Introduction to Soft Matter
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Lecture 29
Three Parameter Model Cont.

So welcome back to another lecture on introduction to soft matter, we were discussing last time, the constitutive equations and the solutions, right. So, we left off at a certain place, where we were looking at the Creep response for the material and its solution. The Creep response for a very, we framed the entire question in the form of a very generic ordinary differential equation and we replaced the constants from the model with just simple coefficient P naught, P 1, q naught, q 1 and why that we did that, it will be clearer to you as we go through the different solutions, right. For the time being that is what we have done we have ordinary differential equation where we have these coefficients.

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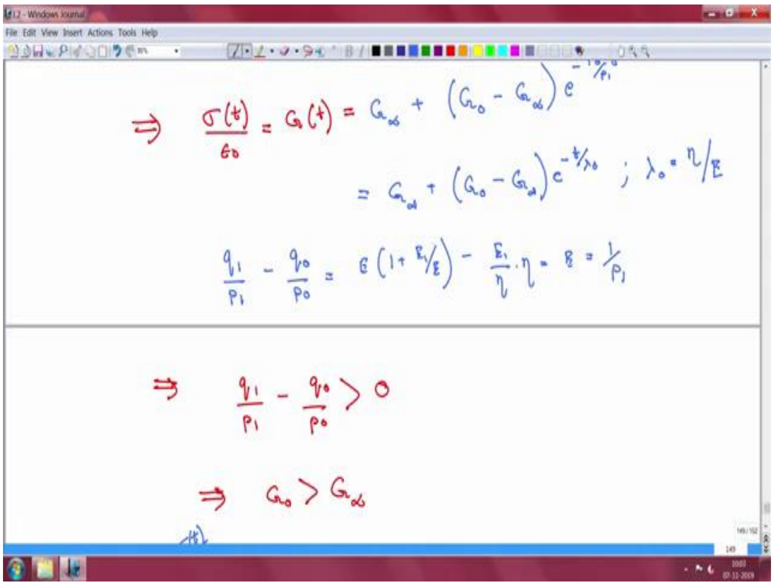

And then we have tried to solve for different cases and we had solved for the Creep response function and you had seen that we get a very nice equation here for the Creep in terms of a G infinity and then is this G naught minus G infinity, e to the power minus t and then we said it was a lambda naught and the lambda naught seems very familiar to us, the expression for this particular characteristic timescale associated with this.

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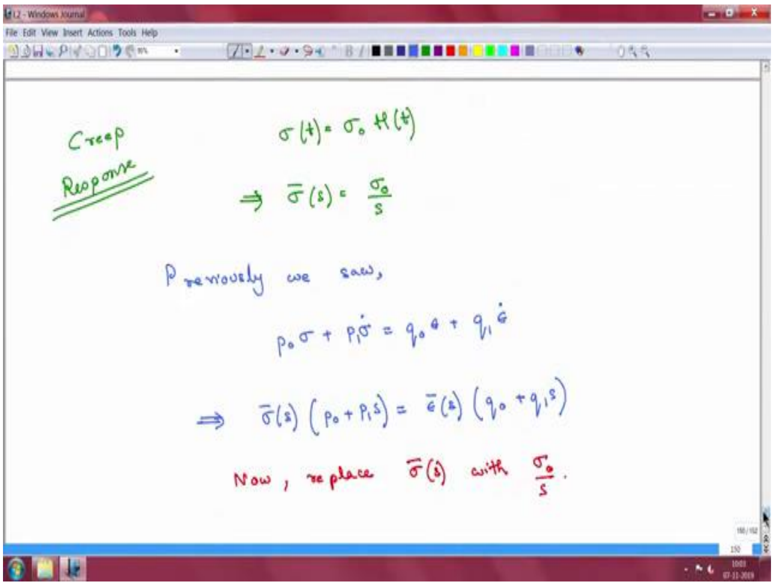

So, where we left off is we said that we will look at the Creep response and that has a very similar formulation, you have the stress and the stress is now in the form of a Heaviside function. When you take the Laplace of that you end up getting sigma naught by S. And we have seen that this is the governing equation that we are trying to solve. So, if you take the Laplace of this on both sides, then you end up with this particular expression. And so, what we want to do now is we want to replace the value of sigma bar s with what we have got.

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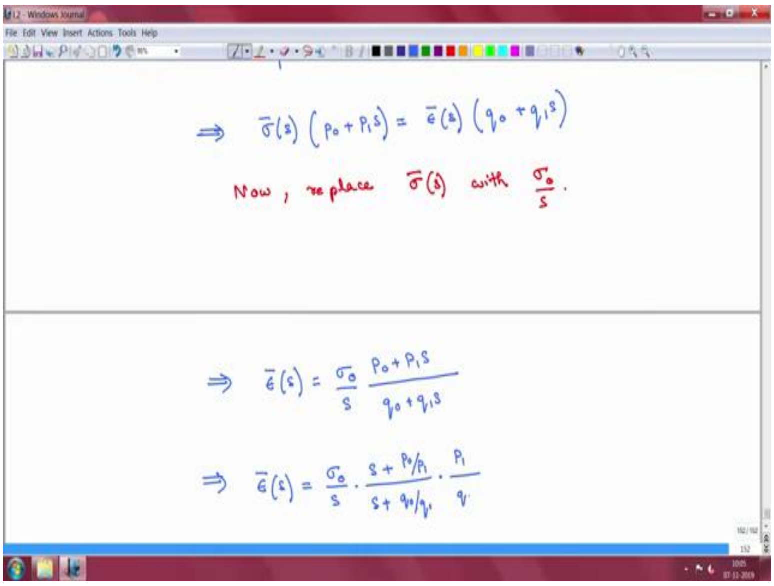

So, that gives us, so, let us carry up from where we left off. So, we are trying to find out the expression for the Laplace of the strain function. So, now we have, we can write this as sigma naught by s, P naught plus P 1 s, q naught plus q 1 s. So, we this is something that is familiar to us now, this is in the form of a rational fraction, we have to convert it into simpler factional forms, so that we can take the Laplace inverse.

And to do that, what we are going to do is I am going to convert the two the numerator and denominator in a form that is in a s plus constant type of form. So, if I have to do that, I will convert it to something like this and the bottom and the denominator you will have q naught, s plus q naught by q 1. And to take into account here I divided by P 1, so I am just going to multiply by this by P 1 and similarly here, q 1.

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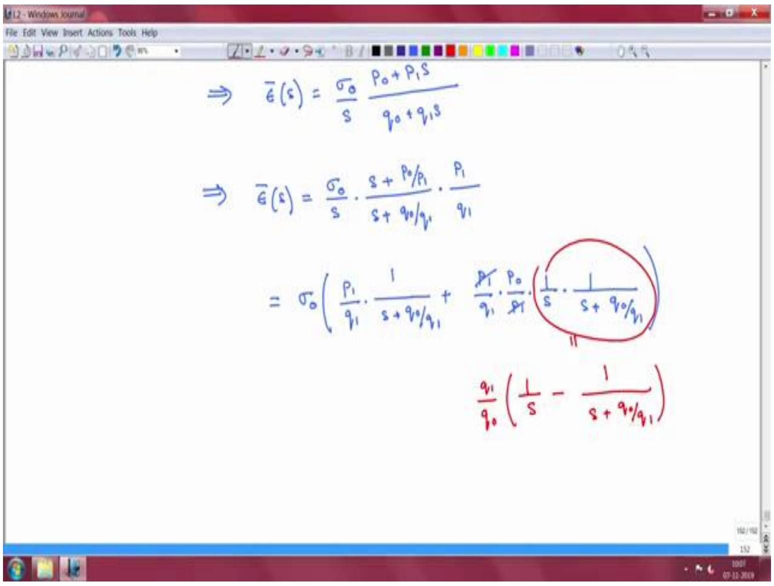

So, now let us say we have sigma 0 and this open a bracket. So, let us take this term s and the let us take the numerator term by term, so you will end up with P 1 by q 1 multiplied by 1 by s plus q naught by q 1 plus P 1 by q 1 multiplied by P naught by P 1 multiplied by 1 by s multiplied by s plus q naught by q 1, so this is, these two cancel out this is now P naught by q 1. Now, what I want to do is I want to expand this particular term into something else.

So, this particular term if I can simplify this is a multiplication, but I can probably write this as some 1 by s minus, so I am just looking at this particular top here. So, this can be written as, s plus q naught by q 1, but when you do that there will be an unbalanced term in the numerator, so, I have to get rid of that and to do that, I have to multiply it with the inverse of the q naught by q 1.

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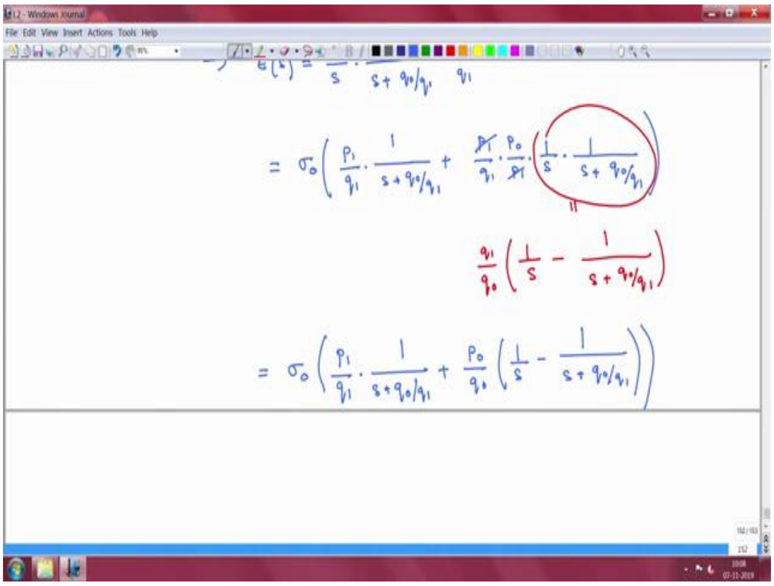

So, here I will have q 1 and here I will have q naught. So, when I insert this back into the previous equation, we have plus, when you simplify this, what you will end up with is, so this q 1 will cancel with this one. So, you will have P naught by q naught multiplied by 1 by s minus s plus q naught by q 1, okay, the bracket ends somewhere else, this bracket ends maybe I and then this entire thing where entire bracket ends.

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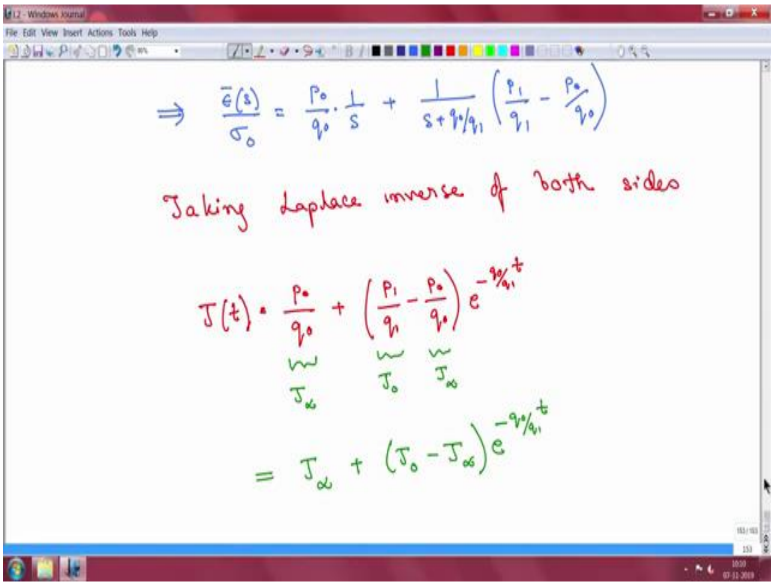

So, now, my epsilon bar s is equal to this, so that implies that if I divide so by epsilon bar s by sigma naught is equal to this nice little expression, I will now on this side I what I will do is I will bunch the different terms. So, we have P naught by q naught into 1 by s plus 1 by s plus q naught by q 1, P 1 by q 1 minus P naught by q naught.

So, when you take the Laplace inverse, this sigma 0 here is a constant, so, when you take the Laplace inverse of this, you will end up with this is by the way, J t. So, when you take the Laplace inverse, so taking, so taking the Laplace inverse of both sides, you have J of t becomes P naught by q naught.

So, we know that the Laplace inverse of 1 by s is just 1, and here you have this is just a constant. So, this will be retained, by the way Laplace operator is a linear operator, so when you have sums like this, you can always take the inverse of these terms individually and add them up and that is the Laplace inverse of the entire sum and this is e to the power minus q naught by q 1 t.

So, I can, so if I say that this term is some J infinity. So, if I denote P naught by q naught J infinity then this is again some J infinity and then I denote this by J 0, then I can write this entire thing in the same form that I had written before, in form of some J infinity that this J t is some J infinity plus J 0 minus J infinity e to the power minus q naught by q 1 t. Now, so let us take a look at what these forms look like. So, J 0 J infinity is P naught by q naught and we had earlier introduced a term called G infinity when we are discussing the relaxation function.

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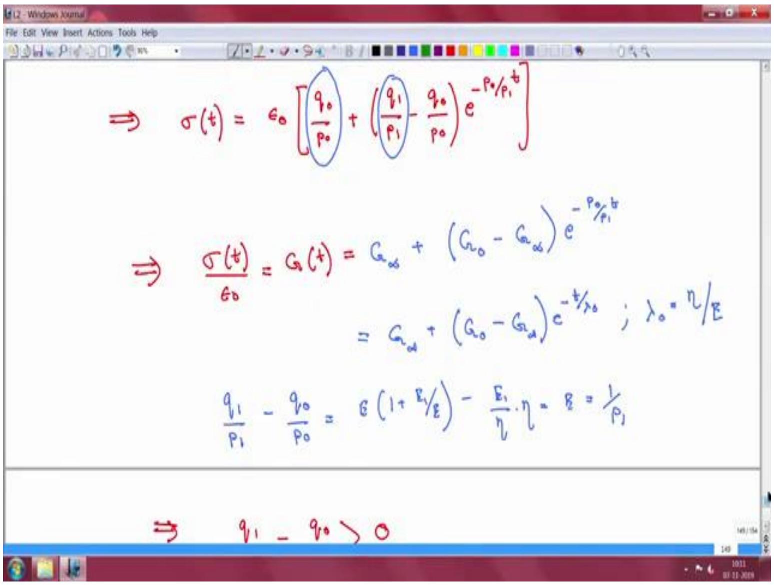

So, let us quickly take a look at what G infinity was. So, G infinity was q naught by P naught, and the J infinity is P naught by q naught.

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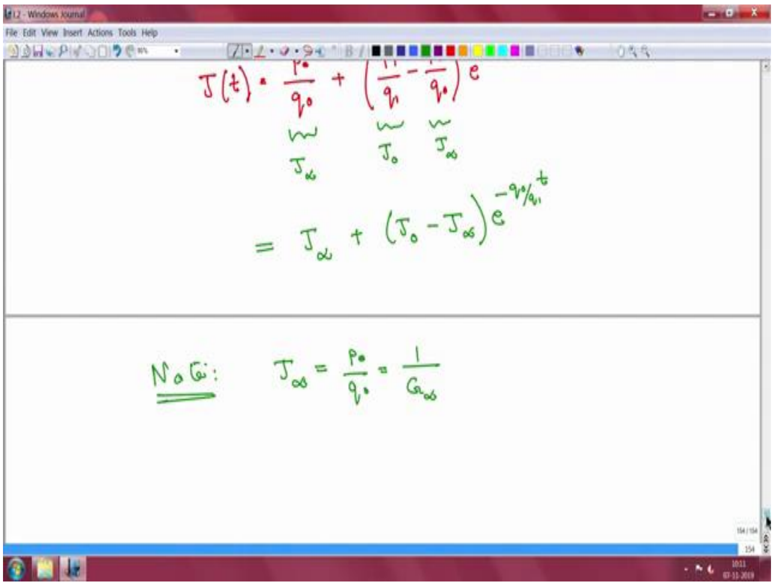

So, J infinity is the exact opposite. Note now, that J infinity is equal to this P naught by q naught and it is exactly the inverse of J infinity or G infinity sorry. Is it clear? Maybe I will just rewrite this thing.

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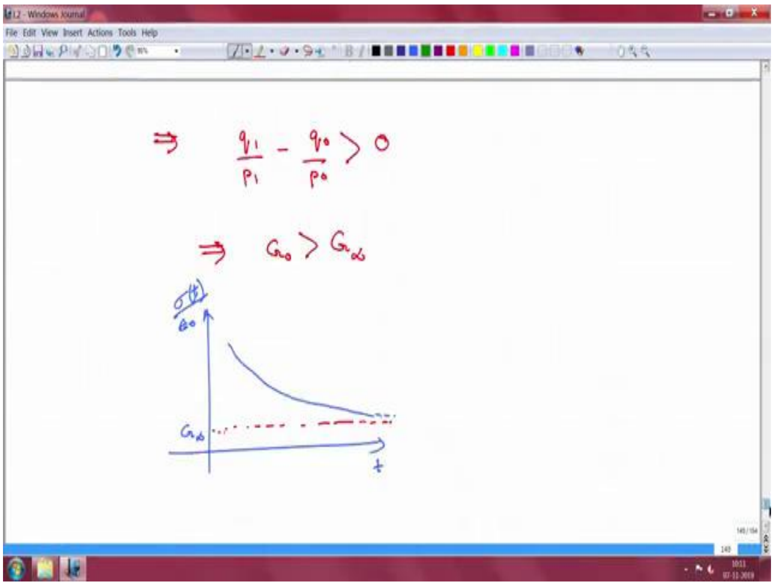

And we have seen before that G naught was greater than g infinity. So, similarly, so this is the second term here was q 1 by p 1 and this was G naught.

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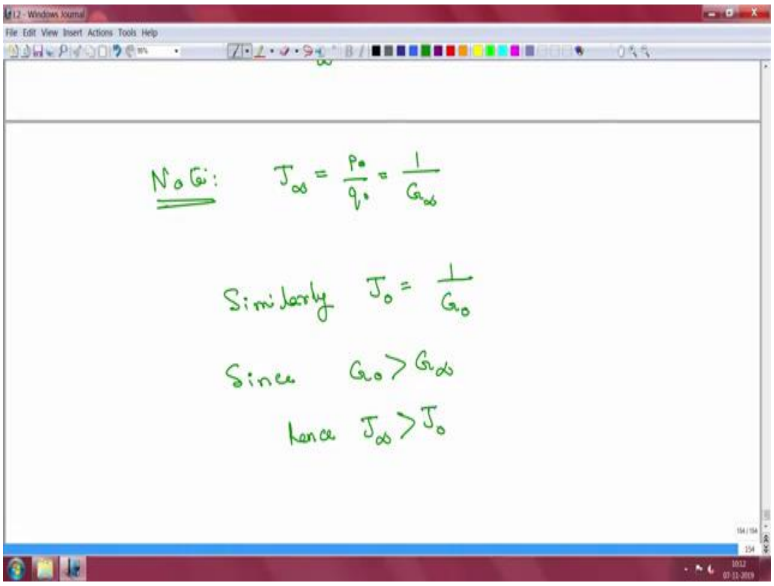

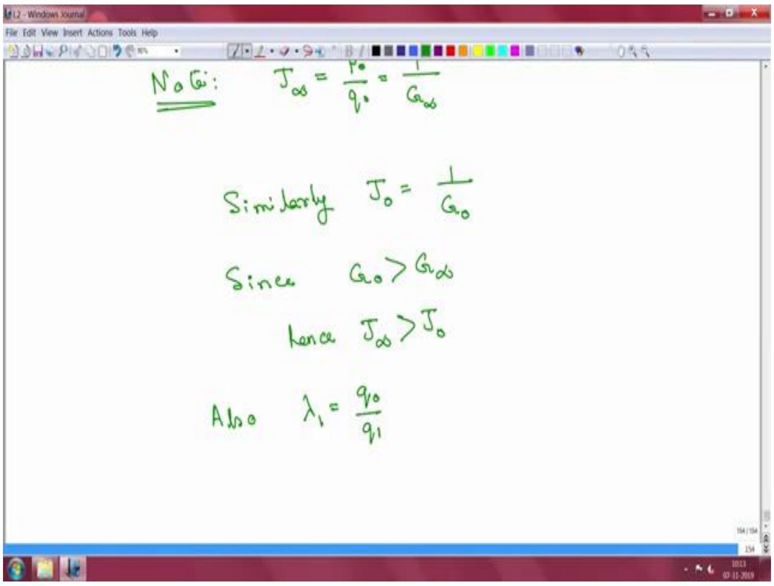

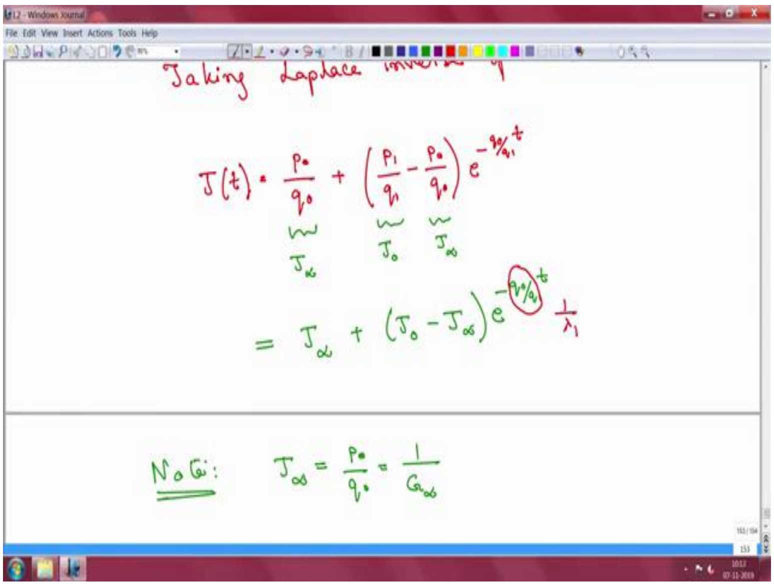

So similarly, G 0 is J 0 is, since G 0 is greater than G infinity, the exact opposite is true for the case of J. There is one more thing to note and that is here I can call this particular term as some 1 by lambda 1, right. So, lambda 1 is equal to q 1 by q naught. So, let us quickly replace the values of q naught and q 1 for the case of the particular model that we are investigating at the moment.

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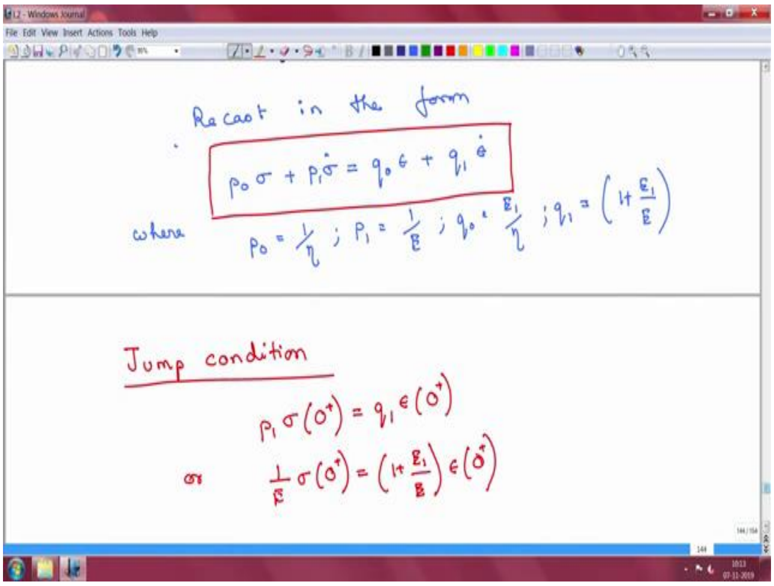

So, let us go back and see what this looks like okay, so q naught is E 1 by eta and q 1 is 1 plus E 1 by E. So, this is so recall this and then we are just going to use this these values below.

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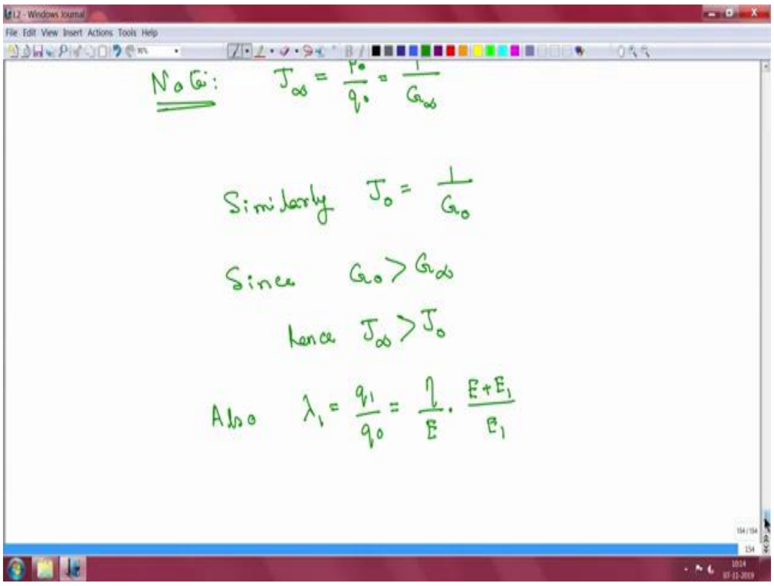

Sorry, here this will be q 1 by q naught. So, this is now going to be by eta by E plus E 1 by E 1. Now, this is another characteristic timescale that is associated with the system. So, that is why when we had written the expression for the relaxation timescale, I had in particular mentioned that this term was I had labelled it as lambda naught, eta by E.

And the reason for that was because that there exists for this kind of a model that exists another characteristic timescale that you can associate with this particular system and this is quite interesting because the addition of one more so, in a three parameter model, you can actually have two characteristic timescales. So, you have to note in particular cases.

So, if somebody asks you for three parameter model, what is the right time scale? You have to ask, what is the particular question? Because there are different timescales that one can use and you have to know the particular context before you can propose the correct timescale in these cases. So, now that we have done these two special cases, it is time for us to put our attention to the general case which is the response to an arbitrary strain history.

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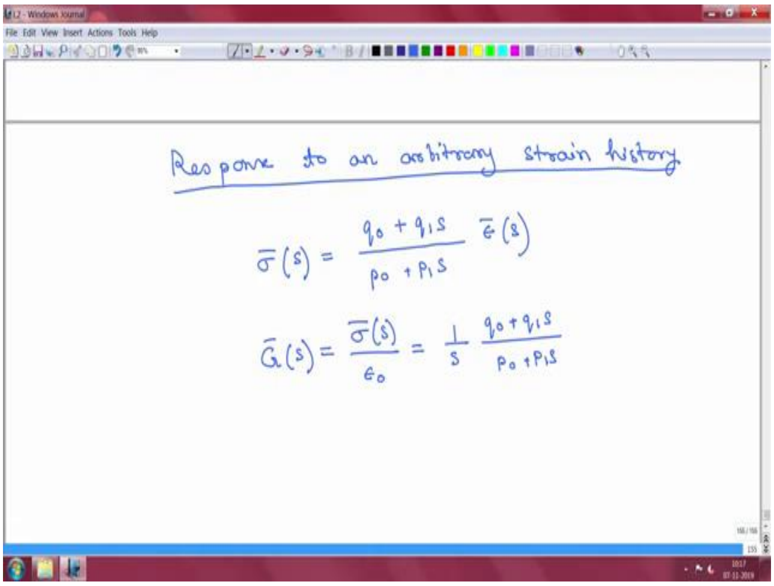

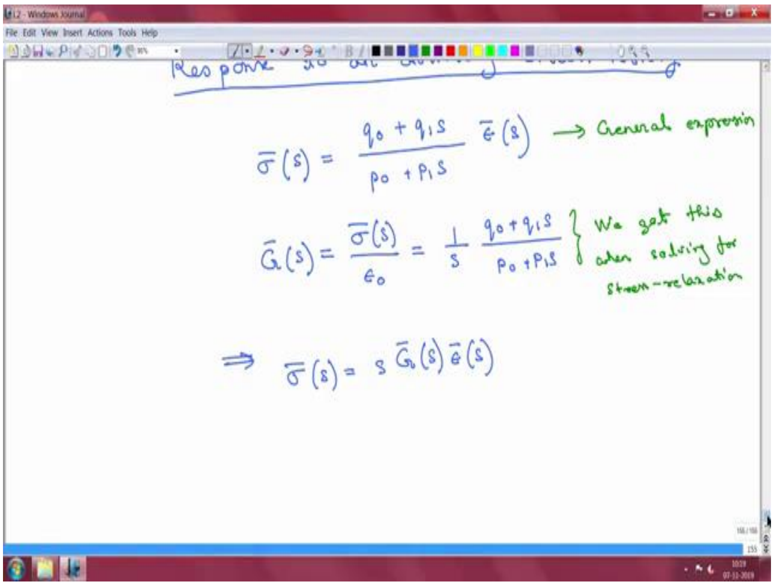

So response to an arbitrary. So, we had said in the beginning that the particular formulation that we are using, we end up getting the two relationships between the stress and the strain or the Laplacian of the stress and the strain in this particular form. Now, if you want to take a Laplace inverse, you can take the Laplace inverse of this whole thing and then get your actual solution, so that will be a little bit cumbersome.

So, the question is can we utilize what we have derived till now, and the answer to that is yes. So, we have seen that G s or the, if you take if you define G s as this quantity. So, when we are discussing the relaxation timescale then this particular quantity was already solved for, this we know that came out to be 1 by s, q naught plus q 1 s, P naught plus s. So, when you input jump condition or a step change in strain, you end up with this particular formulation. And we have already solved for G bar. So, we know the answer to what, how G looks like.

So, if you now apply this particular form over here, then what you end up getting? So, this was the general expression, we got this when solving for stress relaxation and then we took the Laplace inverse of this. So, for the general case, then I can write that sigma bar of s is simply I just use this particular expression and put it back here. So, this will be it will look something like this, please note that both the stress and the strain in this case are general, you want to solve for arbitrary situations.

So, the strain history is unknown and it is it can be any functional form prescribed in that particular case. So, now you have this particular form and we want to solve for it, now here we can take advantage of a particular equation that we had written some time ago, which was the convolution equation. So, we know that a function when it is a convolution of two functions, then when it implies is, is the Laplace domain the Laplacian just multiply.

So, if we can express this expression on our right hand side as just a Laplacian of two functions, then we can just take the Laplace inverse, and express it in the convolution form. The only thing that is preventing us from at the moment is that this extra s is here, otherwise, the Laplace inverse of this is known this is obviously epsilon t just by definition.

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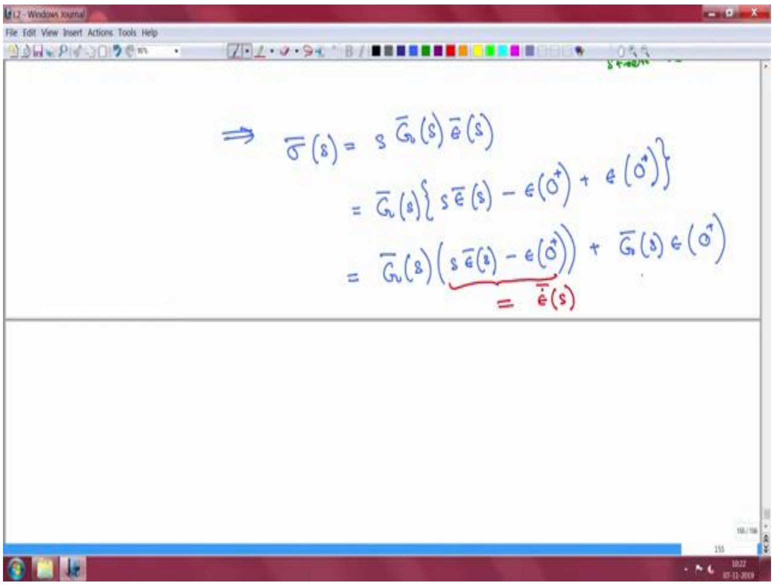

So, let us take the Laplace, so, let us simplify our case. So, because this s is creating the problem, let us see if we can lump this s with this quantity and if that helps with something. So now, if you see this particular form, it probably reminds you of the form of the derivative of a particular quantity and when you take the Laplace of that. So, what I will do is? I will add this artificially here, subtract it here, and then I will add this quantity. So, this now, then we have G bar of s epsilon 0.

And what is this quantity here on the, first term on the left hand, right hand side, this particular quantity should immediately remind you that this is equal to, from the properties of Laplace, this is equal to epsilon dot the Laplace of epsilon dot, and here on the second quantity please note that this is just a constant, okay. So, when you can just go ahead and take the Laplace inverse and get G t here.

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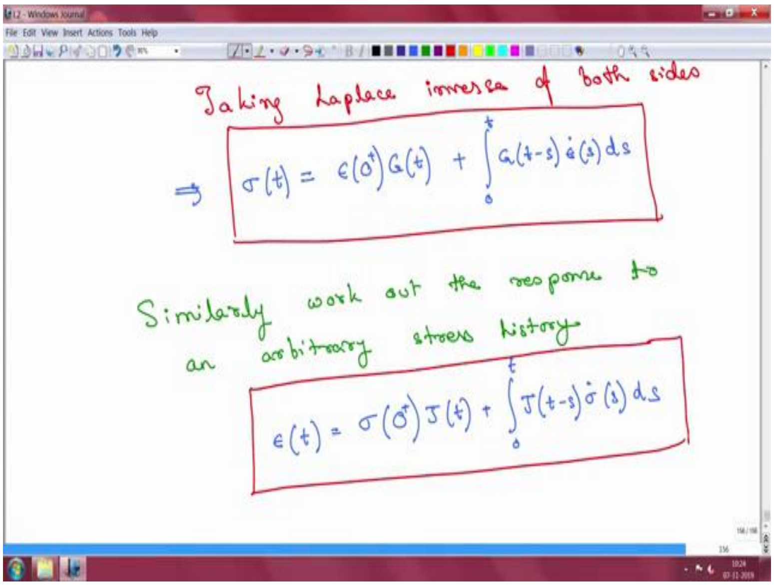

So, now, this implies that, so, now take Laplace inverse of both sides. So, let us just write that, so taking inverse of both sides you find that your on the left hand side you had sigma bar, so the Laplace inverse of that is simply is the sigma t or the function sigma t. Here, let us write this quantity first, so you have epsilon 0 plus the Laplace inverse of G bar is simply G t.

And now you have this quantity here, these are two quantities and two Laplace's multiplied with each other whose Laplace inverses we know. So, the combined quantities Laplace inverse is going to be some convolution, so here it will come out to be and this is our answer. And similarly, so similarly you can work out and I will not going to do this here and is left out as a homework for you.

So, similarly workout the response to an arbitrary stress history and what you will find here is epsilon t equal to sigma 0 plus J t plus 0 to t, instead of G you will have J in this case, and instead of epsilon bar here you will have sigma dot, and the two expressions look exactly the same.

So, now that we have gotten these answers, these are for the case of a special ODE that we were solving for the case of a three parameter model, but when we are discussing the Maxwell model and the Kelvin Voigt models, both these expressions had come up. So this expression, the first expression was also applicable, so maybe we will just quite quickly revisit that.

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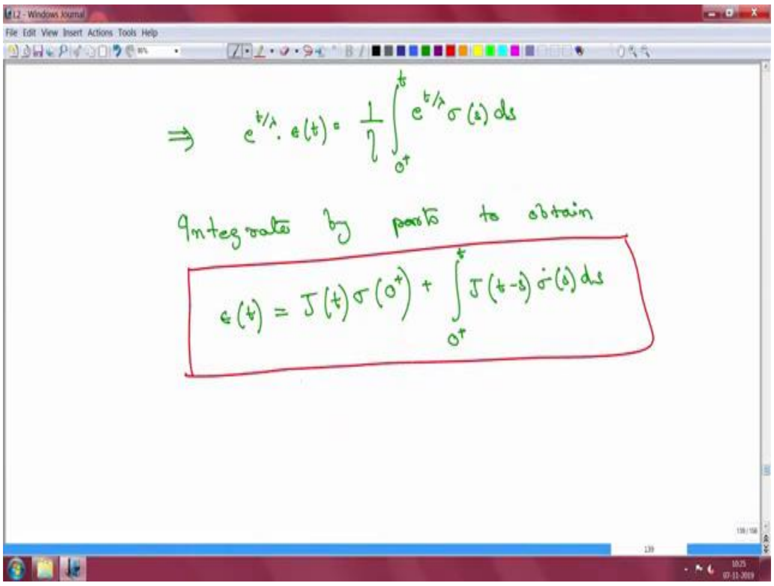

So, this was for example, when we are doing the Kelvin Voigt model, and we found that, when we do the integration, here we end up with this particular expression. And just maybe here I will clarify this is an s. So, we see that this expression, although it was originally derived in the case, you know, in as for a special case of the Kelvin Voigt model, this actually turns out to be a very generic expression.

At the time being at least it is applicable to both the case of the simplest of the models, the simplest of the two parameter models and the three parameter models. So that is an interesting discovery and we will keep that in mind that this seems to be very generic expression at the moment.

So, in today's class, what we saw is that we worked with the general expression for the three parameter model, and we solved further to the relaxation further to the case of the stress relaxation phenomena, we also worked out the case of the Creep function, and we then worked out the general solutions to the general cases. And we were able to find the expressions, the individual expressions for the case of the stress, relaxation and the Creep.

And the important thing there to note is that the Creep function and the stress functions are related to each other. So, there are there is a relationship between the J infinity and the J naught and they are not completely independent of each other. And later on what we saw was that when we looked at the general case, when we solved for the arbitrary histories, then we found expressions that were very reminiscent or the same as the expressions that we had derived previously for the Kelvin Voigt and the Maxwell model. So with that, we will end today's class.