A short lecture series on Contour Integration in the Complex Plane

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Lecture - 09
Examples in contour integrals, ratios of polynomials

Hello, good morning. Welcome to this next lecture on Contour Integration. If you recall, last time we were doing the first example in contour integration.

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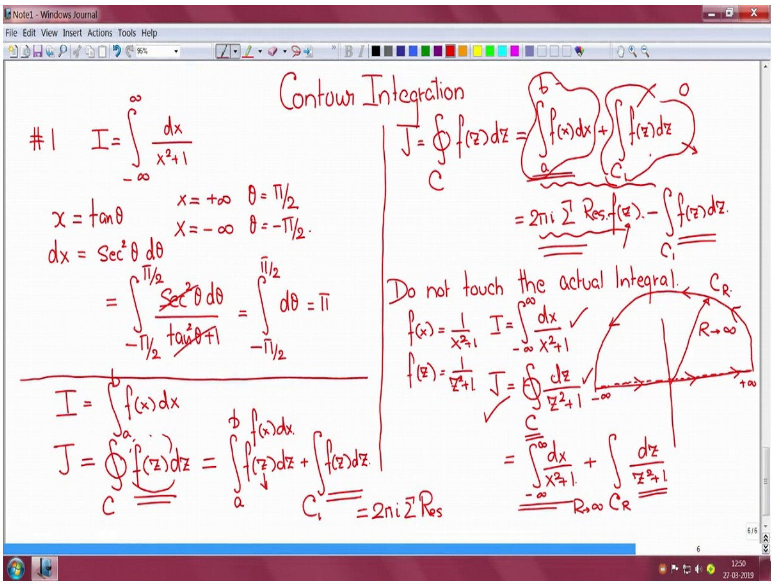

And we left off right here. So, we have this integral dX over X square plus 1 which we are interested in. We replace it with the integral on a closed contour. We call it J. And the X is replaced with z, ok, but we make sure that the contour includes the leg or portion I want, and there is this additional portion which is a semicircular arc integral. So, we have to evaluate this, ok.

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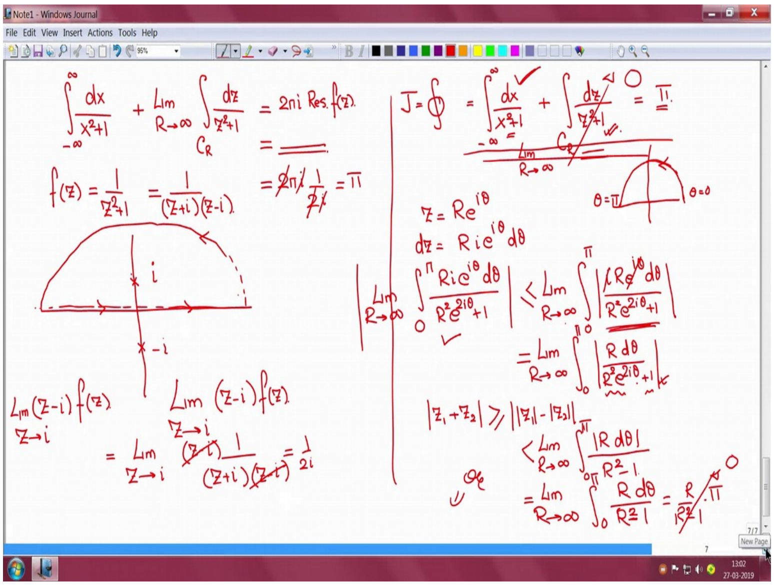

So, what we now have is, integral minus infinity to infinity, dX over X square plus 1, this is the part I want. But additionally, I acquire limit R tending to infinity, integral over the semicircular arc, dz over z square plus 1. Now, that should be equal to twice pi i times, the residues of my function inside the closed contour.

Now, this function of mine: f of z is equal to 1 over z square plus 1. It has two singularities can be written as z plus i into z minus pi, ok; so it becomes singular at plus minus i. So, if I now plot my contour, I have my singularities at i and minus i and my contour goes and closes from above, closes from above and so it does include the singularity i.

So, the right hand side will include the residue calculation. So, you must have seen residue calculations before. How do we calculate residues? We multiply the function f of z, by z minus i and take the limit z tending to i. This is one of the ways we calculate ok. It is gone off the screen, so let me write it here. Limit z tending to I, z minus i into f of z, ok. So, what is it in this case, limit z tending to I, z minus I, into 1 over z plus I, into z minus i. So, this goes with that and my residue is 1 over twice i. So, here I will get twice pi i times the residue which is one over twice i time ok; so twice i, twice i, that is equal to pi, ok.

So, what I now have here is J which is the closed contour integral is equal to the integral I want, minus infinity to infinity, dX over X square plus 1, ok, plus I have integral over a semicircular arc, where the radius is going to infinity, dz over z square plus 1. That is equal to pi. You can already see, because the answer is actually pi and this integral is equal to pi, then somehow this has to go to 0; we can already see that. Now, let us, because the first time we come up we have come across such an integral, we will do it the traditional way, ok. So, we are moving on a semicircular arc, ok, we are moving on a semicircular arc now, ok.

So, let z be equal to R e to the power of i theta, ok, R will tend to infinity, but let z be R equal to R e to the power i theta, which makes dz equal to R is a constant, i e to the power of i theta d theta, ok. Now, if I substitute it in this part of the integral, I get integral, is going to be 0 to pi, ok. I am going this way 0 theta 0, theta equal to 0, to theta equal to pi. And dz which is R i e to the power of i theta d theta, divided by z square plus 1, which is R square e to the power twice i theta plus 1.

Now, we have to find the value, what happens to this as limit R tends to infinity ok. So, what we say is that the magnitude of this integral, magnitude of this integral, ok, magnitude of this integral, is less than or equal to the integral of the magnitude. The absolute value of this integral is less than equal to the integral of the absolute value, integral of the absolute value, which is i R e to the power of i theta d theta, by R square e to the power twice i theta plus 1 ok, this is a known result.

Now, here the absolute value of e to the power i theta is 1, absolute value of i is 1, so we do not have to worry about it. So, this ends up as, I will write equal to limit R tending to infinity, integral 0 to pi, R d theta, by r square e to the power of twice i theta, plus 1. Now, here we use one of these several complex variable inequalities. One of the inequalities is: if I have two complex numbers; z 1 and z 2, ok. The absolute value of z 1 plus z 2 is greater than or equal to the absolute value of z 1, minus the absolute value of z 2, ok.

So, what is the idea here, I am going this absolute value is less than something ok. What I am going to do is, I am going to further raise the value of this integrand such that it is definitely less than that value, which means I will replace the denominator with something smaller, ok. I have two complex numbers, z 1 and z 2 and the absolute value in the denominator, so what I am saying is that is greater than the absolute value of the difference of absolute values, ok.

And so I am substituting this with something smaller in the denominator, so the integrand is now bigger. So, what will I get, I will say strictly less than limit R tending to infinity, integral 0 to pi, R d theta, divided by the magnitude of R square e to the power twice i theta is R square, the magnitude of 1 is 1, so I get minus 1, ok. And R is anyway positive number and theta is going between 0 to pi. So, this is equal to limit R tending to infinity, integral 0 to pi, R d theta by R square, R square minus 1.

Now, R, R square minus 1 they are independent of theta, so they come out. So, I get R over R square minus 1, into pi. With R tending to infinity, R square in the denominator is bigger, so this goes to 0, ok. So, now in here, in here; the right side is pi, in the left I have this portion which I want and this has been sent to 0So, the answer is integral minus infinity to infinity dx over x square plus 1 is pi and this has been done using complex variables. Let us take a second example.

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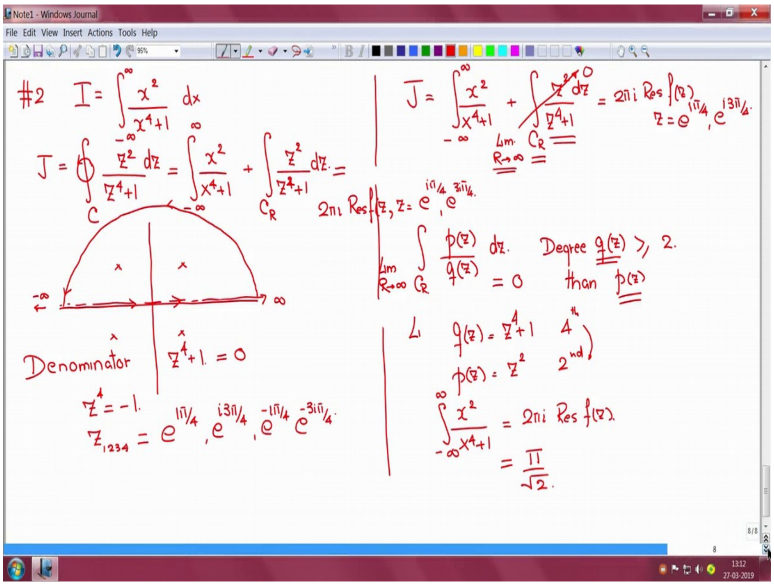

It is very similar, I is equal to integral minus infinity to infinity, x square over x fourth plus 1 dx, ok, this is very similar, so I will not do the full blown derivation of this. So, as is standard now, we will replace this with J, which is a closed contour integral, over some closed contour to be chosen by me and replace x with z. So, we have z square over z fourth plus 1 dz, which is equal to now again I must remember that the portion I want must be part of the contour. So, I have minus infinity to infinity, x square over x fourth plus 1, plus an integral over some other contour ok, z square over z square z fourth plus 1 dz.

I have not yet given you the contour, here it is; it is a very same contour, ok. I am going from minus infinity to plus infinity, plus infinity and then I go the same way again in the upper part, a semicircular arc, ok. Except that now, my function in the denominator, function in the denominator is z to the power fourth plus 1. And z to the power fourth plus 1 goes to 0 in how many places, ok. So, I need the roots of, z to the power fourth is equal to minus 1. So, this has 4 roots z 1 2 3 4, given by e to the power i pi by 4, e to the power i 3 pi by 4, e to the power of minus i pi by 4 and e to the power of minus thrice i pi by 4. So, we have poles here, here, here and here; we have 4 poles with the fourth order.

So, now within the contour, I have two poles, two singularities. So, I have to compute two residues. So, this is equal to, twice pi i times, residues of f of z, ok, computed at z is equal to e to the power of i pi by 4 and e to the power of thrice i pi by 4, ok. So, what we now have, let me write it here. J is equal to integral of the portion I want, minus infinity to infinity x square by x fourth plus 1, plus integral over a circular arc, as the radius is going to infinity, z to the power 2, dz, by z fourth plus 1 and that is equal to twice pi I, times the residue of f of z; computed at z equal to e to the power i pi by 4 and e to the power i 3 pi by 4.

Now, here we use a certain theorem which says that if we have, if we have an integral over a circular arc, of a function, which is the ratio of two polynomials; p of z and q of z, ok, and the radius of the arc goes to infinity. Further, if the degree of the denominator polynomial q z is at least greater than or equal to by 2, ok, than the numerator polynomial, ok. Degree of the denominator must be greater by the degree by must be greater than the degree of the numerator at least by 2, then this integral, ok, this integral here goes to 0.

So, which is indeed the case over here, we have a polynomial in the denominator q of z, q of z which is z to the power fourth plus 1, it is a fourth order polynomial. The numerator is p of z which is equal to z square, it is second order, ok. So, the denominator exceeds the numerator by 2, so it satisfies the theorem and so the integral over the circular arc C R with limit R tending to infinity of that integral is 0, so this is 0, ok.

And therefore, what we have here is integral, minus infinity to infinity, x square over x fourth plus 1, is equal to twice pi I, times the residues of f of z. This is very simple enough now to compute the residues. So, I will let you do the calculation and I give you the answer; the answer is this, it is pi by root 2 ok, its pi by root 2. Anybody has a query then I will solve it in one of the lectures. Next example ok, the next example is a very classic example. Call it example number 3, ok.

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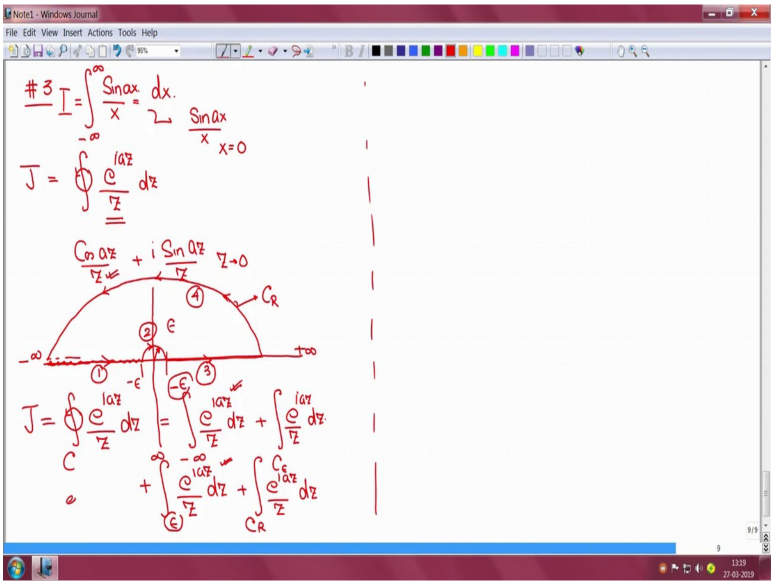

We all must have wondered what it would be to integrate sin x over x, from minus infinity to infinity. So, what we will do is we will integrate. So what I have here a sin a x , ok, sin a x over x, from minus infinity to infinity dx. So, as I said we replace, so let me call this I; we replace this integral by J, which is now a contour integral over a closed contour. And we said, most of the time the x is replaced by z, but occasionally we may take a better function. So, what we choose here is e to the power of i a z by z dz.

Now, in this form which is actually a sinc function, ok, sin a x by x, there is no singularity at X equal to 0, ok. If you use L'hospital rule, limit X equal to 0 of this function, this will acquire a finite value, there is no problem with this function. However, when we choose this form, this form has a singularity at z equal to 0, ok, because this is actually equal to cos a z by z, plus i sin a z by z ok. Again sin a z by z has no problem as z tends to 0, but this has a problem when z equal to 0, this one has a problem at z equal to 0, ok. So, this function that we have chosen has a problem at z equal to 0, ok.

So, now let us see what is the contour we use, ok. We use a contour that comes all the way from minus infinity along the real axis, ok, along the real axis. Now, at z equal to 0, we have a singularity, so we circumvent the singularity, we go around it with an semicircular arc having a radius epsilon. We move forward to plus infinity and we close the contour using a semicircular arc in the upper half ok, so that is the contour we have.

So, how do we now break this up; I have J, which is equal to the integral over the closed contour, which I call C and this I will give it the name C R, e to the power i a z over z dz, is equal to, now let me count the number of portions I have. I have a straight line portion here, I have a semicircular arc portion here, I have a straight line portion here and a semicircle here. I have 4 portions of 4 portions here, ok.

So, now what do I do, I write them down. I have minus infinity up till minus epsilon, this point is minus epsilon, ok, e to the power of i a z by z dz; plus the integral on the semicircular arc C epsilon, ok, e to the power of i a z over z dz ok; plus now integral from epsilon, from epsilon to infinity, e to the power of i a z over z dz; plus the semicircular arc portion C R, with R tending to infinity.

So, let me just say that the portion here and the portion here, they constitute the integral in the sense of Cauchy Principle Value, because both the limits, you have a singularity and both limits are epsilon. So, we will send epsilon to 0 and both the limits on either side are epsilon. So, this integral is in a Cauchy principle value sense ok. We will continue with this example then in the next class. So, I will stop here.

Thank you.