Introduction to Soft Matter
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Lecture 24
Maxwell Model

So, last time what we were trying to do is we were looking at the Maxwell model where spring and a dashpot was placed in a linear fashion and when we combined this two things in a linear fashion we wanted to understand what is the relationship between the force applied on the spring dashpot system and the displacement of the system. Because what we are essentially going to seek is the final constitutive relationship interests of stress and strain.

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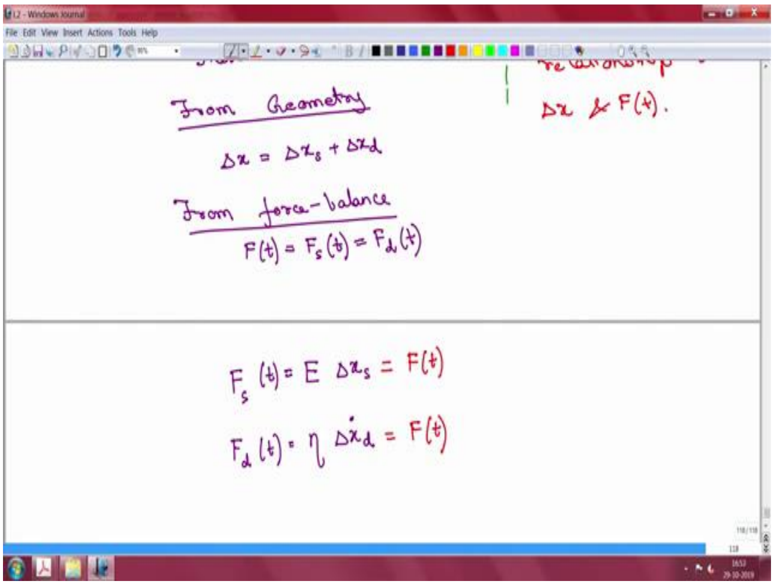

So, where we stopped last time was where we wrote down the equations. So we wrote down the geometric constraint equation how the delta X is and from the force balance we had written down the equation for the force, these are equal, this is rather straightforward equation. Then we had written down the separate relationships for F and this force in the spring and the force in the dashboard. So we will start off from here where we left off.

So, now we know that from the previous equation that this is actually F of T, so let us simplify our life and now we also what we can do is we can take the derivative. So let us say let us take this equation.

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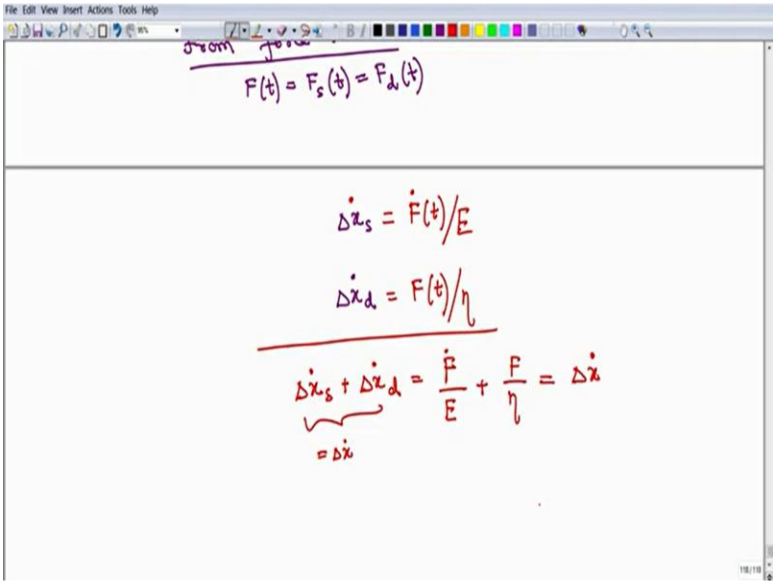

So now I am just going to remove this part because we have taken that out. So, if we take the derivative here and then we add up the two cases and we put this E on this other side. So let us say this E I am going to remove and I am going to put it in the denominator here and similarly I am going to remove eta on this side and I am going to take that here. Then you have your delta XS plus delta XD dot is equal to F. So I am going to not use this T because that is sort of understood.

So, just simply going to write my this is F dot and you have F by eta, but we know from geometric constraint that this quantity right here this is equal to your delta X dot. So now what you have is that you can find that this quantity is there equal to delta X dot and this is the relationship that we were seeking in the first place set as we had said that we wanted a relationship between delta X or the total displacement F that the system was looking at and the net force on the system.

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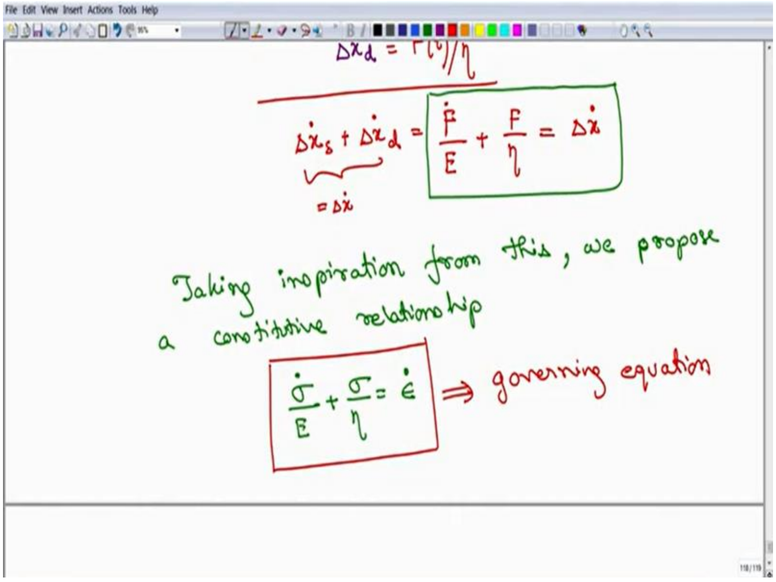

So this becomes so this is our important equation right here and this is in terms of the force and the displacement and from this what we say that in taking inspiration from this we can say. So taking inspiration from this equation, we can we propose a constitutive relationship which is of the form sigma dot by E sigma. This will be epsilon dot.

So, this now becomes our governing equation and remember this is we are discussing it for the one-dimensional case. We are assuming that the displacements are small so that it is easy to define epsilon and the stresses and at the same time it is being applied in a Lagrangian sense. So this is our governing equation.

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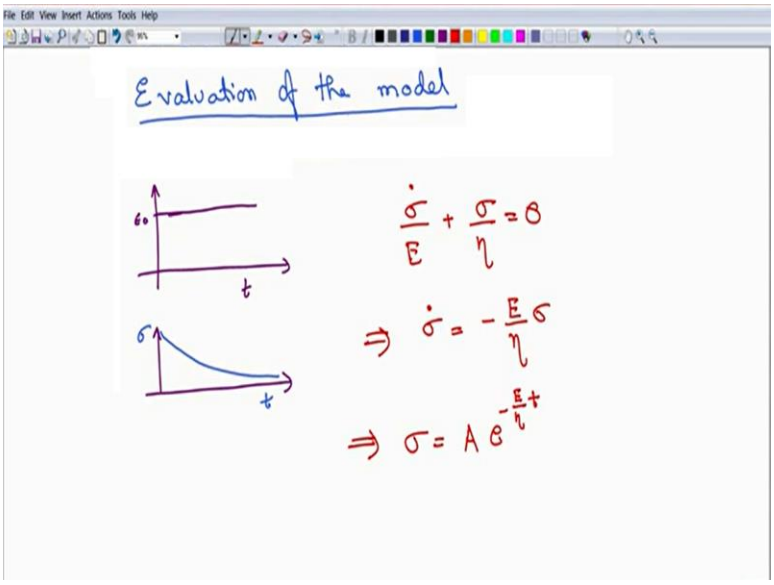

So, now what we want to do I, do is evaluate this model. So let us consider the situation where you are going to apply a given strain rate let us say epsilon 0 and then the stress we know from our introductory classes that the stress should decrease with time. It should be some decreasing function. So what is the district reason function in this case of this kind of constitutive equation? Well so for that we know that in this section your strain is constant so your epsilon dot is 0. So this quantity on your right hand side that disappears.

So, in this particular situation where you are applying a constant strain your governing equation becomes rather simple equal to 0, which implies our sigma dot is minus E times of eta sigma and this is a very simple ODE for which the solution should be sort of known to you. So this is only possible when your Sigma has an exponential behaviour. So from that we can write from very elementary ordinary differential equations, we can write that is some constant into e to the power minus E by eta times of tau of T.

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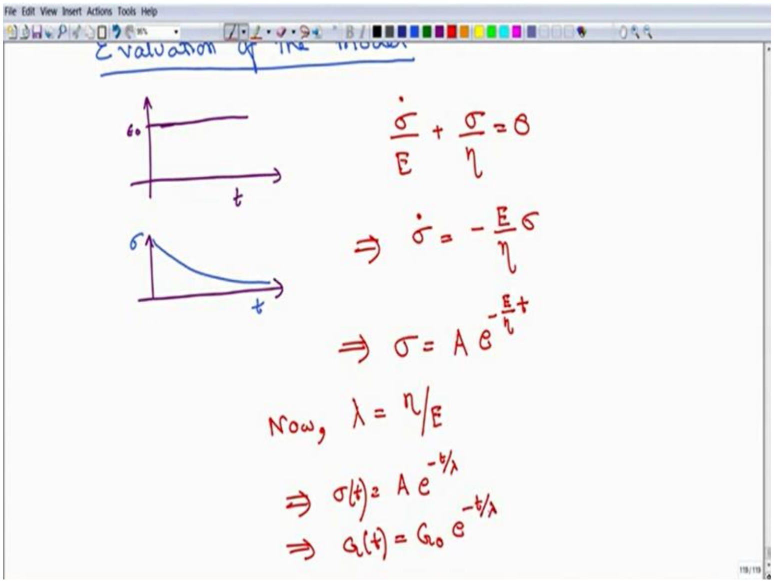

Now, if I want to write, now I introduce a quantity called lambda which is equal to eta times of E then the sigma is basically some constant times e to the power minus t by lambda. So, your GT so this is by the way a function of time maybe I will just make it explicit here and this implies that your stress relaxation function has a functional form of G naught into e to the power minus t by lambda and lambda here becomes a relaxation time scale.

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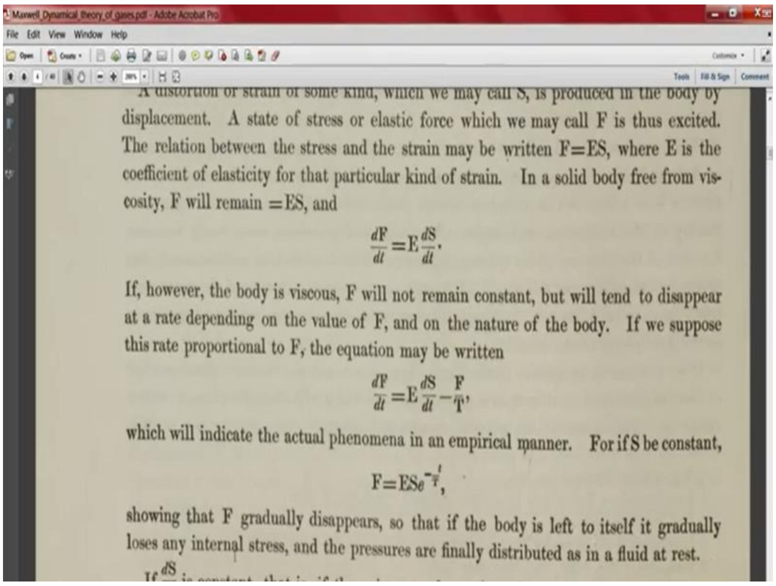

A distortion or strain of some kind, which we may call S, is produced in the body by displacement. A state of stress or elastic force which we may call F is thus excited. The relation between the stress and the strain may be written F=ES, where E is the coefficient of elasticity for that particular kind of strain. In a solid body free from viscosity, F will remain =ES, and

$$\frac{dF}{dt}=E\frac{dS}{dt}.$$

If, however, the body is viscous, F will not remain constant, but will tend to disappear at a rate depending on the value of F, and on the nature of the body. If we suppose this rate proportional to F, the equation may be written

$$\frac{dF}{dt}=E\frac{dS}{dt}-\frac{F}{T},$$

which will indicate the actual phenomena in an empirical manner. For if S be constant,

$$F=ESe^{-\frac{t}{T}},$$

showing that F gradually disappears, so that if the body is left to itself it gradually loses any internal stress, and the pressures are finally distributed as in a fluid at rest.

So, now let us go back since we had originally said that we are using the equation of proposed by Maxwell you can see that we have found this particular equation is something that we had, we have been able to derive. Here T is a time scale, is the relaxation time scale. So you have to take that into account and then we have found that this is the, this is how the body itself gradually loses all internal stresses. So good what we have done is things agree with the original Maxwell's derivation.

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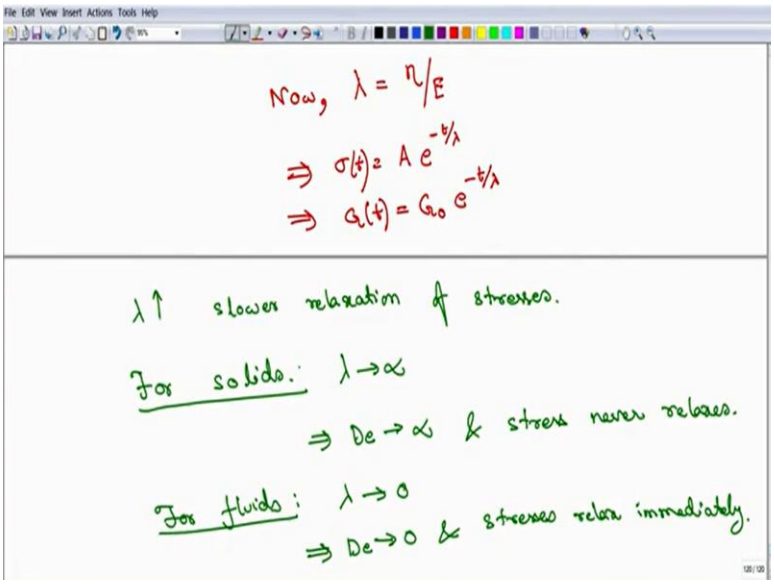

Now, let us just do a quick sanity check. So in this quick sanity check, we want to ask whether the material will behave the right way if you take different limits. So if the material is more solid like, so if lambda increases then the stresses should relax slowly. So slower relaxation of stresses, and is this consistent? So for solids, we have what should be lambda? It should increase, it should be a very-very large value.

So, for solids the lambda goes to somewhere close to infinity which basically says that in our case the Debra number also goes to infinity and the stress never relaxes. For fluids, your lambda should tend toward 0 which means that the Debra number tends towards 0 and stresses relax immediately and that is what we expect. So this is completely consistent with the kind of behaviour that we would expect.

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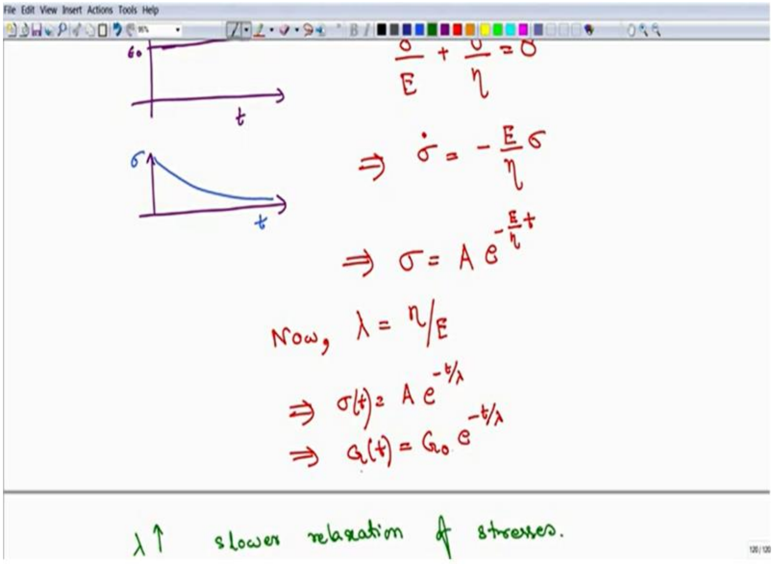

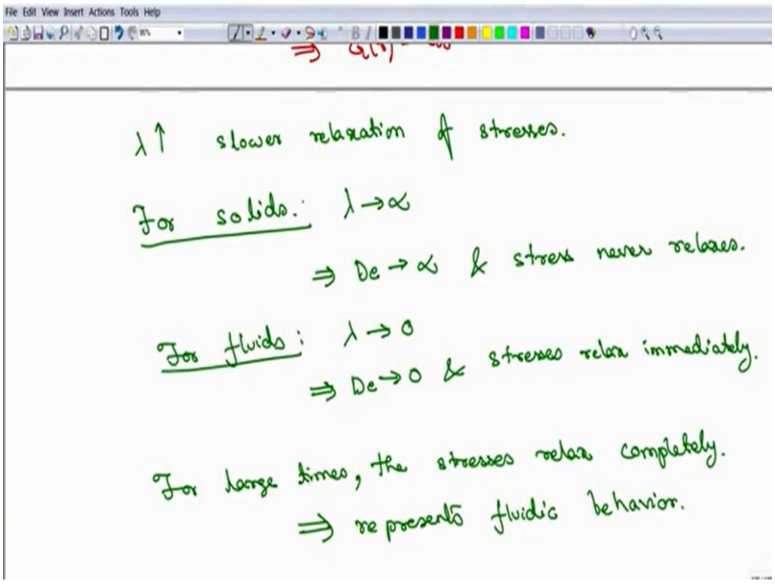

So, our model does have physical characteristics. It behaves as something as a material behave in the real world and then finally one more thing that you can observe from this is that as t goes to infinity this quantity goes down to 0. So this stress relaxes totally and that is indicative of what kind of behaviour? A fluid behaviour, so stresses for large timescales for large times the stresses relax completely.

Now, quickly recall from one of the previous classes what we had discussed. This represents a fluidic type of model, so represents fluidic behaviour. So the Maxwell model is appropriate to describe certain fluids and not solids. Now when we go back here this constant A we were not able to determine and the reason we were not able to determine because you need one more condition to evaluate what A is, but we have not given that to you and the reason that is because you have to evaluate it in a slightly more involved fashion.

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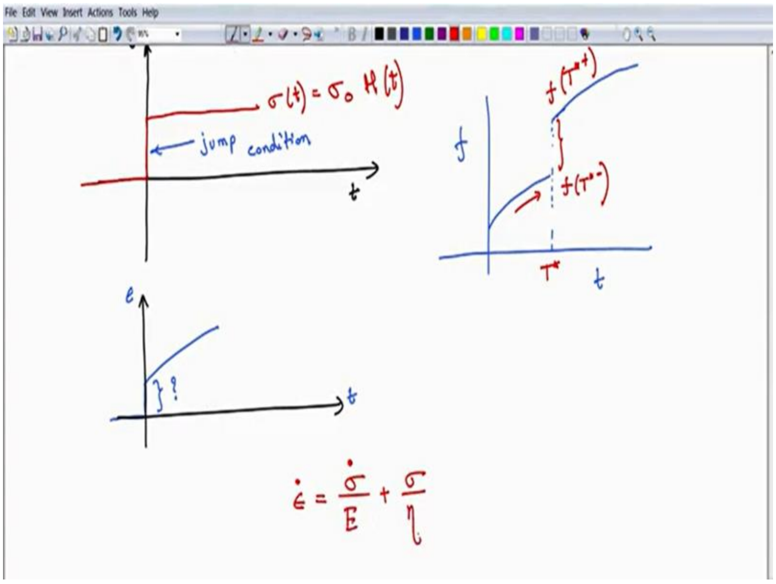

Now let us say that in your case you are applying some stress with respect to time and the situation that we have created for ourselves for the various tests is such that the history at minus infinity or till 0 is 0. So you will be applying nothing till that time and suddenly at t equal to 0 you will be applying some known amount of stress or vice versa. So if this were strain you would be doing the same thing.

So, you have this sigma t has a jump at t equal to 0. So your sigma t you can also write it as sigma naught here into H of t, H being the Heaviside function and we discussed what Heaviside function definitions and by the way MATLAB defines may define Heaviside function in a slightly different way so there are alternative definitions so you should be careful or when you are implementing Heaviside function in a numerical case how you should do that.

So, the question is that if you are applying this then there probably be some jump here and what is this jump? Now because when you are not applying anything, the system response should also be 0, but when you are applying a sudden jump condition at t equal to 0 there will be a sudden jump in the system also and then the system will behave in a certain continuous fashion, but what is this jump in the response?

So, this is what is which is or this is a situation which is also known as a jump, so this is your jump condition and to be just a mathematically a little bit more elaborate a jump condition implies that if you have a function f that you are plotting then you can have a situation where the two limits from the two sides, so if I if this value is let us say a T-star.

So, if I start approaching T time from the left-hand side then I end up with a different limit condition and this is usually called minus from the other side and if you approach the same point from the right-hand side then you end up with a different limit and this is you jump, this represents that jump condition.

So, our problem is that if let us say given a jump in the in the signal, you have to figure out the jump in the response and the jump and the signal can be either the stress or the strain and the corresponding response then will be the other one. So our equation was the equation that we are working with is, so all I have done is I have used this equation back. So I have this should be E this is eta.

So, so let us, so the solution that I am going to show you is not very mathematically rigorous, it is somewhat hand waving but it still gets you the right result. The full mathematically rigorous solution is not in the purview of this course, so the slightly hand waving argument will still lead us to the correct answer, but I think I should point it out that it is not exactly very rigorous whether full proof proceeds on very similar lines.

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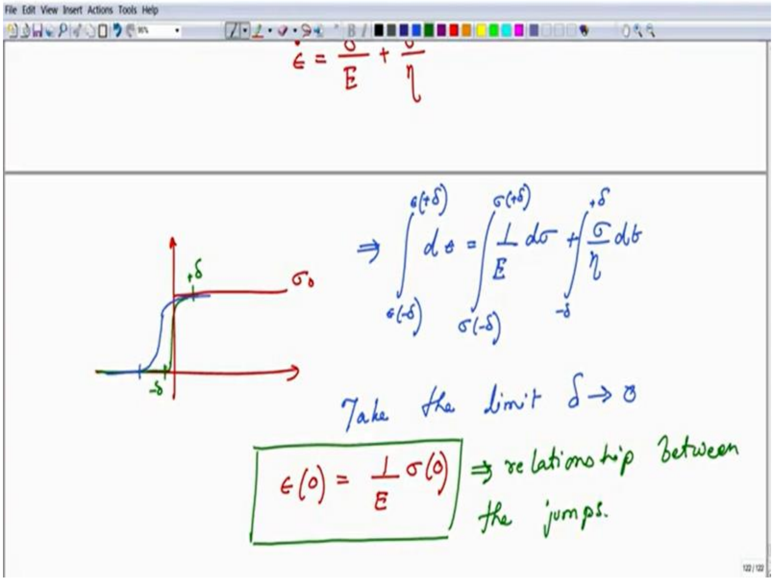

So, what we want to do the problem is being created by the fact that you are applying a discontinuous function. So instead of this discontinuous function, what we will do is, we will assume that there are certain continuous functions that are fully integral and this continuous functions go and merge into your original signal and there is a small gap of let us say plus minus delta.

So, this is minus delta this is plus delta, where this signal is slightly deviating from your jump condition type of signal that you want and basically what you can think of is there is a whole series of different functions that are there and then you are slowly are approaching the condition of this jump and basically this delta is something that you are going to slowly-slowly shrink.

So, you have constructed this set of continuous functions such that they are, they merged into your original signal at plus minus delta, but then this small gap they deviate and then you will apply the condition that delta goes to 0 and then you will end up with the jump condition type of situation.

So, we have our equation we have written it right here, so this implies that if I just want to write it in the differential form, I will get, so let us only integrate within this small region of plus minus delta, so integrate over this small region, so now we will integrate, so we put the integration sign here and we are integrating in time from plus minus delta and in this case you have sigma at minus delta sigma at plus delta you have sigma so you have the appropriate mission is and then take the limit delta tends to 0.

So, what you will end up with is the jump condition. So now this integration will give you the jump in epsilon at t equal to 0 because till 0 you have you should have response should also be 0, but suddenly at plus epsilon you will have the response. So this integration is basically going to give you epsilon at 0 and this is your 1 by E times.

The jump in stress and what happens to this last term? So this is some stress some value some numerical value into 2 delta, but this quantity I am taking delta as in the limit tends to 0, so this third quantity is contribution here is 0. So this now becomes relationship between the two jumps.

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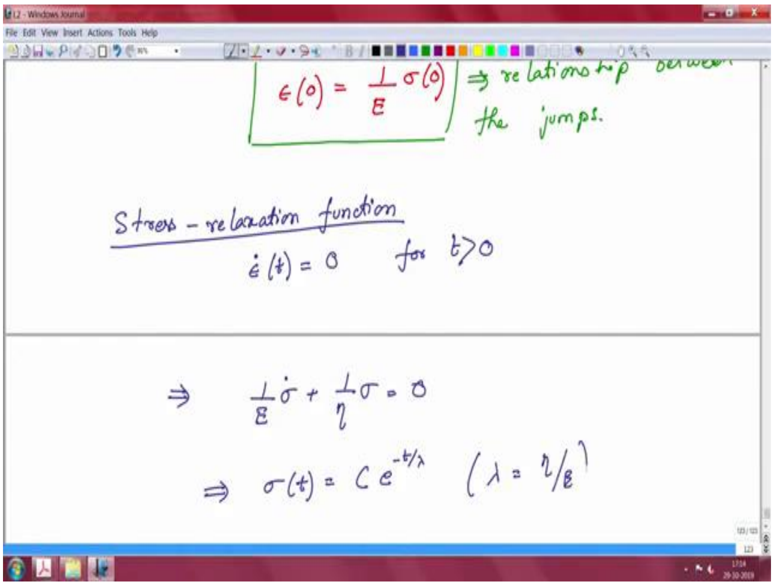

So, now that we have this let us go back to the derivation of the stress relaxation function. So we were looking at the stress relaxation, so our situation is that epsilon t is equal to 0 for t greater than 0 and our corresponding equation was 1 by E of sigma dot plus 1 by eta times of sigma equal to 0. So this sigma t is some constant times e to the power minus t by lambda where lambda was this eta by e.

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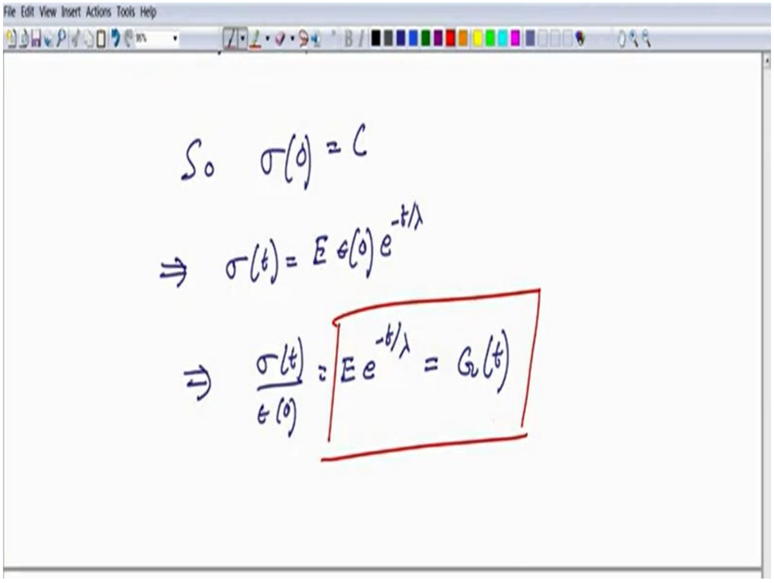

So, sigma 0 is equal to your constant and we know that sigma 0 is the relationship between the two is this. So this epsilon 0 is the given quantity is a quantity that we provided to you. So you can write it as sigma t equal to E times epsilon 0. So your stress relaxation function is actually this quantity. So this is your, so with the jump condition now you are able to get a proper equation for G.

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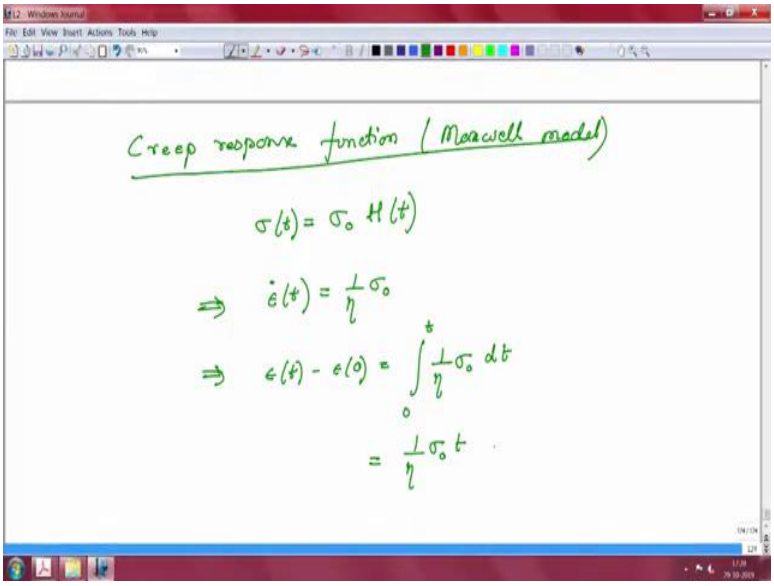

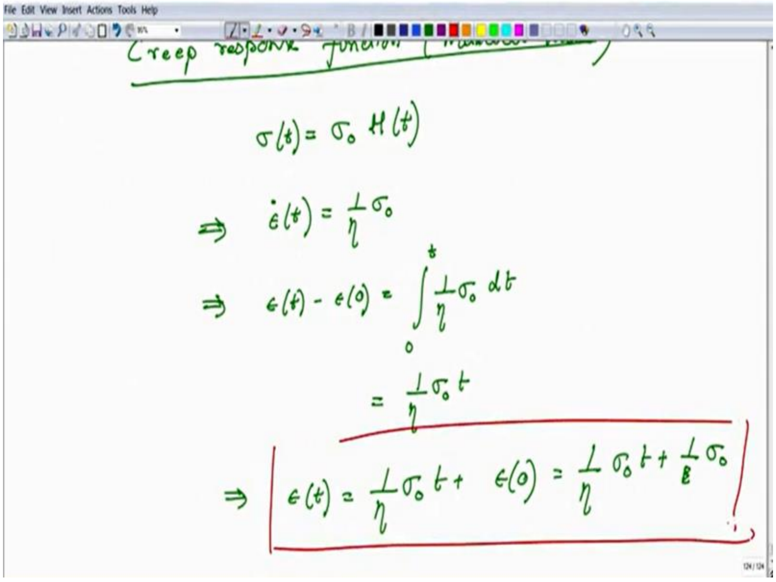

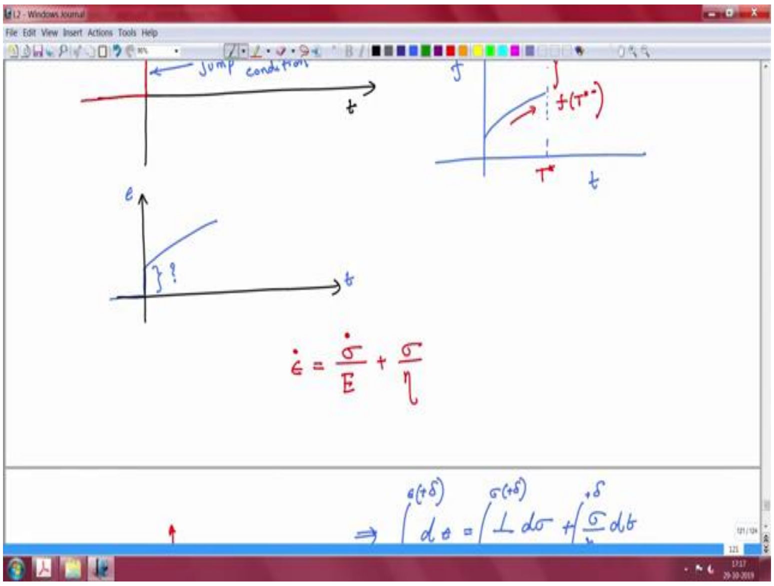

So, we have looked at the stress relaxation function but we know that the other important function is the Creep relaxation function. So we should also look at that this so look at this take a look at the Creep response function for Maxwell model. So here what you are going to do is you are going to apply a stress which is a Heaviside function. So, if we go back to our question here, the stress part this goes away to 0 because you are going to apply a constant stress. So your epsilon dot is now going to be this sigma variable.

So, in this governing equation so when you apply a constant stress when we look at the governing equation this term which contains the time derivative of stress that will vanish and instead you will only be left with one term on your left hand side and one term on your right hand side. So that implies that my epsilon dot is simply 1 by eta times x.

So, my governing equation is that your strain rate, the derivative of the strain is given by this quantity on your right hand side and from very elementary ODEs, we know that when you integrate this what you are going to get is a linear function. So if you take this quantity and then you integrate this, you will have you, so we can see that the Creep function, so this is your final solution and the Creep function is actually linear in nature.

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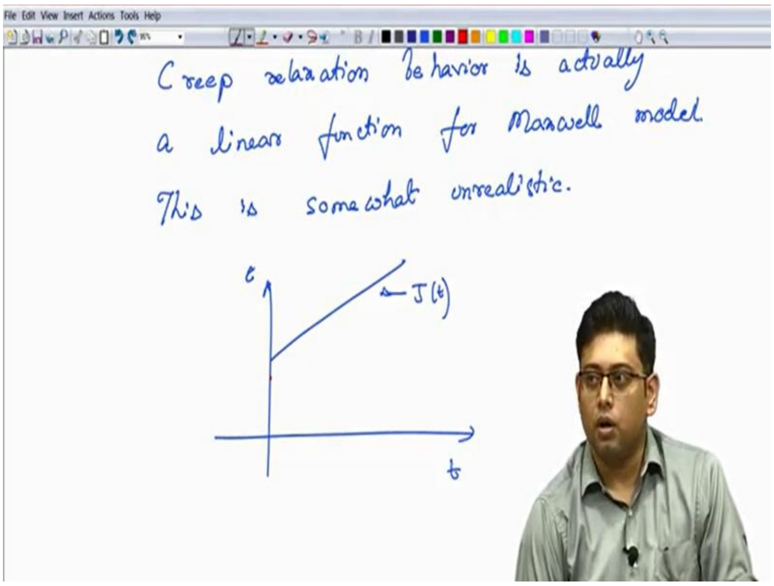

So, Creep relaxation behaviour is actually linear, a linear function for Maxwell model. This is not very realistic, this is somewhat unrealistic. So if you plot this it comes out over a straight line so your Creep function will look like this this is your GFT. For a Maxell model, the Creep response behaviour turns out to be a linear function which is not very realistic. This does not compare well with the graphs that we had drawn initially if you recall those graphs for our Creep functions should look like this doesn't compare well with them, but it is still ok.

So, what we did today is we looked at we started off with the Maxima model and we looked at its stress relaxation function and the Creep function. The stress relaxation function turned out to be a very nice well behaved exponential function which also showed us that the model essentially is representing of kind of a fluidic behaviour.

So, that is where the domain of application of Maxwell model has to be that you have to apply it to viscoelastic fluids and we also looked at the Creep response and we saw that it has a linear behaviour which is not very satisfactory but in many cases it might still work. So we will stop here and we will pick up from here in the next class.