A short lecture series on Contour Integration in the Complex Plane
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Lecture - 07
Implications of Cauchy Gorsat Theorem, Cauchy Integral Formula

Good morning and welcome to this next lecture on a Complex variables.

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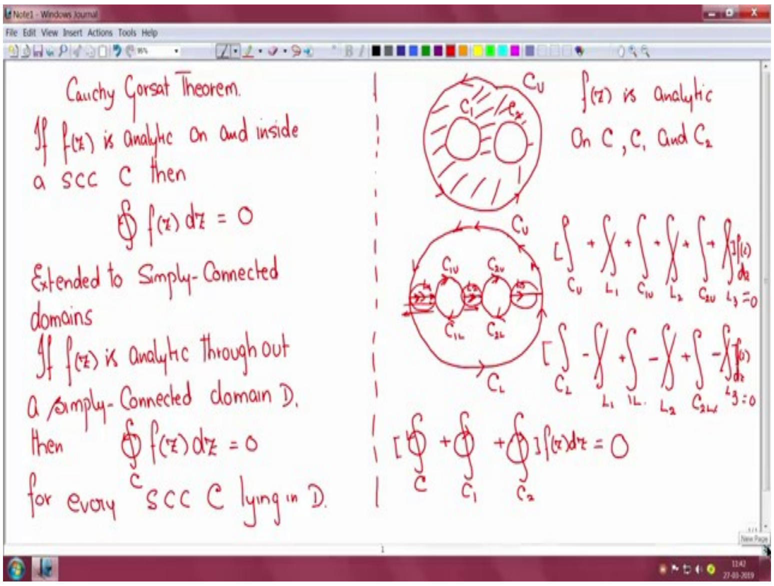

Last time we had seen the Cauchy Gorsat theorem. We had seen this in some detail, so I will not repeat it. In brief it says, if f of z is analytic on and inside a simple closed contour named C, then integral over the closed contour f of z dz is equal to 0, the contour navigated in the positive anti clockwise direction.

Now, this can be very easily extended to simply connected domains; simply connected domains because interiors of closed contours are involved. So, we can also say that if f of z is analytic; if f of z is analytic throughout a simply connected domain D, then integral over C a closed contour, f of z dz is 0 for every closed contour; every simple closed contour C lying in D. It's a pretty straightforward extension to simply connected domains.

Now, this has very important consequences. Now consider this situation. We have a closed contour C taken in the positive sense. And suppose I have two more regions. Let us call this contour C 1. Call this contour C 2. Let us not yet give any directions to them. So, this is the extension of Cauchy Gorsat theorem or the implications of Cauchy Gorsat a theorem. So, if now I break this situation in the following way: let me first say that a function f of z is now analytic on C, on C 1 and C 2 and in between the region that exists between C, C 1 and C 2. The regions inside C 1 and inside C 2 are excluded, ok.

So, now I am going to apply the Cauchy Gorsat theorem here. Before that what I do is I close these contours in the following manner. I will call this the upper part ok, so let me draw this again, ok. I will call one piece the upper. This is C 1 upper, C 1 lower, C 2 upper, C 2 lower, ok. The direction on C upper and this part is C lower. And I take I call this bit L 1. I call this bit L 2, I call this bit L 3. the directions are given. I am sorry this will be this way.

So, I have two contours now: C upper comes this way gets into L 1; L 1 joins C 1 upper, joins L 2; L 2 joins C 2 upper, joins L 3 and then I close. Similarly, I have another closed contour with C L, so we will see what happens now. So, I say that this function f of z, ok, with integrals C upper, plus L 1, plus integral C 1 upper, plus integral L 2, plus integral C 2 upper, plus integral L 3, ok, this forms this closed contour, this forms this closed contour, ok.

Now, given this statement that f of z is analytic on C; on C 1; on C 2 and in between, ok. So, I have the statement made for Cauchy Gorsat theorem: a function f of z is analytic on and inside a closed contour. All these little bits form this closed contour on the upper side. So, now, this integral on f of z is equal to 0, of z dz is equal to 0, ok.

So, let us see now I have gone so far into the border. Let me just say here, that is equal to 0; is equal to 0. Similarly, for the lower side what I will do? I will take integral C lower this one, then I have L 3, but in the opposite direction. So, I will take minus L 3 here, minus L 3, ok, then I take C 2 lower; I take C 2 lower, plus integral C 2 lower, ok, then I have L 2, in the opposite direction. So, I do minus L 2; L 2, then I have C 1 lower, plus C 1 lower ,then I have L 1 in the opposite direction minus L 1, ok.

So, these integrals, these line integrals again form an integral of f of z on a closed contour which is on the lower side. Now, the function f of z is well behaved in these regions between the circles, ok. So, here if I have an integral going rightward and an integral going leftward, they will cancel out, because the function is well behaved. So this is also 0, f of z dz here is also equal to 0. Now, I am going to add the two. If I add the two, plus L 3 and minus L 3 cancel out, plus L 2 and minus L 2 cancel out; plus L 2 minus L 2 cancel out, similarly plus L 1 minus L 1 cancel out, ok.

So, what am I left with? I am left with C upper and C lower, which forms a counterclockwise, so that makes it integral counterclockwise C, ok. Then I have C upper and C 1 upper and C 1 lower, which is plus integral clockwise C 1. Then I have plus integral C 2 upper and C 2 lower, which makes it again a clockwise closed contour C 2. So, this set of integrals operating on f of z dz equal to 0.

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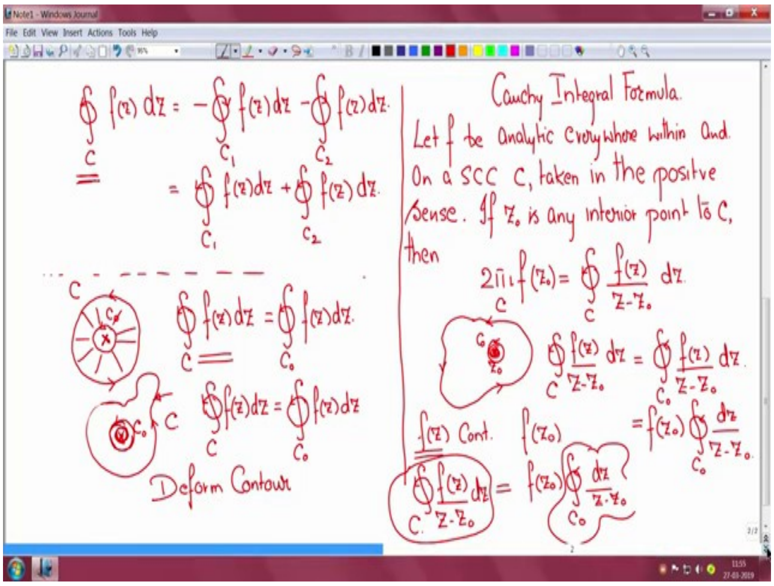

Now, what I will do, we need to go to the next page. I will bring C 1 and C 2 to the other side. So, I will have integral counterclockwise on the C, f of z dz is equal to a negative integral clockwise C 1, f of z dz, negative integral clockwise C 2, f of z dz. Now, if I change the signs of this, I can change the directions of the contours. So, I change the sign and change the contour direction, now this is made counterclockwise C 1, f of z dz plus integral control clockwise C 2, f of z dz.

Now, you see the integral of f of z on the outermost contour is the sum of integrals of f of z along C 1 and C 2 which are contours inside, ok. So, if I have a large number of these regions inside C, where the function interior to these contours is not analytic, but its analytic on those contours and between the outer contour and these circles. Then the integral over the outermost contour is equal to sum over the integrals of all the contours inside.

Now, a further bigger consequence is, suppose there was only one. So, this is the outermost contour C traversed in the counterclockwise direction and there was one inner contour let us say C 0. And if a function was analytic between these two contours and on the two contours, but not inside, then the integral counterclockwise on C, f of z dz is equal to the integral counterclockwise C 0, f of z dz. This is very big, this result is very big and very significant.

What it implies, is that suppose I have an integration to be done of this form on a closed contour and perhaps the contour is somewhat inconvenient or something, then I have a singularity inside somewhere, I have a singularity, suppose this is a situation some contour that is has an odd shape.

And I have a singularity inside and I am supposed to do the contour integral on C ok. What I can do is I can deform this contour into whatever convenient form I want perhaps a nice circle and take it down as long as I do not touch the singularity. So, I can deform this contour C into C 0 of any shape that I want as long as I do not touch the singularity. So, this integral, outer integral C, the value f of z dz will be identical to integral over the convenient contour of my choice C 0, f of z dz ok.

So, we can deform the contour as long as we do not touch a singularity or cross a singularity. Now, this can be formalized and the formal statement is there in the book by Churchill. Now, let us proceed to the next theorem ok. The next theorem is called Cauchy integral formula, ok. In the spirit and with the pace I am going I will give a very sketchy proof of this. The rigorous proof is there in the book by Churchill and Brown ok. So I will refer to that book.

So, what does this Cauchy integral formula say? Let f be analytic; f be analytic everywhere within; everywhere within and on a simple; on a simple closed contour C, taken in the positive sense, counterclockwise sense, ok. If z 0 is any interior point to C, then 2 pi I, f of z 0 is equal to the integral over the closed contour taken in the positive sense, f of z by z minus z 0 dz.

So pictorially, there is this contour C, closed contour C taken in the positive sense and there is this point z 0, ok. So, let us see how, so as I said I will do a not so rigorous sketchy proof. The epsilon delta type proof is where in Churchill. so I would recommend that students look at it. So, now, let me enclose this point z 0 by a small circular contour and call it C 0 taken in the counterclockwise sense. So, now, from what we have seen previously, our function f of z is analytic everywhere on C on C 0 and in the in between region.

So, this particular integral f of z by z minus z 0 dz, counterclockwise on C is identical to the integral taken counterclockwise on C 0, f of z by z minus z 0 dz. Now, from the earlier portion we have seen, that I can go as close as possible to the singularity, as long as I do not cross it or touch it. So, here also I can go closer and closer; and closer and closer to z 0, as far as possible without touching z 0, ok.

And since f of z is a well behaved function; f of z is a well behaved function and suppose f of z is analytic and hence continuous, what will happen is as C 0 is made smaller and smaller around z 0, it will acquire a constant value of f at z 0 or it will approach that value, because f of z is continuous.

So, on the right side I will make C 0 tighter and tighter around z 0. So, that as f of z approaches or z approaches z 0, I can put f of z 0 outside the integral. It is acquiring a constant value as I make C 0 tighter. So, this is now dz over z minus z 0. As I say this is a rough proof ok, but it is useful, ok. So, now, what I have is, f of z 0 and an integral over C 0, dz by z minus z 0.

So, what is this? This is the value of the integral we began, f of z by z minus z 0 dz over the original contour C which we should not forget. We are trying to compute the value of this particular integral, ok. So, now, we have to evaluate this part; we have to evaluate that part, ok.

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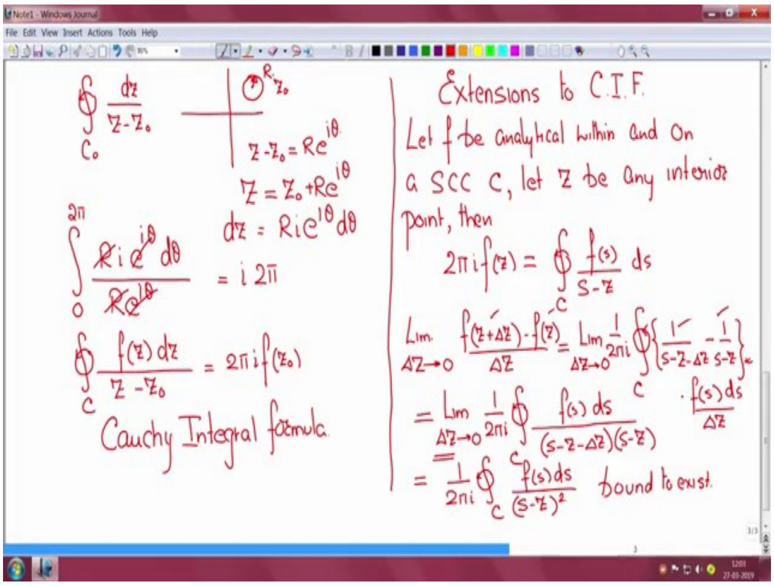

So, let us see what is integral over C 0 dz by z minus z 0, ok. I say that we are going around a small contour around z 0. Let us say over a radius R. So, my z minus z 0 I will replace with R e to the power of i theta. z then is equal to z 0 plus R e to the power of i theta, ok. Then dz is equal to: z 0 is constant, R is a constant, so R i e to the power of i theta d theta.

Now, I replace everything in z with everything in theta and so my limits on theta become 0 to 2 pi, 0 to 2 pi. dz is R i e to the power of i theta d theta divided by z minus z 0 which is R e to the power of i theta. So, R goes out. e to the power i theta goes out. i d theta from 0 to 2 pi is i times twice pi, ok.

So, if we add what is on the previous page, the integral I want: counterclockwise over C, f of z dz by z minus z 0 is equal to twice pi I, f of z 0. This is called Cauchy integral; Cauchy integral formula, formula. Now, the Cauchy integral formula has consequences, ok. Extension,s extensions to Cauchy integral formula.

Now, what is the extension? So, what we say here is: let f be analytical within and on a simple closed contour by the name C and here let z be any interior point ok. Then we do it, we write this in a different set of variables. So, twice pi I, f of z is equal to, integral closed contour C, f of s by S minus z d S. We have written it in terms of z. Earlier we wrote it in terms of z 0, ok.

Now, watch what happens. What I will do is, I will take this limit delta z going to 0, f of z plus delta z minus f of z, divided by delta z, ok. How do we what happens on the right hand side? Obviously, I will get the limit delta z going to 0. Then I have 1 by twice pi i. Then I have the integral over the closed contour C and here I have 1 by S minus z minus delta z, minus 1 over S minus z ,into f of S d S, divided by delta z, ok.

This is for f z plus delta z; this is for f z, ok. So, now, what happens. Let me write it here: is equal to limit delta z tending to 0, 1 by twice pi I, integral counterclockwise C. I do the LCM of this and what I get is, f of s d S by S minus z minus delta z, into S minus z. I have taken an LCM of this and brought f s d S on the top. The LCM just gives me delta z on the top which cancels with this delta z in the denominator here.

Now, the point to be made here is that, f of s is well behaved everywhere inside. So, I can straight away take the limit delta z going to 0 in here. It is not going to cause any issues. I am not violating singularities or any creating any singularities. So, delta z can be straightaway taken in. So what I get is, 1 over twice pi i, integral counterclockwise C, f of s d S by S minus z whole squared. And this is bound to exist ok; this is bound to exist ok. We will extend this further for a general case. So, I will stop here for today we will continue in the next class.

Thank you.