Introduction to Soft Matter
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Lecture 26
Kelvin Meyer Voigt model

Okay, so welcome to one more lecture on Introduction to Soft Matter. Last time what we were discussing was Kelvin Voigt and the Meyer model and we derived some of the fundamental responses for that, we also derived, we derived first the constitutive relationship and from the constitutive relationship we were able to derive how the Creep response and the stress relaxation function for these, for this particular model looks like. So, till now we have introduced two of the simplest possible models when it comes to viscoelastic continuum models.

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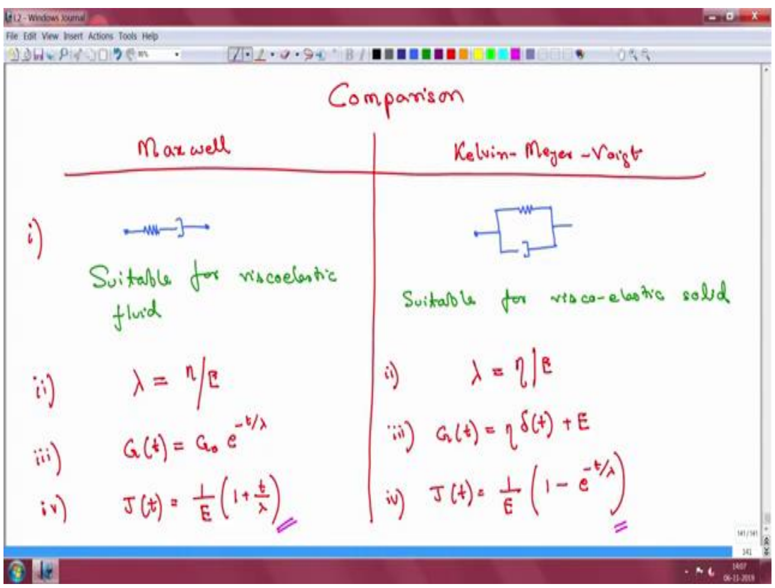

So, let us just do a quick recap, because we want to understand some of the key principles that are involved. So, let us just do a comparison. So, one side we have done the Maxwell model and the other one is Kelvin Meyer Voigt model, both of them are continuum models which means that we are using analogues disregarding the exact content atomistic basis for the viscoelastic system. So, we have not asked ourselves what is it at the atomic scale which makes this particular system viscoelastic.

Rather, we are saying that we will phenomenologically introduce a couple of different models, and we will see whether or not they are applicable in our case. And this need not be restricted to only to polymers this can be applied to any other system, where there is a phenomenological match between our model and its curves or its responses with the real material response.

In both cases, we took a spring, a single spring and a single dashpot. In one case, we put them in series and in the other case, we put them in parallel. And both cases, the spring and the dashpot were massless. The other important issue here is that when we place them in a series configuration this results which saw that it is suitable for describing a viscoelastic fluid right? So we will just quickly make a note.

So, the Maxwell model we found was suitable for the viscoelastic fluid. Similarly, the Kelvin model, or the Kelvin Meyer Voigt model was suitable for a viscoelastic solid. But why did that really happen other than the mathematics which shows us the way. Is there a physical and intuitive manner in which we can explain these results? The answer is yes.

So, if you look at the Maxwell model. Now, see, when you apply a finite amount of force, a spring will only stretch by a finite amount, but when you apply a finite amount of force through the dashpot, it need not stretch only by a finite amount, it can continuous stretching till the strain rate is itself capped or bounded. So, the model for the dashpot only suggest only constrains the strain rate rather than the amount of strain.

So, when you apply a small amount of force, this dashpot could keep on increasing and increasing, which is representative of a fluid type of behavior where we, even if we apply a very small amount of sheer stress to a fluid body, it should keep on moving, so it should keep on getting displaced. But when you come down to the Kelvin Voigt model, the fact that they are in parallel here implies that the displacement of the two have to be shared.

So, the displacement that has happened in the spring has to be shared by the displacement in the dashpot. So by applying a finite amount of force, you are only going to displace the spring by a finite amount and the dashpot hence is constrained to displace only a finite amount. So, that is why this is suitable for a viscoelastic solid. Obviously, the mathematics also confirms our intuitive understanding.

We saw that in both the cases we could define a certain timescale given by eta by e and in this case, again a time scale which is again eta by e. So, the response time scales for both cases look quite similar. In the case of the Maxwell model, you have relaxation function; the stress relaxation function is a exponentially decaying function.

Whereas, for the case of your Kelvin model, your stress relaxation function was a delta, a Dirac delta function plus a constant. Finally, your three function for this was one by on plus t by lambda whereas for the Kelvin model you instead had an exponentially a function that was exponentially achieves an asymptote value, an asymptotic value.

Now, neither of these two are very, or rather I would say that both these models have their own shortcomings. For example, we saw that for the case of the Maxwell model, this function the Creep function is not very satisfactory. Similarly, for this particular case you have prediction of an infinite response at t equal to 0 that is also not very satisfactory.

Even the Creep response for the Kelvin model is not very satisfactory, because, in the beginning there is no elastic response, even though this is a model that is suitable for a viscoelastic solid, the beginning we do not have a strong in the Creep portion we do not have a elastic response in the beginning.

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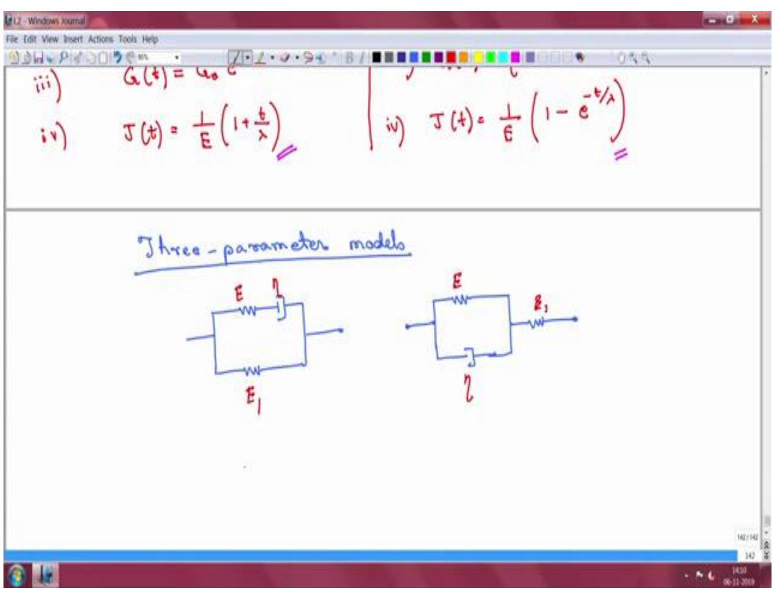

So, now what we can do is, but these were simplest possible models, which means that we can go on and build more and more complicated models. So, the obvious next step would be to build a three parameter model, we had only two. So, let us introduce one more complexity. So, there are different three parameter models that are possible.

And the three parameter models are quite interesting because they can show a better behavior in terms of the different functional response functions. So, let us see how we have many different types that we can build obviously, let us build two which, just a second. So this is, for example, one, another one we can build. So, maybe I will just quickly label them, label these. So, let us say this is E, this is eta, this is e 1. Similarly, this is E, this is eta this is E 1.

These are two different possible models. There are, there is one more possible model that is a fluidic model. But let us take a quick intuitive look, the reason I have, I have started out with this is because I just want to give you an intuitive response or intuitive idea. I want to give you an intuitive idea about how these models, whether these models are appropriate for a solid or a fluid. Now, let us take a look at the image on your left hand side.

Now, see in the upper side you have basically what is a Maxwell model. So, that is a fluidic model, and then it has been placed in parallel with another spring. Now, irrespective of whether the Maxwell model behaves like a fluid or not, the spring will constrain for finite forces the displacement in the maximum model.

So, this entire thing will act or will probably is appropriate for viscoelastic solid. You can draw a similar comparison in the other one, so this we I we can understand that this probably is better for viscoelastic solid. So, let us proceed with this as an example. We will also look at the other three parameter models in some time. And this is also often known as the standard linear solid.

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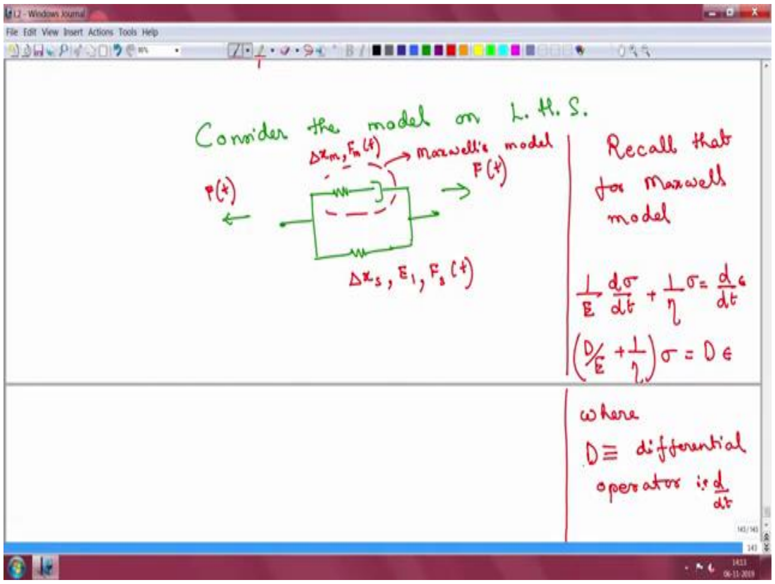

So, let us consider only this particular one, consider the model on the left hand side. So what do we want? I am now is going to start labeling these. So there is some force, let us say Ft that is being applied to the entire system. Let us say this is your spring, your displacement in the spring, and we know that the modular here is the E1.

Now, this basically is going to behave like a Maxwell model, so this entire thing is a Maxwell's model. So, let us denote the displacement here by XM and the force it experiences as F mt. And here you have the force here will be Fs. Now that you are using the Maxwell model, recall what for Maxwell model we wrote the equation 1 by E d sigma by dt plus eta sigma d dt of epsilon.

So, let us rewrite this as D by E plus 1 by eta times of sigma equal to D of epsilon, just continue it here, where d is the differential operator. I am actually simplifying for the sake of algebra. So, basically that is here, d by dt. So, I just want to make my life a little bit easier. That is why I am using this kind of a notation.

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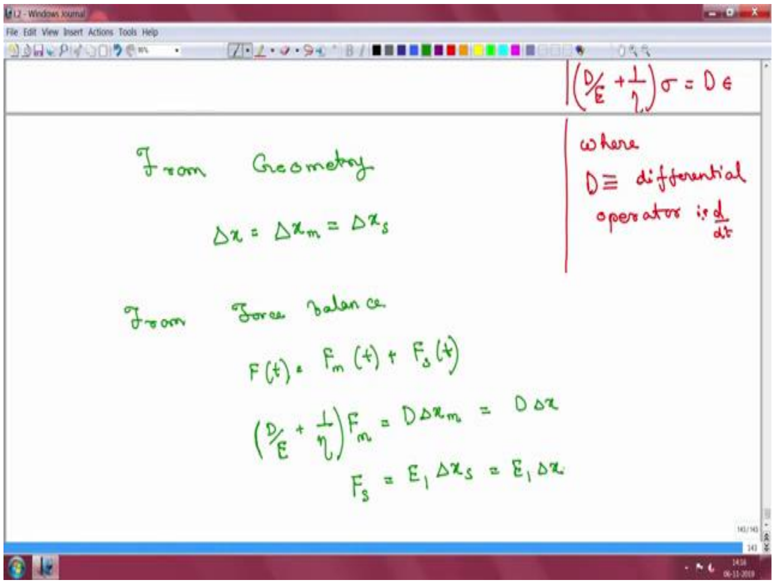

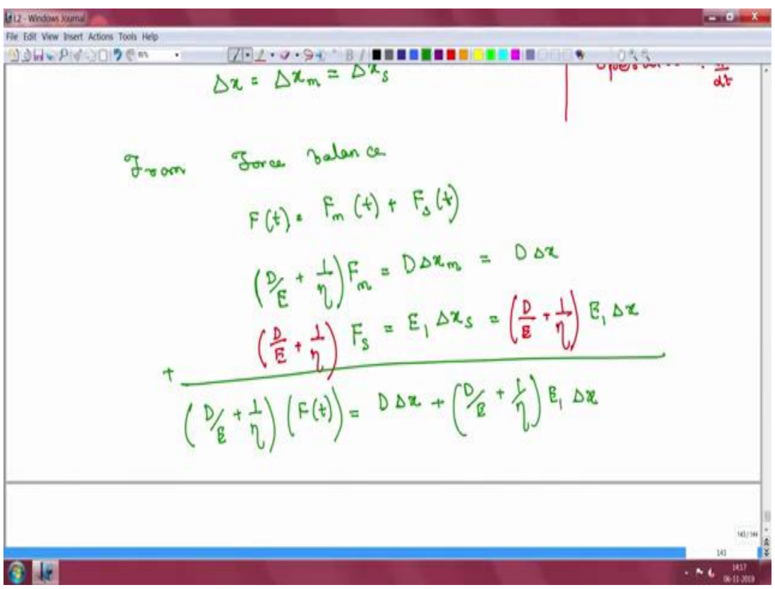

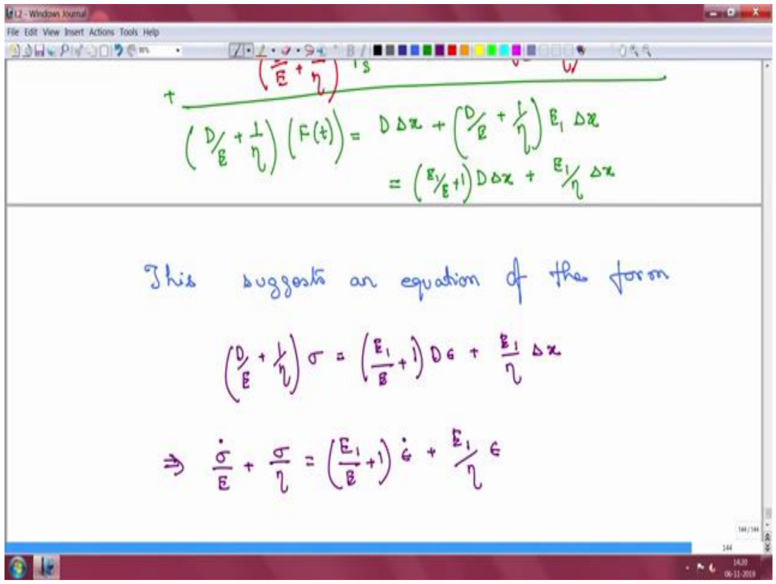

So, when it comes to the first is the Maxwell model or rather for the entire system, from geometry we can say. So, the geometry as we discussed earlier that is going to constrain the displacement, so that the displacement of the entire system is the same as the is going to be shared by the two individual systems.

So, if we denote the displacement of the system as delta x, then that displacement is equal to the displacement that is suffered by the Maxwell model or we can say delta Xm because that is what we are using for the Maxwell model and that is also equal to the displacement that is encountered by the spring. Now, from force balance we again have Ft is equal to Fm, the two forces are simply to be added.

Now, our relationship for the force when it comes to the Maxwell model is, so for D by E. So, if I want to write the individual relationships, that is the force in the Maxwell model is this quantity multiplied by equal to D of the displacement suffered by the Maxwell model. Similarly, the force in the spring is E 1 time's delta Xs which is by the way, remember please that this D is differential operator. So this is delta X.

Now, if I have to add these two together, so to get my force, I already have the right quantity on my right hand side, I already have the system displacement. But to get the force I need to be able to add these two quantities. So, I can multiply both sides of this equation and here, I am just going to just erase this because I have to change the order, so I will introduced this here also or delta x, this s is not required.

So, now I can add these two together and I end up with D by E plus eta, the two added together is just Ft. I can simplify this a little bit more by clubbing the proper terms together. So this is equal to my left hand side is equal to E 1 by E plus 1 D times of delta x. Because this is our differential operator, I am always writing it in front of delta x, E 1 by eta delta x.

So, this implies or this suggests, this suggests an equation of the form, the constitute equation of the form D by E, 1 plus eta of sigma equal to E 1 by E plus 1 D of epsilon plus E 1 by eta delta x. Or basically this is my sigma dot. I just take this operator and I because this is DDT, I am just changing the form of this. And then here you have E 1 by E plus 1 epsilon dot E 1 by eta epsilon.

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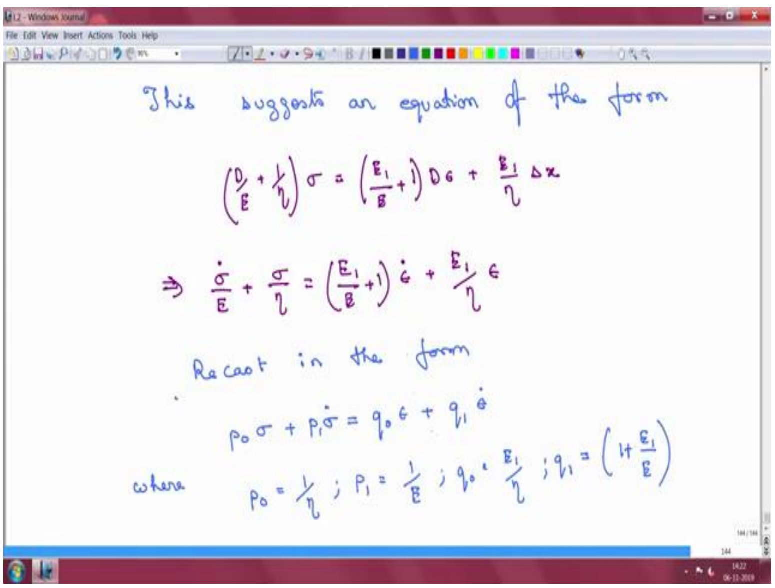

Now, I am because of a certain reason. But I am going to do is. So now we again have another first order equation. This is, it involves derivatives of both stress and the strain in this case. So you can think of this as some simple linear polynomial on this side, if you wanted to write it in just in the form of, if you wanted to write this just the form of differential operators, you could just write this on the left hand side on the right hand side a simple polynomials linear polynomials in D.

And, for reasons I will go into, I will show you why that will be important later on. I will recast this particular equation in the form, so recast in the form P naught of sigma plus P 1 plus sigma dot equal to q naught epsilon plus q 1 epsilon dot where this P 0 in this case is 1 by eta, P 1 is 1 by E, q 0 is E 1 by eta and q 1 is 1 plus E 1.

So, I have not done anything I have just recast this original equation in this particular form. Where these are some constants and we can solve the equations for the keeping the constants in the particular form and then later on we can evaluate what the final solution is by putting the values of the P 1 and the q 1 that we already have.

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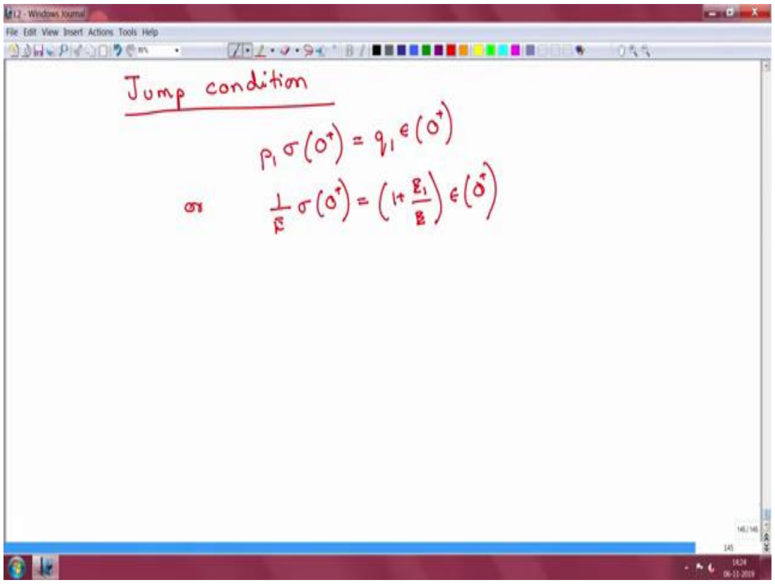

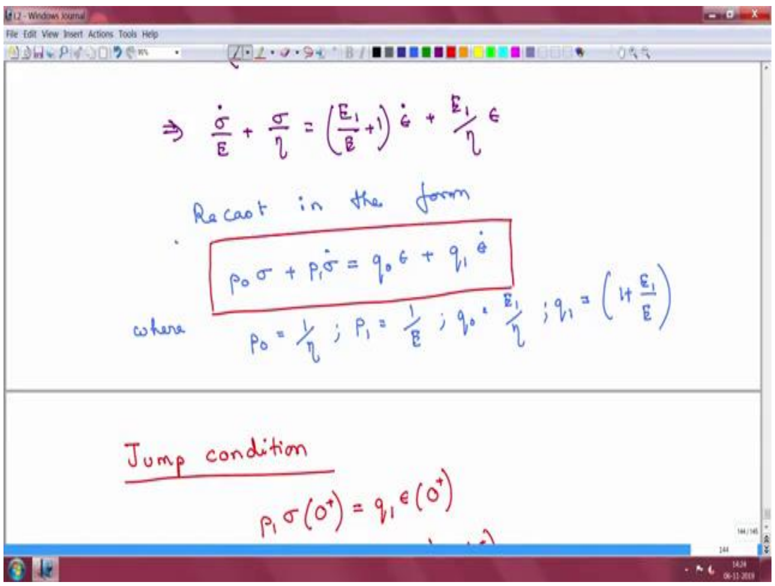

So, for both the Maxwell model and the Kelvin model, for the both the Maxwell model and the Kelvin model, we had discussed jump conditions and I had gone through details of how the jump conditions are to be found. In this particular case, I am not going to go through the same thing again. I am going to leave this as a homework for you. So, the jump condition in this particular case you can find it by yourself.

Jump condition will obey the form P 1 of sigma 0 plus equal to q 1 of epsilon 0 plus or in this particular case you have 1 by E sigma 0 plus equal to 1 plus. So, this is a more general result. So, irrespective of what P 0 P 1 q naught and q 1 are, this result sort of holds assuming that none of these other quantities are 0, you can derive, so if P naught, P 1 and q naught and q 1 or not nonzero then this relationship holds true.

So, now we have to solve this particular equation, this particular equation right here, maybe I will box this because this is going to be very important. And there are different ways of solving this for analytical purposes. Probably the one of the easiest ways to apply this is Laplace transforms. So, we will do a quick recap. We are going to apply Laplace transforms for quite a quite a bit.

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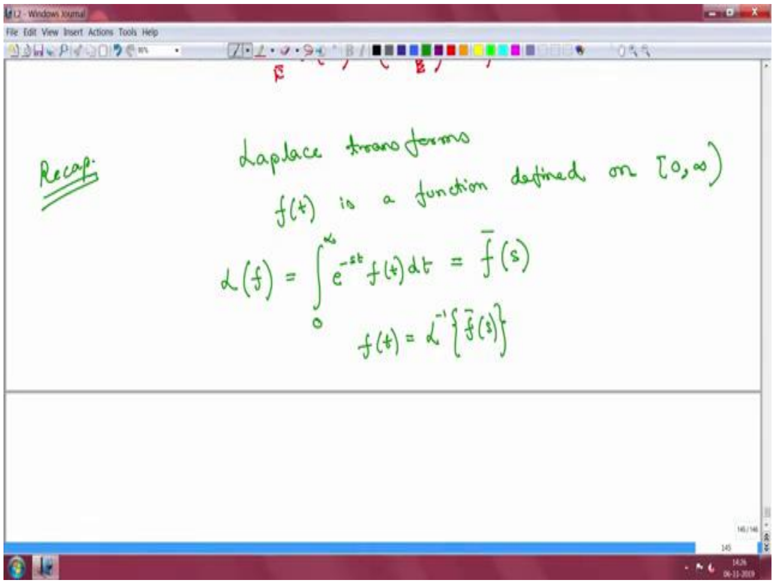

So we will just do a quick recap of Laplace transforms. So, Laplace transform, Laplace transforms for a function f. So let us say ft is a function defined on 0 to infinity, 0 to infinity then Laplace transform, the Laplace transform is usually denoted as L of f is given by this particular integral 0 to infinity, E to the power minus st ft tt.

For the sake of our solutions, what we are going to do is, we are not going to adopt this L of f because this will make things a little bit clumsy, but we will adopt this particular notation so we will use f bar and we will say it as S. So f bar denotes a Laplace of f. And instead of it being a function of time, it is now going to be a function of the Laplacian variable s in this case, please do not confuse this s with a previous s where that we use to a dummy variable in the integrals.

And when the Laplace exists the, we can show that the function ft can be recovered by what is known as performing the inverse Laplace of the Laplace function. So maybe we will stop here today. And from the next class we will look into a lot of more detail about the Laplace function and how to apply this to solve this particular form of the differential equation that we have. So, we will stop here for today.