A short lecture series on Contour Integration in the Complex Plane
Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 08
Implications of CIF, converse of CG theorem

Hello good morning, welcome to this lecture on complex variables. Last time if you recall we were looking at extensions of the Cauchy integral formula.

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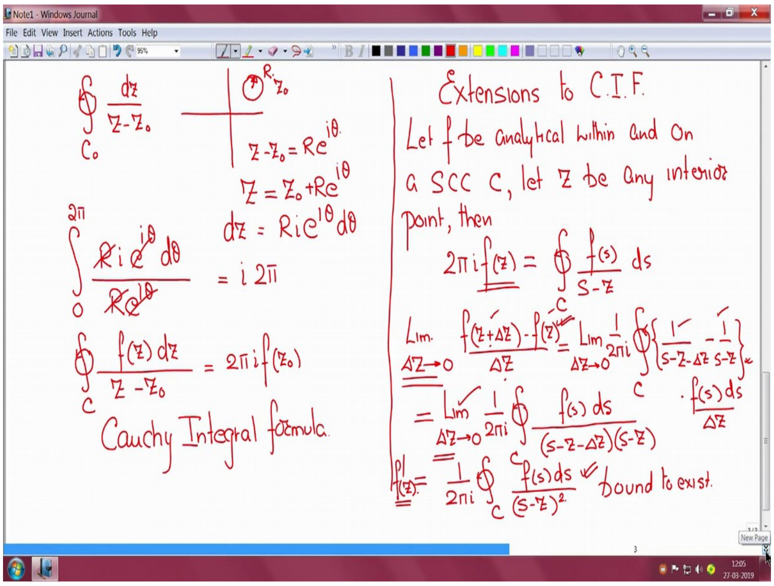

And this is where we left off, ok. We formulated this limit delta z tending to 0, f of z plus delta z, minus f of z, by delta z. And we put it in terms of the original f of z. We took the limit delta z tending to 0 and we found that it happens very smoothly. There is nothing preventing this limit from existing.

So, this is bound to exist and this is f dash of z; this is f dash of z. This is the derivative of f of z. Similarly, we can write a next level derivative, ok.

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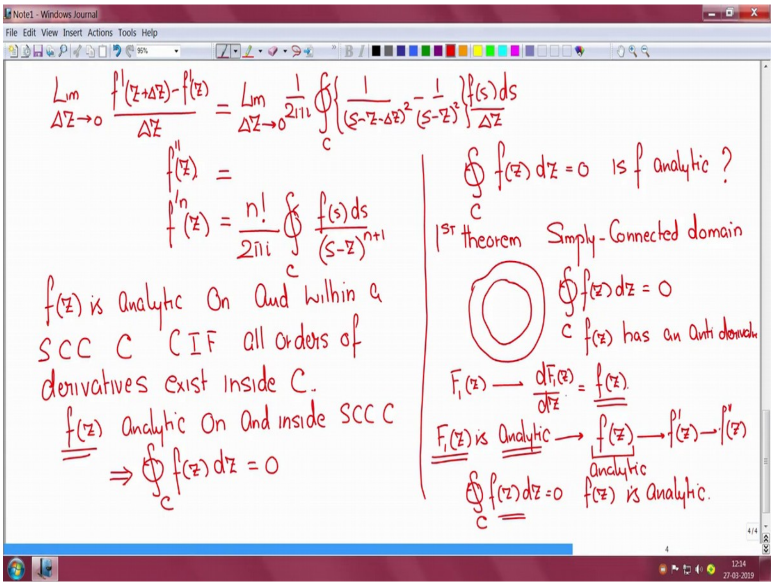

Which is: limit delta z tending to 0, f dash z plus delta z, minus f dash of z, divided by delta z, ok. And how will this look like? This looks like limit delta z tending to 0, one over twice pi I, integral counterclockwise C, 1 over s minus z minus delta z, whole squared, minus 1 over s minus z, whole squared, f of s d s, by delta z, ok.

And again this will also exist; this is f double dash of z. So, if we continue like this, the nth derivative will also exist: f dash n, f dash n, the general form will turn out to be, n factorial by twice pi I, integral counterclockwise C, f of s d s by s minus z, to the power n plus 1. So, if a function is analytic, if f of z is analytic on and within, on and within a simple closed contour C, what Cauchy integral formula tells me is that all orders of derivatives, all orders of derivatives exist inside C, ok.

It is also very important. So, once a function is analytic at a point, all orders of derivatives of f of z exist at that point, ok; this is also very big result. Now, we said earlier, that if a function f of z is analytic on and inside a simple closed contour C, then integral closed contour C, f of z dz is equal to 0. We said this. This was the Cauchy Goursat theorem, ok. And we said, at this stage, it is one way. f of z is analytic implies this is true ok. We did not state that the other way is true.

If I suddenly find, ok, if for a closed contour integral f of z dz is equal to 0, is f analytic there or not, is f analytic in that domain, we did not say anything. So, now what we have here, we will use the very first theorem we stated and for now let us look at a simply connected domain, ok.

Now, suppose in a simply connected domain I find that integral; closed contour on a simple closed contour C, f of z dz happens to be 0, happens to be 0, in a simply connected domain on a certain simple closed contour, I find that f of z dz happens to be 0, ok. Then the first theorem says that in this simply connected domain, f of z has an anti derivative, ok. That means what? There is a capital, let us say capital F 1 of z, ok; which is such that d F 1 of z dz is equal to f of z. There exists a function whose derivative is this function f of z.

So, because there is a derivative of F 1 of z; in this entire simply connected domain, F 1 of z is analytic. But we have just seen the consequences of Cauchy integral formula, that once a function has a derivative at a point, all orders of derivatives exist. So, F 1 of z has a derivative which is f of z ok, then f of z has a derivative which is f dash of z, then f dash of z has a derivative which is f double dash of z and so forth.

And therefore, f z becomes analytic, ok. So if in a simply connected domain you find: now this is a complete converse; if in a simply connected domain you find that integrals over closed contours f of z dz are 0, then f of z is analytic in that domain. So we have a complete converse now, ok. If in a simply connected domain, the function is analytic on every simple closed contour on and inside, then integrals are 0. And the complete converse: if integrals over every simple closed contours are 0, then in that simply connected domain f of z is analytic. It is a complete converse.

Now, the next topic. We have covered more or less the theorems that we need, ok. From here on our orientation would be to take examples of contour integration because that is the main focus, ok. And that is the reason I have quickly gone through the relevant theorems, ok. The one main item now remains is this Cauchy principle value, Cauchy principle value.

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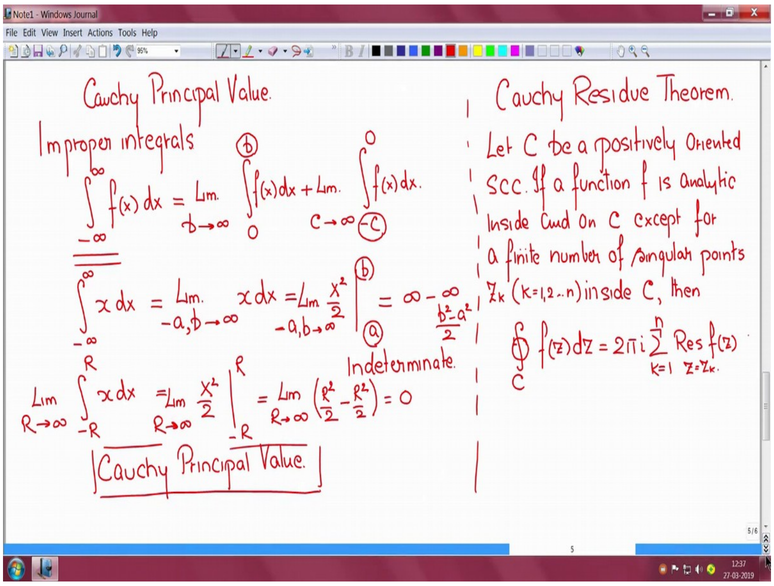

The Cauchy principle value is this; when we do improper integrals, integrals where the limits go to infinity or there are singularities within the domain, ok. For example, if we have an integral of this form: infinity to infinity, minus infinity to infinity, f of x d x let us say, ok.

Then what is implied is that this is equivalent or how it is computed is that this is equivalent to limit b tending to infinity, integral 0 to b, f of x dx, plus limit C tending to infinity, integral minus C to 0, f of x dx, ok. This b is a different number, the C is a different number and they are sent to infinity on their own, ok. This is how the improper integral is implied, ok. Now let us take this example; I have an integral, integral minus infinity to infinity, x dx, ok. So, if I send the upper and lower limits to infinity on their own.

So, I write this as limit, say minus a comma b, tending to infinity x dx and what happens over here is that I get x square by 2, with b and minus a, and limit ok. Limit a comma b tending to infinity, ok. I can put a over here; let me do that. I take minus a, minus a comma b tending to infinity, ok. So, what this gives me is an infinity minus infinity; b goes to infinity on its own, a goes to minus infinity on its own.

So, I get the answer as infinity minus infinity. So, it is indeterminate. Instead, what I will do is, I will say that integral is minus R to R, limit R tending to infinity x dx, ok. Then I get x square by 2, R minus R, limit R tending to infinity, ok. So, this gives me limit R tending to infinity, R square by 2, minus R square by 2, which is 0, ok. Here I got b square minus a square by 2, ok. Here I got b square minus a square by 2 whereas here i get R square by 2 minus R square by 2; so this is 0 first. So, this is called the Cauchy principle value.

So, we will be seeing integrals in our future lectures and they will be done in the Cauchy principle value sense ok; that is to be noted. We need one more last theorem which is this, which is very famous Cauchy Residue Theorem; which many of you should have seen in your earlier classes, ok; which says let C be a positively oriented simple closed contour. If a function f is analytic inside and on C, except for a finite number of singular points; of singular points z k; k going from 1 to n, inside C.

Then integral over the closed contour C; taken in the positive sense, f of z dz is equal to twice pi i, times the sum of residues of f of z, ok; z is equal to z k. This is a theorem all of you must have seen before, ok. Now, I will not prove it; we will accept it as it is and this is very useful when doing our contour integrations. So, now, we are ready to take a look at our first example in contour integration, ok.

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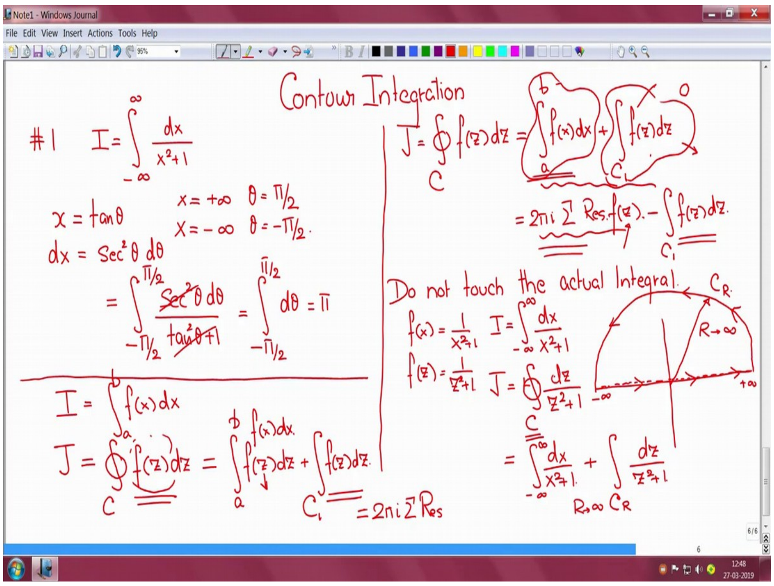

So, contour integration. There will be other theorems and corollaries that one would require. I will introduce them as and when we need it without proof, ok. For detailed proof one can again go back to the text by Churchill and Brown. So, let us see the first example contour integration first. I have this integral I given by integral minus infinity to infinity, dx by x square plus 1. All of you must have seen this integral at one time or another. How would we do it traditionally? We take a substitution; x equal to tan theta so that my dx is equal to sec square theta d theta. When x is plus infinity, when x is plus infinity; theta is equal to pi by 2 and x is at minus infinity, theta is minus pi by 2.

And so after I take the substitution, my integral limits will change: minus pi by 2 to pi by 2, sec square theta d theta, by tan square theta plus 1. tan square theta plus 1 is again sec square theta. So we can cancel these two. So, we get integral minus pi by 2, to pi by 2, d theta, which is equal to pi, ok. Now, we would like to use contour integration in the complex plane to achieve this result.

Now, the main idea in doing contour integrations in the complex plane is that I have an integral I ok. I have an integral I, let us say a real integral I; let us say it is some integral of some function f of x dx real variable, ok. I want to use the principles of complex variables to find the value of this real integral.

So, now what I do is I create another integral J, ok, which is an integral over a closed contour, ok; closed contour C of my choice, ok. And some other function or a function now of a complex variable d z mostly x gets replaced by z, but occasionally function itself changes, for convenience, ok; so, we replace that integral with this. Now what is this contour like? This contour, let us say this integral is from some a to b, ok. So, this contour must include that part of a to b, ok; f of z dz plus other portions f of z dz, ok.

So, in the portion a to b; z also must be equal to x, so this is actually f of x dx; that means, z becomes real in this leg of the integral. And then because we have a closed contour; in addition there will be a portion some other contour C 1; which makes a to b and C 1 together a closed contour. Now, we have a closed contour over a complex function. So, that is going to be equal to twice pi i times the sum over residues, ok; here comes the residue theorem.

So, what is happening? I have replaced my original integral with an integral over a closed contour, ok. As I said, the same function which has z as argument or slightly different function, but main point is the portion I want to be integrated that is a to b, f of x dx must be one part of the contour. And in making it close, there is another portion that is coming in, ok. And this together, because now this is a closed contour C; let us call this C 1, I will have the answer equal to twice pi i times the residue theorem or residues of f of z.

Now, the contour is so chosen that I should be able to find the answer to this portion, ok; sometimes there can be more number of integrals in this portion. So, I should be either easily be able to evaluate this integral or find that it is going to 0; very often it goes to 0, ok. So, then what happens is, if this is not 0 it comes on to this side and becomes whatever the value it acquires, ok. Then I compute the residues of my function; now having a complex argument. So, the integral I want is equal to the residues minus whatever additional portions have come up.

So, basically I do not touch, I do not touch; I do not touch the actual integral, that is the whole idea. I do not evaluate this integral which is my goal, directly. I am going to do it indirectly, that is the idea of contour integration. So, now, let us see in this case, I have a function, f of x equal to 1 over x square plus 1, ok. So, I replace it with f of z which is 1 over z square plus 1, ok. z is now suddenly the complex variable, ok. So, if I call this integral the real integral the I; I have integral minus infinity to infinity dx over x square plus 1.

And then I replace it with J, which will always represent my closed contour integral; closed contour to be chosen and this in this case is exactly replacing x with z. So, what is this contour that I have to choose? The choice I make is this ok; I come from all the way from minus infinity, ok, I come from minus infinity; I go to plus infinity. I take a semicircular contour in the upper half of the complex plane, come back and join at minus infinity.

So, the direction is counterclockwise, ok. So, the whole contour I will denote by C; this portion I will denote by C R, because this is a semicircular arc at a radius R; R tending to infinity. So, you can now see that this closed contour integral includes minus infinity to infinity and along this my z is equal to x. So, I have dx over x square plus 1, ok, but additionally I have an integral over a semicircular arc C R; where the R is tending to infinity, ok. Here, the variable is complex dz by z square plus 1. So, this example I will take up in the next class and we will finish it; I will close it here for now.

Thank you.