Introduction to Soft Matter
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Lecture 25
Maxwell model Cont.

Okay, so welcome back to one more lecture Introduction to Soft Matter. Last time, we started looking at what is called a Maxwell's model, Maxwell's viscoelastic model. And, we started deriving some of the simple formulas that are a consequence of the model.

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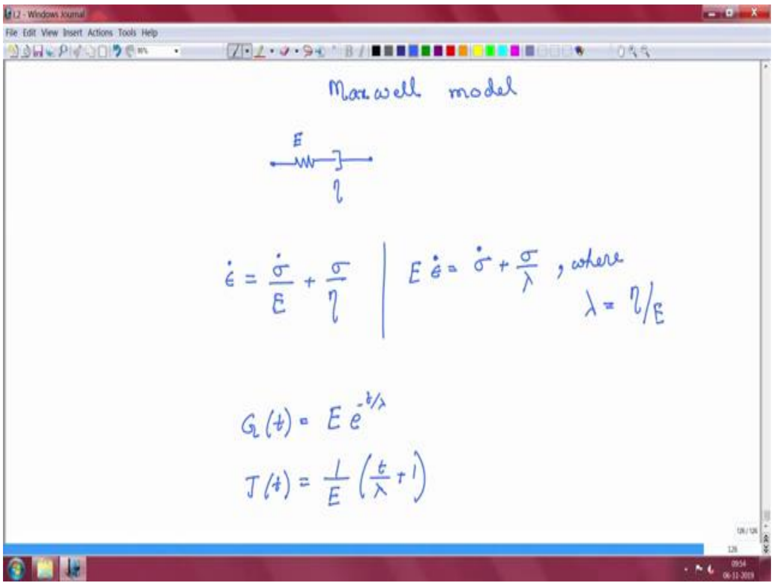

So, just to refresh our memory, what we will do is, so in the Maxwell model is a very simple model where you have 2 over a spring and you have a dashpot in series and these two are considered to be massless and then we use this and relationships that are integral to a spring and a dashpot to derive the constitutive equation. And the constitutive equation came out to be of the form, we can write it in a couple of different forms.

So, we say, you can write it in this way or if you want you can write it as in this other fashion also where this lambda is a relaxation time scale is given by the ratio of eta and E. And we saw that for the stress relaxation function for this particular situation, that comes out to be of a very simple form is given by the E, which is a modulus. And there is an exponential decaying function.

Similarly, we derived the equation for a Creep function and then that also came out to be a simple form and however in this case, now to remember that, while the stress relaxation is exponential in nature, the Creep turned out to be linear in nature and this linear nature of creep is not extremely realistic, but something that can be still worked with in different situation.

It depends on the particular situation whether or not that is a proper adequate model. So, the stress response or the strain response, the stress response function and the Creep function. So, the stress relaxation function and the Creep function, they are obtained as special cases of an applied strain or an applied stress.

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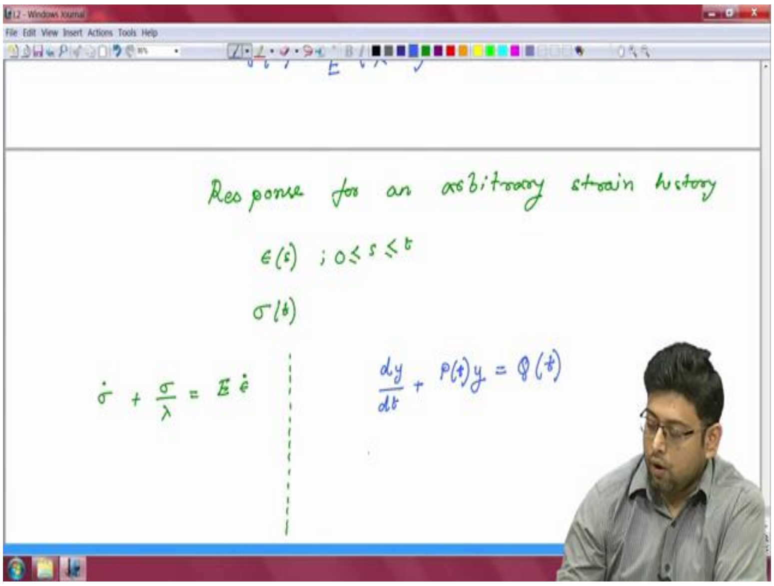

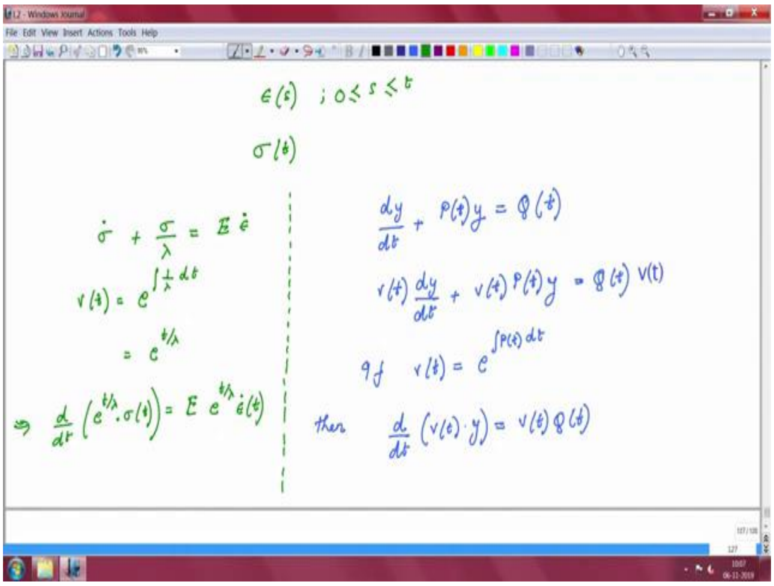

So, the next question is once we have gotten these expressions is what happens if you have an arbitrary stress history or a strain history? So, let us first take a look at what happens if you have when you have a response for an arbitrary strain history and here what we imply is that

that is strain history for example is known, so epsilon s for that set of s is, which is less than the current time, but greater than 0. This is given to you.

And then you are asking the question, what is your current state of stress? So, if the strains are known to you then we know that a viscoelastic material has memory. So, what we will do is to solve this particular problem. So, to solve this particular problem, what we will do is that we will write this equation such that we have the stresses on our left hand side, so we have the stress so, we are writing this ODE.

So, this sigma is what you want to figure out and epsilon is some, given to you. Now, to solve this equation, you probably have realized that this is a first order linear ordinary differential equation. Now, when you have an equation of this form and you want to solve for the general situation, then a method which is called the method of integrating factors is usually applied.

So, just let us do a quick rehash of what the method of integrating factors is. So, let us say there exists a different ODE which is given as dy by dt plus P some function of time multiplied by y and on the right hand side, you have a function of time. This is a linear ordinary differential equation, when Q t is equal to 0 then you end up with a homogeneous linear differential equation.

If Q t is identical equal to 0 always then you have a homogeneous equation, if you have Q t as a nonzero variable then it is a non-homogeneous equation. Now to solve this particular case, we introduce another function called v t and we multiply both sides by v t. And now if v t is of this particular form, then this allows us to write this previous equation as and this final form formula you can solve by integrating.

So, this v t is called integrating factor. So, for our case, let us find out what the integrating factor is. So, what is v t for us? We have integral of and here you have for y you have sigma. So, you have, you end up with 1 by lambda dt. So, you end up with e to the power t by lambda. So, this is our integrating factor.

Now, if you use this and then multiply both sides by the integrating factor you will end up with a situation where you will have the time derivative of e to the power t by lambda multiplied by your stress, current stress. And on this side, you will again have E, you have the integrating factor and then you have the derivative of the strain.

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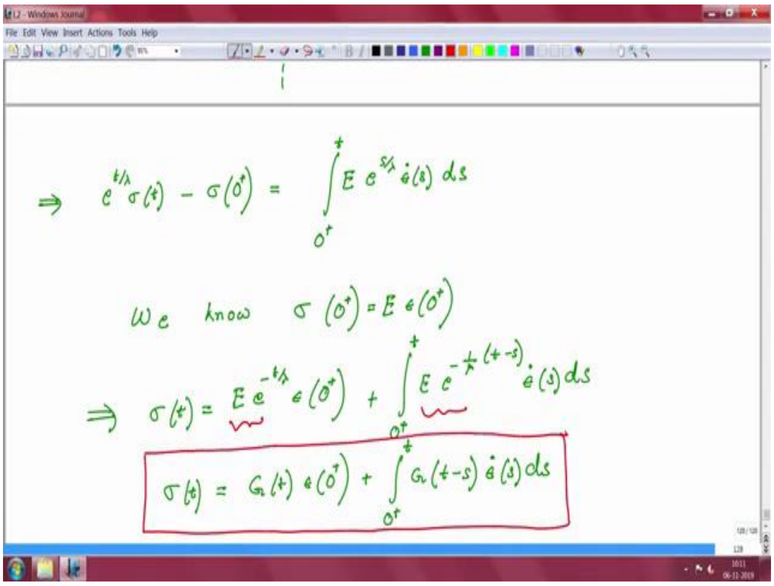

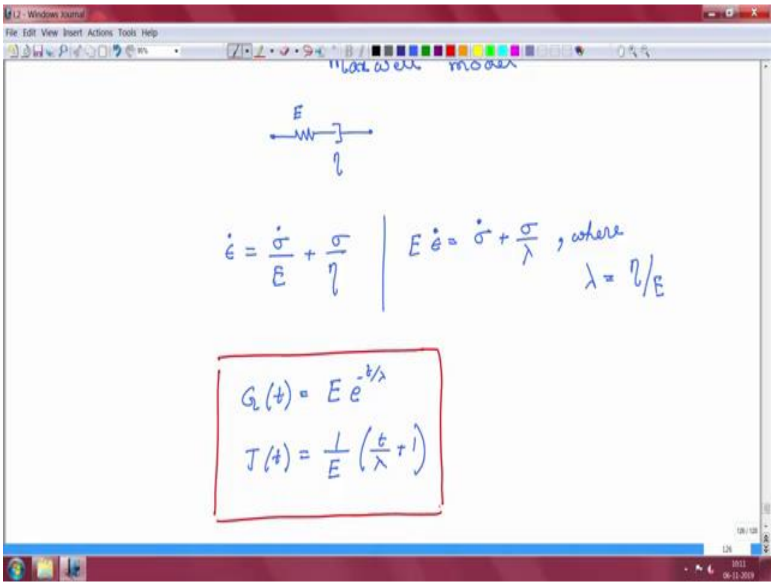

Now, to solve this you integrate both sides and you integrate from 0 plus to a time t. So, the left hand side then if you do this integration and it you end up with, it is your t by lambda sigma of time at time t, which is the variable that you seek minus sigma at 0 plus. So, the epsilon here e to the power t by lambda factor here becomes just 1 in this case. And on the other side you will still have 0 plus this is where integrating over E.

And now, you are integrating this quantity from 0 plus to some particular time t. So, I have already used up the time the variable t, so I am just going to use some other dummy variable let us say s here. And now, we know that sigma 0 plus and epsilon 0 plus the share a relationship. And the relationship is if we go back to this original equation, so the jump between the 2 variables epsilon and sigma are related to each other.

And they are related by, so if you introduce this back into this previous equation, you will end up with the stress at the current time. And I am only having that on the left hand side. I am going to move this back to the right hand side. So, you will end up with e to the power minus t by lambda epsilon 0 plus, I have taken this factor and I have moved that the exponential factor I have also moved on to the right hand side.

You have the integral to the second quantity E to the power minus by lambda t minus s. Now, what does this quantity remind you of? This is your stress relaxation function, so we can just write this rewrite this previous equation in the form of the stress relaxation function. So, this becomes an important equation for us.

I am just going to, so we have seen 3 important equations till now, the stress relaxation function, the Creep function and then we have looked at the stress which comes out as a result of previous provided strain history. This is a very important equation because we will see that this comes up again again, there is a particular reason why I put it in the form of the stress relaxation function. We will get to know that a little bit later y.

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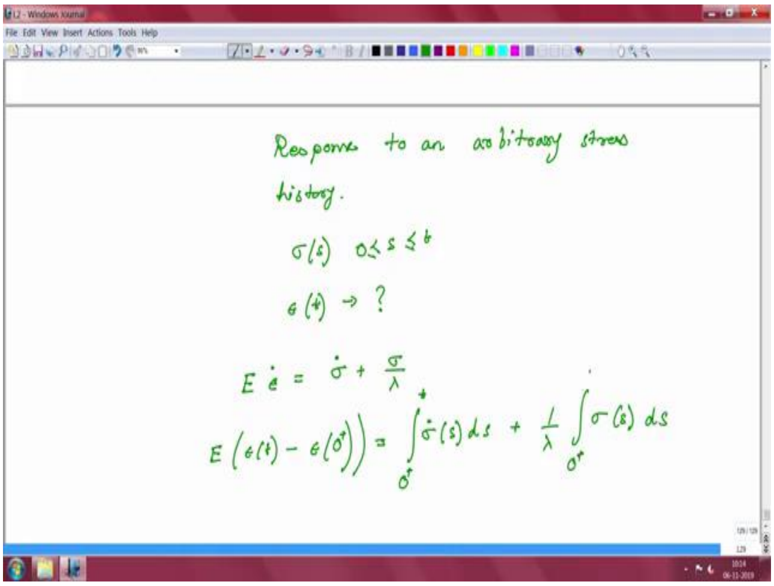

So, now that we have looked at the response to an arbitrary strain history, then we should be looking at the response to an arbitrary stress history also. So, let us take that response to an arbitrary stress history. So, here, the question is flipped, you have been provided, let us say the stress as a function of time. And the question you are trying to ask is what is the strain at the current time. That is the question.

And as we know, again, just like the previous case, the entire stressed history is going to influence the strain at the current time. And the current time, strain should be a functional form that takes into account the entire history of imposed stress on the material. So, we just, so we rewrite our equation, but we will flip it a little bit, because we want to be sure which is the variable we want.

So, since epsilon, is the variable I want, I am kept it on the left hand side and we just rewrite this equation. Now, you integrate both sides, if you integrate both sides from 0 plus to the current time t, then you end up with, you have 2 integrals on this side, both going from 0 plus to t, you have a sigma dot. And then again, I am using a different variable, a dummy variable s here, because I have already used the variable t plus 1 by lambda and let me put the limits.

So, now, you have a functional form which has a derivative of the stress here, but you have another one which is just has a derivative. So, here we can actually convert this second term on the left hand side. The right hand side sorry, the second term on the right hand side can be converted into an analogous form with sigma, if we apply integration by parts.

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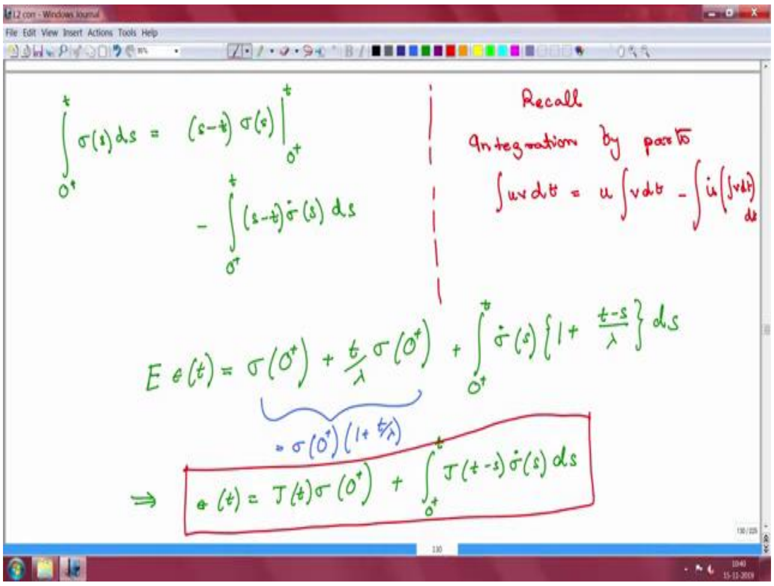

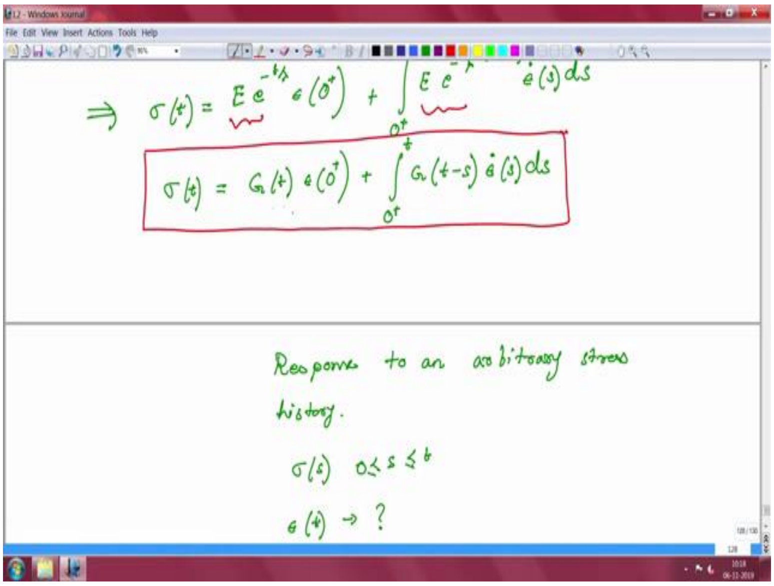

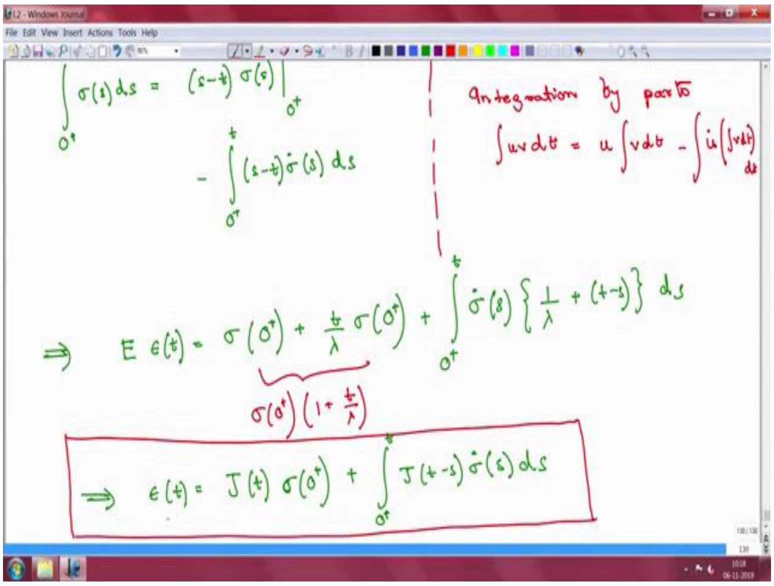

So, just to remind ourselves what integration by parts is, we are just write it on the side, recall integration by parts. And in this you have an integral u, v dt, let us say, if you are computing this, then you can rewrite this as in u into the integral of v dt minus u dot integral of v dt. And there is another dt here.

So, using this integration by parts, what we have to do is we have to simplify and I want to get rid of the pure stressed term and converted into a derivative form and I am going to write this as, see I can choose this quantity here. And here this is variable s and this variable s this is being evaluated from 0 to t and you have minus integral again plus t, s minus t sigma dot, so I will just apply the integration by parts here.

So, by applying the integration by parts and rearranging the terms, we can rewrite the left hand side as being this quantity. And on the right hand side, you will now have sigma 0 plus and then you have the quantity under the integration side which goes from 0 plus to t and here you will have sigma dot s and then the bracket you will have 1 plus t minus s by lambda ds.

And you can see that this quantity right here, this is basically this can be summed up and we can write this as 1 plus t by lambda. And then once you bring this E over onto the right hand side, then this equation simplifies further and this we end up with epsilon t is, this is now the J of t and this multiplied by sigma 0 plus and now you have the quantity under the integration side and you can see that this quantity is the Creep compliance written appropriately.

So, this against the limits of the integration still remain, the Creep compliance now takes on the form t minus s and you have sigma dot s ds. So, this is an important equation. So, you can see that things have sort of gotten flipped. So, when you are looking at an arbitrary strain history, then you have here the stress relaxation. So, if you have an arbitrary strain history, then you have the stress relaxation function appearing here whereas, if you have an arbitrary stress history, then you have the Creep functional form coming in here.

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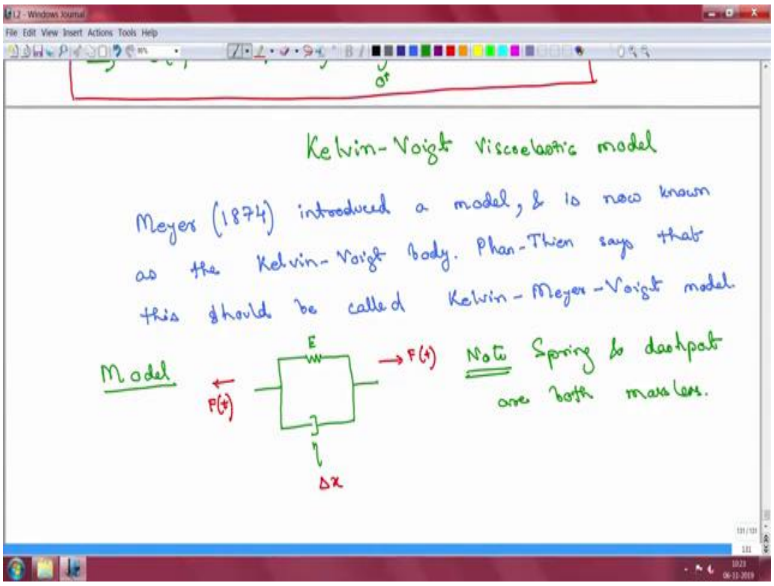

So, with this we have sort of looked at some of the very-very important results for the Maxwell equation or the Maxwell‘s viscoelastic model. Now, there is another important model in viscoelasticity, so Maxwell model is one of the simplest possible schemes that you have, but there is another one and that is called the Kelvin Voigt Viscoelastic model. So, we will look at now Kelvin Voigt Viscoelastic model.

This model was introduced in 1874 by Meyer, so Meyer 1874 introduced this model, introduced a model which is a model which was different than the Maxwell’s combination of spring and dashpot and it is now very much known as the Kelvin voigt body and so and is now known as the Kelvin Voigt.

I have not introduced the model I will introduce that in a second body as far, so unfortunately Meyer’s name is missing, but Phan-Thien in his book, understanding viscoelasticity, he clarifies that this should be called, the Phan-Thien says that this should be called Kelvin Meyer Voigt model. And a good reference to look up this is the book called Rheology and Historical Perspective and that is written by Tanners and Walter and it was published in 1998.

So, if you are interested in the history of this reality and the various models that have come up, you can refer to their book. Once again that book is called Rheology and Historical Perspective by Tanner and Walters. So, let us look at this particular model. So, what is this model all about? So, this model is, now instead of placing the spring and the dashpot in series, the spring and the dashpot are placed in parallel.

So, this is your E, this is eta and the 2 bodies have been placed in parallel, you have the forces on this, when you apply a force f t is undergoes a deformation let us call it delta x. And then the question is what is the relationship between f t and delta x and that is how we have been dealing with this, we have been applying force balance. Once again let us remind ourselves that these 2 are massless. So, that is a very important point, so just we will make a note of it. Note, spring and dashpot are both massless.

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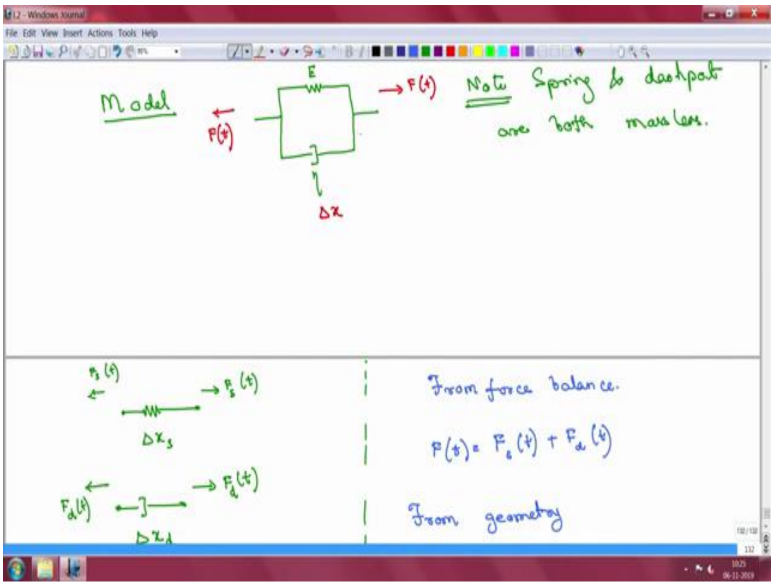

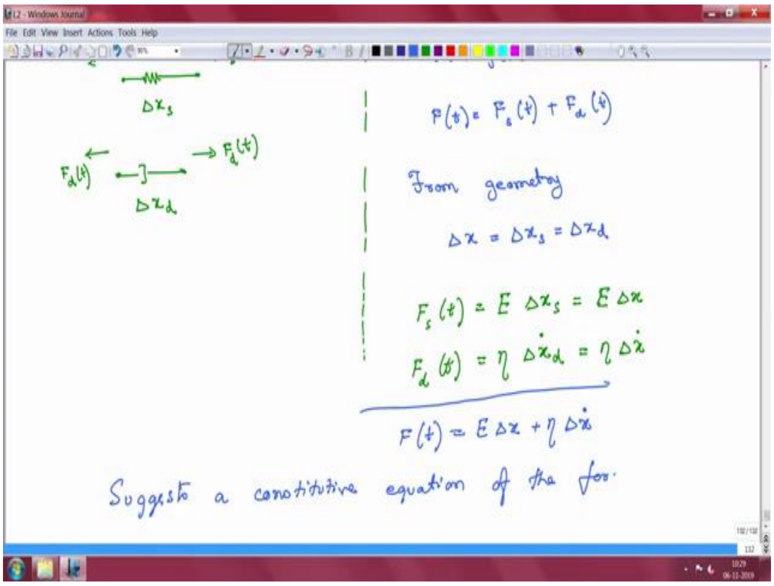

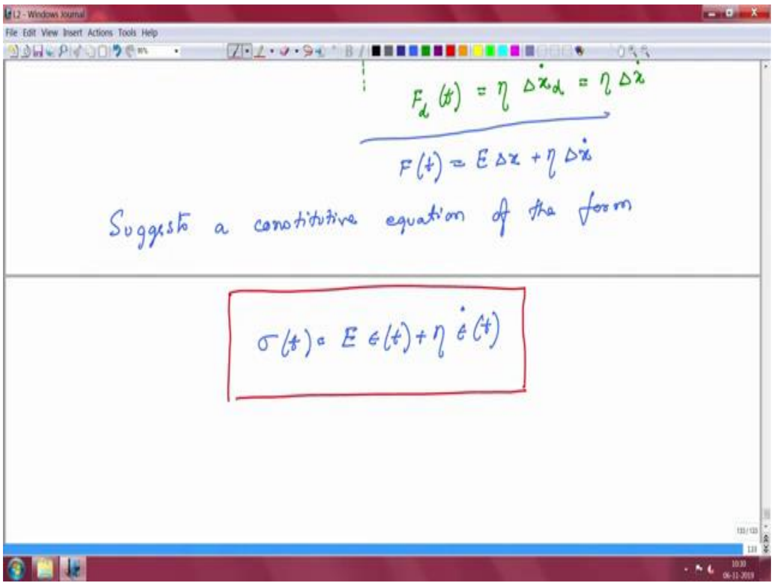

So, individually I think I need a new space. So, what we are going to do, let us break up this space into 2 parts. So, you have, let say the spring, let us say the force in this is experience that the spring is experiencing is F s and the information it undergoes is delta x s. Similarly for the dashpot, so let us say we have a force F d, sorry, I just made a mistake. So, this is, we should put F s in the subscript this is time, this is subscript.

So, this is d subscript, d standing for the damper the r, dashpot delta x d. So, from force balance, for the model, if you look at this model for force balance, we can say that your F of t is equal to force in the spring for the force in the dashpot. From the geometrical constraint, now geometry here plays an important role because you have placed this in parallel. What is implies is that the deformations or the extensions in both the spring and the dashpot have to be the same.

That is why, that is the meaning of this particular representation. So, from geometry we know that the deformations are going to be equal in the 2 cases sorry, so from geometrical constraint tells us that the system deformation is same as the definitions being seen by the spring and the dashpot. So, now let us write the individual equations that apply to the spring and the dashpot.

So, the individual equations should be that that should just be a result of how spring behaves, we have, so now if I add these 2 equations, this equation here right over here, then what we end up having is F s plus F d, which is we know from force balance is equal to F d. So, my addition gives me and that these 2 by the way, these are equivalent to writing it in the form of delta x, which is the, what this entire system is seeing.

So, I can just write F t as E times, so where this process of addition I can write it as, so now we have a relationship what we are seeking initially a relationship between the force and the system displacement. So, this suggests a constitutive equation, equation of the form sigma t is equal to E times epsilon t plus eta times epsilon dot and this is the governing constitute relationship for the Kelvin Voigt model.

Now just compare this to the Maxwell model, we had a derivative and stress in the Maxwell model which is not existing here. But, we have now have we have the derivative and strain still there, but we have an extra strain term without the derivative that also appears. And it can be solved by almost the exact same methods that we have solved for the Maxwell case. Now, let me, so we will end here for today's class and then we will discuss the solutions to this oriental differential equation in the next class.