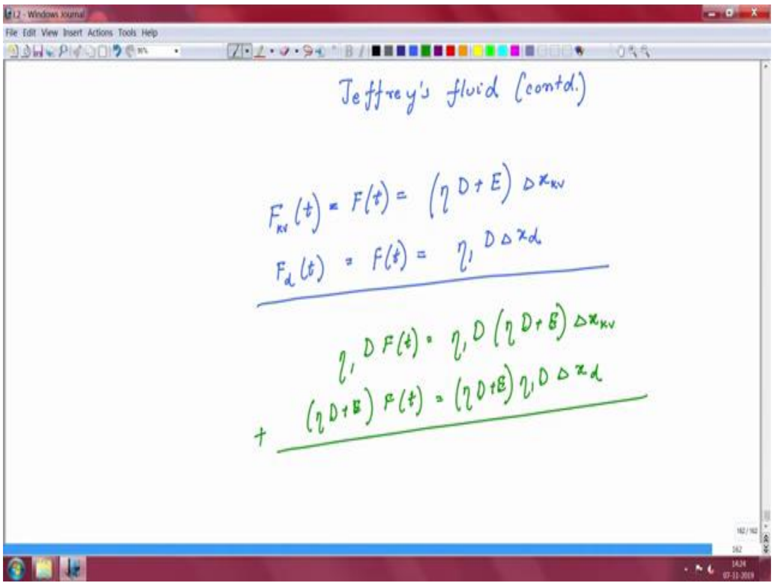

So, let us do that. So, we have the force for the Kelvin Voigt and this is as said, as we discussed is equal to the force in the system. And this in terms of our operators, this is a eta into D. D being the differential operator plus E and multiplied by this is delta XKV. Similarly, for the other case, we had the force in the dashpot is equal to the net force on the system anyway and that is again equal to so, eta 1 that is that belongs to the other dashpot into D times, where D is the operator XD.

So, what we want to do is, we want to add, we able to add these two together but something is preventing us at the moment because we want to add the displacements and make them the, to end the total displacement for the system. So, what we can do is we can multiply and what is missing is this part. So, what we can do is we can multiply it the two equations with appropriate factors, such that the addition is possible.

Now to do that, you will recognize that we have to multiply the first equation by eta 1 D F of T eta 1 D, sorry, this is eta, this belongs to the other dashpot eta D plus E delta XV, KV. And then we have to multiply the other, the second equation with the appropriate one which case we are going to multiply it with this particular factor. So, we apply eta D plus E, times of FT equal to this is eta D, delta Xd dashpot. Now you can add these together. So, the addition is not possible.

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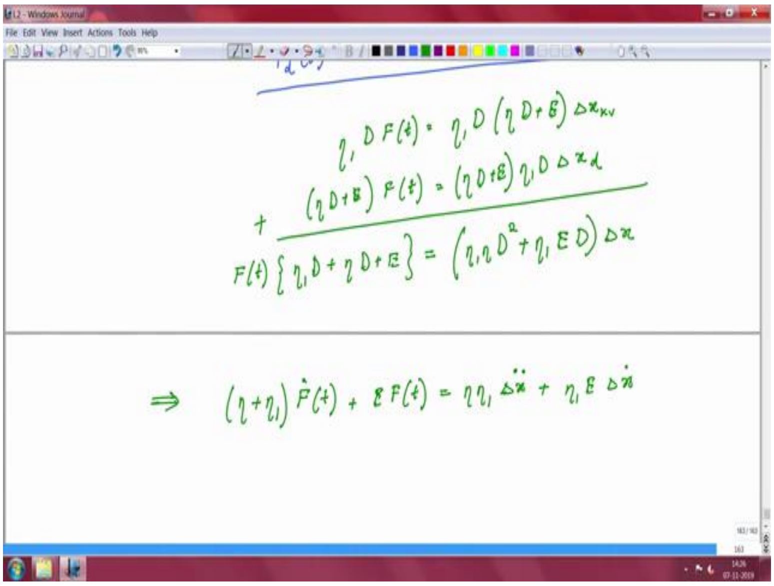

So we add and here we will get FT eta 1 D plus eta D plus E and this is equal to, if you simplify, if you open all the brackets, then what we find is you will have an eta 1 D square plus eta 1 D, let me just write if ahead because D is the operator. So, I choose to write it last into delta X. These two got added together and they become clicks.

So this is basically nothing but if I want to write it in the form of the dot, if I now change my operator to just say, to the dot form, then we have eta plus eta 1 F dot of T plus E times of FT and this is equal to, on this side we have a double, so there is a double dot, double derivative of delta X. And the factor in front of it is eta into eta 1 and this is delta X double dot plus eta 1 E delta X dot.

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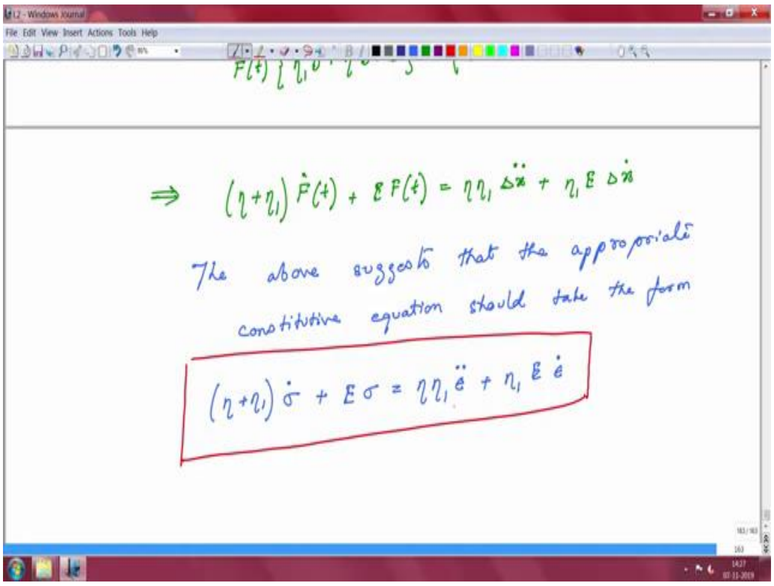

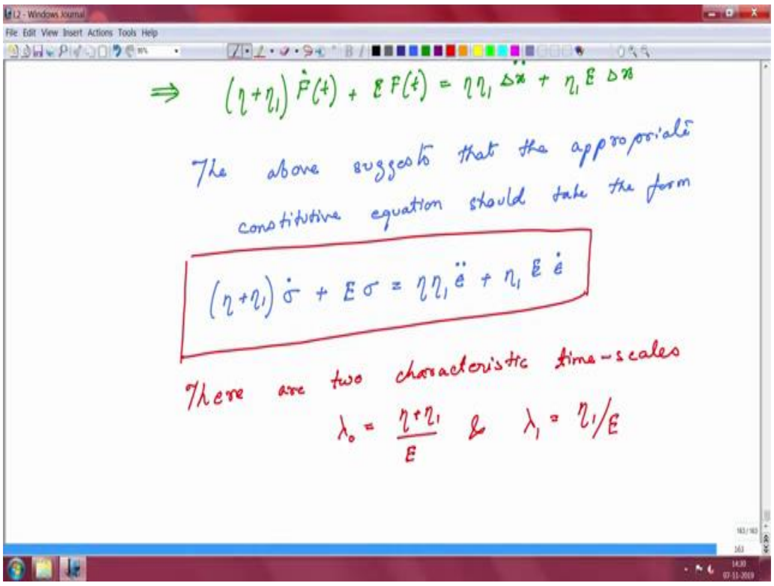

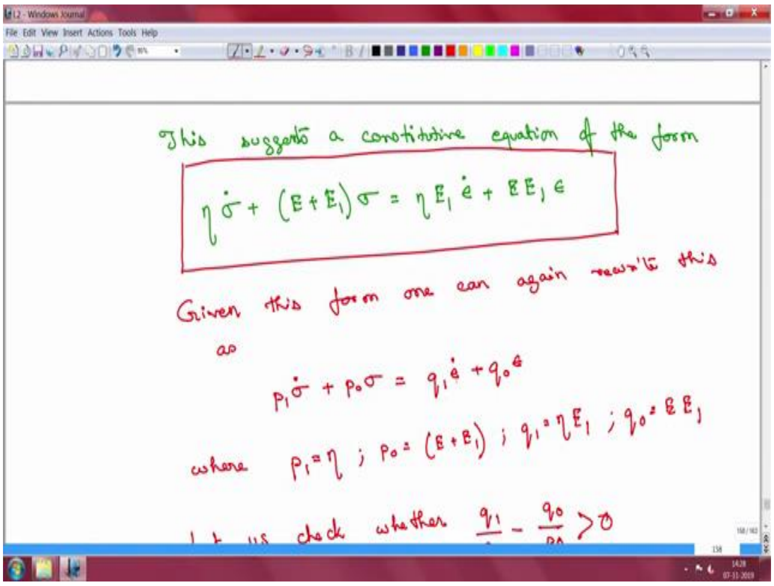

So, this now suggests, the above suggests that the appropriate constitutive equation should take the form and then we are just going to create an equivalent system where we say. So, this basically suggests that your stress strain relationship can be given by E times sigma and on this side, we have eta, eta 1 epsilon double dot plus eta 1, eta epsilon dot. So, this is a question that we are, this is the constitutive relationship for the Jeffery's fluid.

Now, you will recognize that this equation has a double derivative and not, and hence does not fall into the form that we had solved for previously. So, this all our solutions that we had done previously were only for this particular case where you had a single derivative of the stress and a single derivative or the first derivative of strain. But here you have double derivative. So, this cannot be solved in the same process.

But obviously, you are going to have to use a similar, you can use a similar process of taking Laplace transforms and solving for that. But this is quite interesting that the fact that the double derivative appears itself is very interesting here. So, one question we should ask is, what are the other situations in which double derivatives can exist? And that should lead us to a window into try understanding a generalized equation when you have many, many of these different strings and dashpots, all of them put together in a big circuit.

Finally, what you are doing is you are just putting all these different, different elements to model the complexity of real life. So, if a real fluid exists, it is likely that it will have many different especially in cases where polymers where there is polydispersity and etc. The polymers will have, will behave, will not behave in a manner that you can represent them by a single E or single eta.

So, a realistic model should have many different springs and dashpots together probably and hence, we should now look towards a more general case. And before I end this particular discussion, I just want to note that there are two characteristic timescales here, are two characteristic timescales in Jeffrey's fluids. And the first one is eta plus eta 1 by E and the second one is eta 1 by E.

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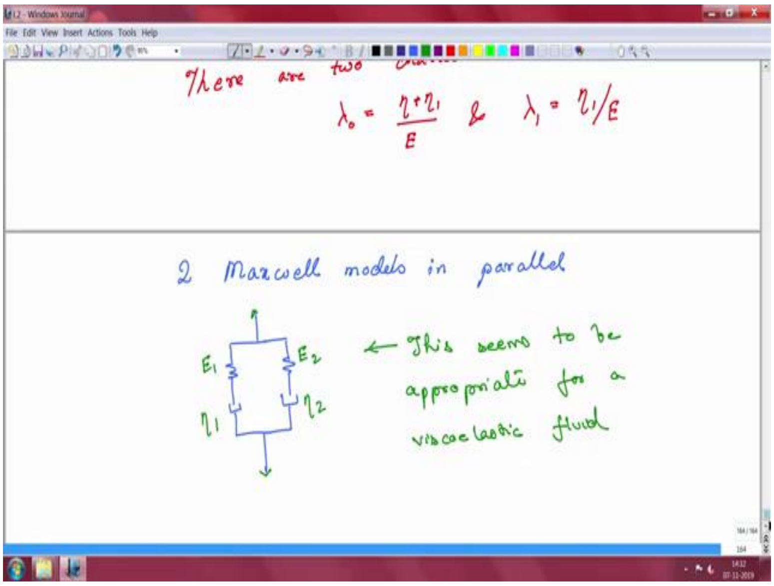

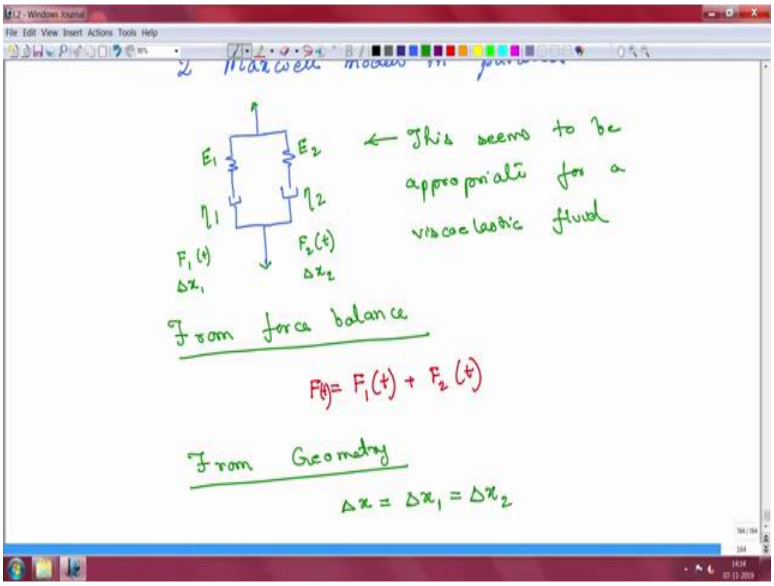

So now we march towards a little bit more complexity. So, the first thing we will do is two Maxwell models in parallel. And we want to check out a hunch whether two Maxwell models in parallel will end up giving us a double derivative for both cases or not, we will see what happens. So, when you have two Maxwell models, let us just draw this. And now that all of these are going to be Maxwell, both of these, I am going to label this E1 and E2 for easy demarcation and eta 1 and eta 2.

Before we proceed further on this can we use our simple understanding of such systems to conclude whether this is a viscoelastic fluid or viscoelastic solid? See, if you apply a force on this system, there will be some force that we shared by the first Maxwell and the second Maxwell model. But whatever the forces we have a small; you will always have a dashpot that is going to keep on giving you displacement. So, this both of them independently are going to behave like a fluid and the system itself is going to behave like a this classic fluid. So, this is appropriate, this seems to be appropriate for viscoelastic fluid.

So now, we have to figure out the correct constitutive relationship for this. Now, because they are in parallel from force balance we can say. So, let us say that the forces in them individually are F1 T and delta X1 and here you have F2 T delta X2, there is no more need to label these subscripts and any more complicated way because they are both Maxwell's model. So, I am just using the labels X1 and X2.

So, the total force here is obviously just like the previous case going to be a sum of the two forces. Again, similarly, similar to the previous case from geometry will have delta X equal to, because it is geometric, both of them are dramatically constraint, the displacement of the system has to be equally, equal and be shared by the two cases. So, we will write that relationship as this.

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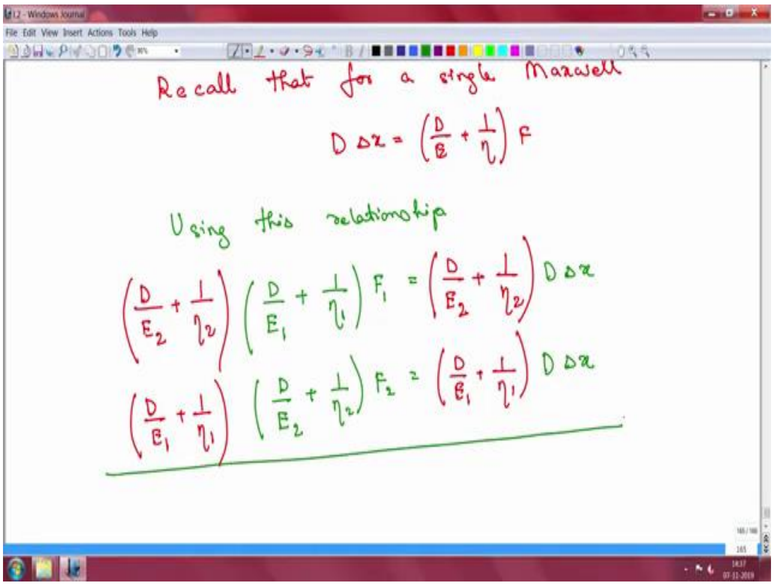

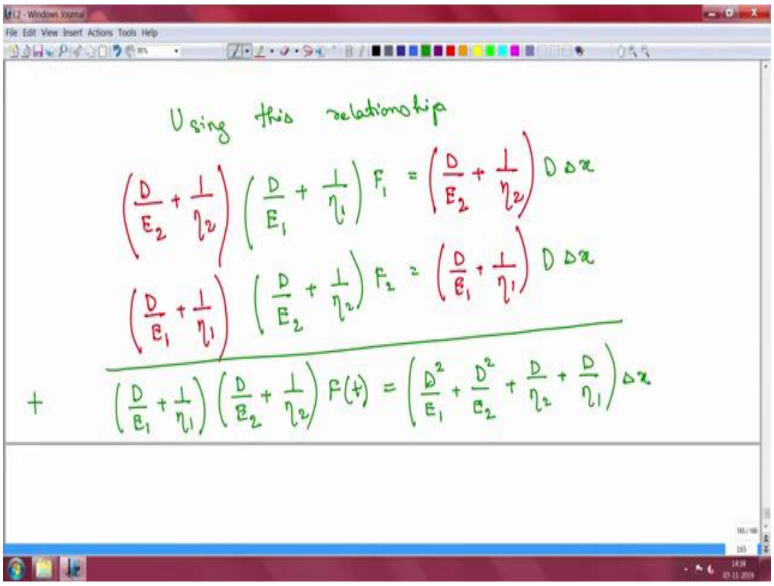

Now, recall that for a single Maxwell, the equation that we used to write would be, so far, a single Maxwell load, write D of delta X equal to D by E1 plus eta times of F. So, so we will use this here. So, using this relationship, so, using this relationship we can say that D by E1 plus 1 by eta 1 times of F1 is equal to D by delta X1. Similarly, for the second Maxwell we have D by E2 plus 1 by eta2, F2 equal to D times of delta X2.

Now the delta X is our shared, so I can just as well drop the subscript. And if I drop the subscript, life becomes a little bit easier for us. But we have these F1 and F2 and we have want to be able to do our work here. So, in order to do this, we have to multiply both sides by an additional factor. And that additional factor here if you see is going to be, so multiplied this said, but I have to also multiply the other side.

So, what I am going to do is I am just going to create some space for myself. So, we have this factor here now, this is delta X. Similarly, I have to multiply this side now by E1 and I do the same thing here, sorry, this is eta 1 times of delta X. So, now this is in a situation where I can add these two together. So, you just add them and the resulting equation will look something like I will this in the factored form and here you will have.

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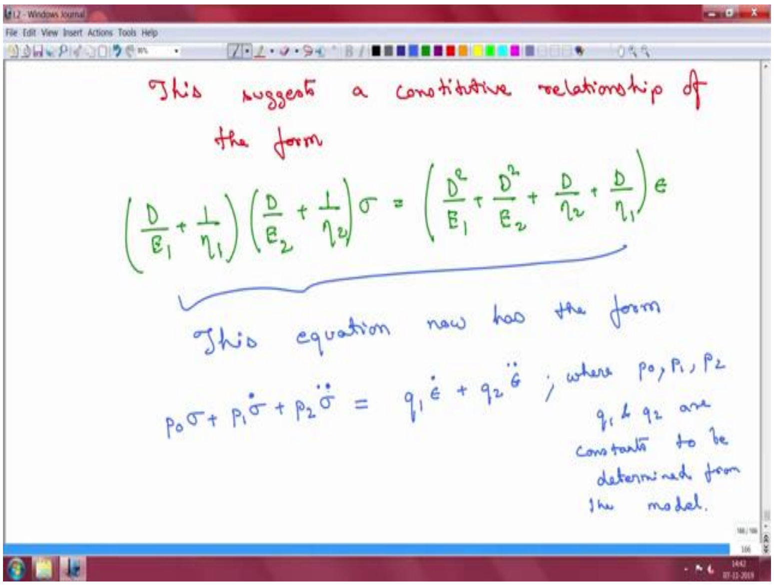

So, what does this suggest? So, if I, now if I have to replace the F, so it suggests a constitutive relationship of the form. So, we can replace the F with the stress of just writing the prefactors once again and then you have the stress here and then you have, sorry, now I replaced the displacement with epsilon. So, this is now of the form.

So, we can see that you have the double derivative of both sides I just left the left-hand side in the prefactor form because, in factored form because it is just easier to leave it like that. Because the final point is that this equation, this equation now has the form. So, here you will have see, you will have a double derivative of stress, you will have a single derivative of stress and you will also have one term with only stress.

So, this side, the left-hand side is of the form P naught sigma plus P1 sigma dot plus P2 sigma double dot. But the -hand side has a double derivative of the strain. It has a single derivative of strain but does not have any other constant term. So, this side will have it looks like Q1 epsilon plus Q2 epsilon dot, there is a epsilon dot here. So, Q1 epsilon dot, where these are constants. So, P1, P2, Q1 and Q2 are constants to be determined from the model.

So, we see that we have by having two Maxwell elements juxtaposed with each other, we have created a situation that leads us to an ordinary differential equation once again to determining the ordinary, ordinary differential equation relating stress and strain, but this time you have, because you have two models, you also have a double derivatives. Now, this probably helps us, suggest a platform or a model or a mechanism by which we can generalize this idea to let us say N-Maxwell models in parallel.

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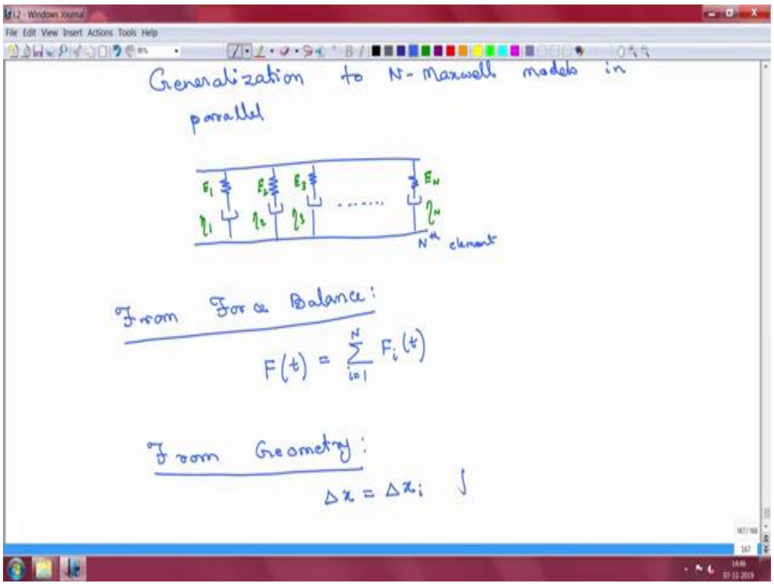

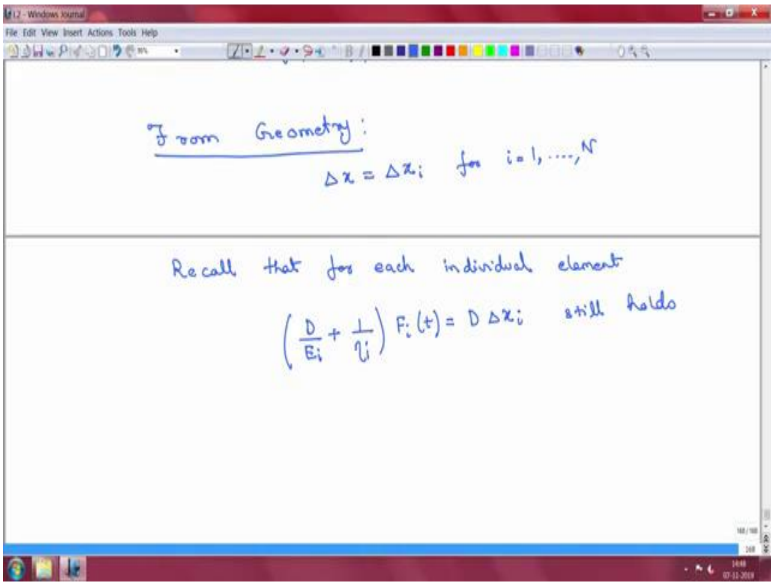

So, let us consider next the generalization to N Maxwell models in parallel. So, we have a situation where, we have many of these Maxwell models, actually N in total number. So, these are, this is the Nth one, Nth element. And each element is individually a Maxwell model. So, if you have to write the constants, we will just start for E1, this is E2, this is E3, and this gone and this will be EN.

Similarly, you will start here with eta 1, eta 2, eta 3 and all the way till eta N. So, before we proceed with this, is this still a fluid or has the behaviour changed? So, if you think about this particular model right here, then see if you apply know a certain amount of force to this particular system, then that force will be divided into each of these separate elements.

And each element will experience a certain amount of force, which is going to be a fraction of the total applied. Now, each of these elements, however, small the forces the dashpot will result in a motion such that dealt, the strain or the displacement of the dashpot will not be constrained with time, it will keep on moving as long as the force is applied.

So, that means, that each of these elements separately individually will keep on acting like fluids. So, this entire thing to get, taken together will also act like a fluid. So, this model is appropriate for a viscoelastic fluid. So, the question is now that once we have this Nth, N number of Maxwell models, what should be the correct partial? What should be correct differential equation relating stress and strain?

So, we have already seen how to do this. So, we are just going to repeat the same procedure. But now there is just one complexity that we have N number of elements that is all. So, let us write down from our previous experience, what are the force we always want to start with the force balance equation and the displacement equation.

So, what will those be for this particular case? So, from force balance, so once again, the system will experience a net force, which is going to be a sum of all the individual forces. So, in this case, your FT is nothing but a sum of going from i from 1 to N. Similarly, from geometry you will have the constraint that the system displacement is the same as the displacement of all the individual elements.

So, I can write the delta X is equal to delta XI for all i from 1 to N. So, now, the individual, so recall that for each individual element the relationship that we had discussed earlier, which was D by E i plus 1 by eta i Fi, D of still holds for all the different I. Now these relationships are individually still true.

So now our question is how do we figure out a constitutive relationship for all of this put together and we have to follow the process, a procedure which is very similar to what we did before. And this I would like to leave as a homework problem or just a self-work problem for you, in the next class and we will pick it up exactly here. So, we will stop here today.