A short lecture series on Contour Integration in the Complex Plane
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Lecture - 06
Cauchy Gorsat Theorem

Hello, good morning. Welcome to this next lecture on Complex Variables. So, if you recall last time we had seen a theorem.

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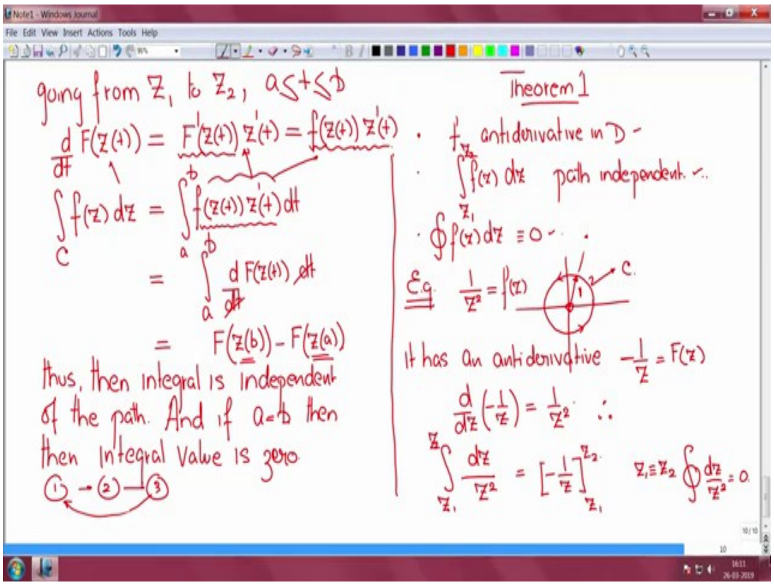

We will not repeat the entire theorem, but main statements. It said that about a certain function which was continuous, if any one statement is true, the others are true. So, one was that f had an anti derivative; anti derivative in D, then integrals of f between any two points z 1 to z 2, was path independent and lastly integrals on closed contours was 0, ok.

So, if any one statement is true the other two are also true ok. And, assuming the statement one to be true, we saw the truth of the second and assuming the second to be true we saw the truth of the third, but 3 to 1 we did not see. I said it is complicated then one can see it in a proper complex variables class, ok.

Now, to further explore this idea let us look at; so, there is it let us take a look at an example to understand this theorem. So, the example is this. I have a function 1 over z square. It is obvious that this function is not well behaved at the origin ok; at z equal to 0, this function is not well behaved. And let us consider a simple closed contour which is a circle of radius 1, the circle of radius 1 ok. This is my C, ok.

Now, the domain of this function is obviously, the entire complex plane minus the origin. Except the origin, the function is well defined all the way up to infinity and it has an anti-derivative. It has an anti-derivative. That means, we have suddenly found that it has an anti-derivative. That was the nature of the theorem. You suddenly find that any one of the statements is true about this function. So, about this function there is an anti derivative, ok.

So, what is that anti derivative: which is minus 1 over z. This is the anti derivative. Why is that? Because d by dz of minus 1 over z happens to be equal to 1 over z square ok. In the same domain. And therefore, the integral from z 1 to z 2 of d z over z square is equal to the anti-derivative, minus 1 by z, evaluated between z 1 and z 2, ok. And, if it is a closed contour, like we started off, the value is 0. If z 1 happens to be equal to z 2, then the integral is 0; integral over this closed contour dz by z square equal to 0.

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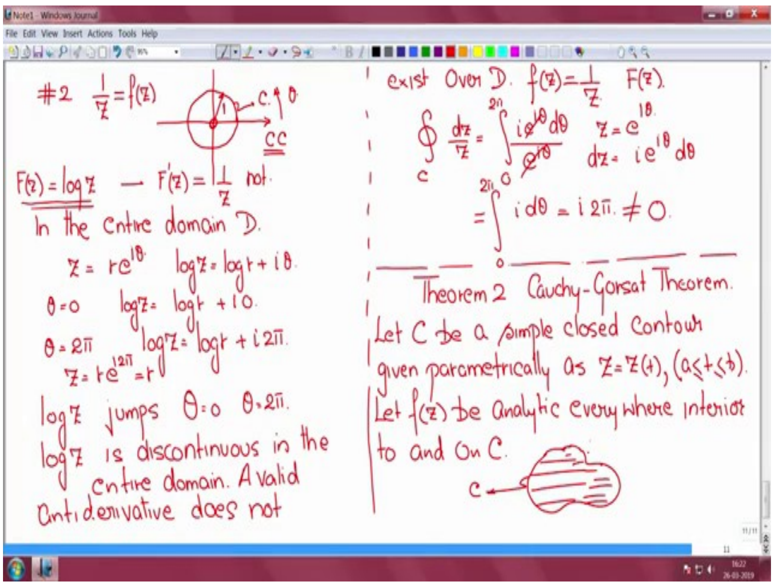

Next, so, we will see a contra example. The next example is the contra example ok. So, there is this function 1 over z and again we will consider this closed contour C with the radius of 1 ok. For this function also, the domain is the entire complex plane except the origin. The function is ill behaved only at the origin ok.

Now, supposedly it has an anti derivative which is log of z, ok. It has an anti derivative F of z equal to log of z, such that F dash of z is equal to 1 over z, ok. But not: this anti derivative is not in the entire domain D; we will see why not in the entire domain D, ok. Now, suppose we use this contour which we have and we start, since we are moving on a circle, z can be represented as r e to the power of i theta; r is the radius which is a constant. So, in this case 1 in this case it is 1; then log of z is equal to log of r, plus i theta, ok. So, we are examining the anti -derivative. How it is behaving we are looking at the behavior of the anti derivative, ok.

So, log of as we move on the circle of radius r or radius 1. So, log of z on this circle with radius r is log r, plus i theta ok and theta we start measuring from the real axis. We measure theta from the real axis and for us counterclockwise is positive always ok. In complex variables counterclockwise is always positive.

So, now, if I start at theta equal to 0; when I, when theta is equal to 0, then my log z, log of z is equal to log of r, plus i 0. I go full circle on this contour and I come back to the same point of z ok; the z has come back to the same point when theta is 2 pi. What is z? z is equal to r e to the power of i 2 pi and hence r. It is back to the same point. And, what is log z? log z now is equal to log of r, plus i times twice pi.

So, the function log z has taken a jump, as we go from theta equal to 0 to theta equal to 2 pi, which as far as z is concerned is the same point, ok. So, the function is discontinuous; log z now is discontinuous. If we consider the entire domain, in the entire domain, and therefore, a valid anti-derivative does not exist over D.

So, our f of z; f of z which is equal to 1 over z valid in this particular domain that excludes the origin does not have a valid anti derivative F of z and therefore, what happens if I integrate now over this closed contour C dz over z. And what is z? z is let us say the unit circle e to the power i theta parameterized in terms of theta. Then dz is equal to i e to the power i theta d theta. Then this ends up as integral over theta going from 0 to 2 pi, dz is i e to the power i theta d theta, by z is e to the power of i theta. So, this is integral 0 to 2 pi, i d theta, which is equal to i into twice pi, ok. So, an integral over a closed contour not equal to 0. Why? Because it does not have a valid anti derivative. So, we have explored theorem 1.

Now, the next theorem is a very important theorem. It is called; so, let us call it theorem 2 which is the Cauchy Gorsat theorem, ok. So, let me strictly go through the theorem and then we will see the implications. So, let us begin, let C be a simple closed contour SCC, given parametrically as z equal to z of t, where a less than equal to t less than equal to b ok; t is a real variable extending between a and b.

Next, let f of z be analytic: we should now know what analytic means, analytic everywhere interior to and on C. So, there is the function, there is the simple closed contour in the complex plane C and this function f of z is analytic on and inside this closed contour; that means, it has derivatives at every point.

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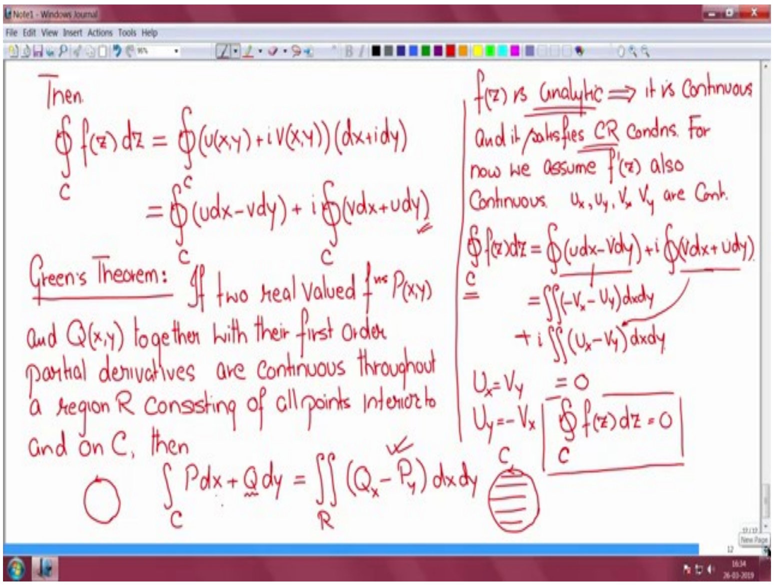

Then what? Then we have; then integral on this closed contour f of z dz is equal to integral, I will put the circle when we are doing a closed contour. So, this one should be familiar from your complex variables, ok. u of x comma y, plus i of v x comma y, then dz is dx plus i dy, that is now equal to: I will separate the real and imaginary parts.

So, we have integral, closed contour, C, u dx minus v dy, plus i, again integral over the closed contour C, v dx plus u dy, ok. So, you can see that when I put this circle, it is a closed contour, integration on a closed contour and I also put the direction which is counterclockwise; counterclockwise being considered the positive direction, ok. This familiarity I will assume you have from your earlier complex variables ok.

So, we have now reached a certain point, we have made a certain statement. Now, we need a certain result from calculus which is a Green's theorem which says that if; so I will write a little below, if two real valued functions, real valued, if two real valued functions P x comma y and Q x comma y, together with their first order partial derivatives are continuous throughout a region R consisting of all points interior to and on C, ok.

Then, integral P dx plus Q dy is equal to integral R so, C this is the line integral, this is the area integral; this is the line integral, this is the area integral Q partial derivative with respect to x minus P partial derivative this to y dx dy, ok. This is the result ok. So, you can see if you have P dx and Q dy, it is Q partial x minus P partial y: is a very useful result ok, the line integral done in the counterclockwise direction; so, this is the line integral going in the counterclockwise direction. Now, how do we use this result for us, ok.

So, we started out by saying that f of z is analytic for us; f of z is analytic for us which implies that it is continuous and it satisfies the CR conditions. Remember these are necessary conditions or CR equations, ok. Now. For now we have only said f of z is an analytic, but for now, we assume for now, for now we assume that f dash of z is also continuous. We do this because here P of x, y, Q of x, y along with their first partial derivatives are continuous.

So, we are assuming that f dash here is continuous for the moment. That means, since f of z is composed of u v, so, what we have is u partial x, u partial y, v partial x and v partial y are continuous. That is the meaning. Now, let us see. I have integral over the closed contour, counterclockwise C, f of z dz, and we found out the full form earlier which is integral u dx minus v dy; u dx minus v dy, plus i times, v dx plus u dy, it is right here, ok.

Now, this part is a real integral and what is it? u is a function of x comma y, v is a function of x comma y and their first partial derivatives are continuous. Same here, v is a function of x comma y, u is a function of x comma y, their first partial derivatives are continuous. So, the Green's theorem now applies.

So, what happens to this? This ends up as an area integral over the region that C encloses, ok. But now you see what happens we have Q x minus P y, ok. So, this is minus v x minus u y dxdy, plus i times, the area that this closed contour encloses, ok. Now, u x minus v y dxdy. This for this part, this for this part using this theorem. But f of z is analytic and therefore, it satisfies CR conditions and therefore, u x is equal to v y and u y is equal to minus v x.

And, therefore, now you can see what happens here, both these terms go to 0. So, that is equal to 0, ok. That means, my integral over the closed contour taken positively of f of z dz is equal to 0. So, very important theorem, ok. So, if a function is analytic on and inside a closed contour C, then the integral of the function, along that closed contour is identically equal to 0 ok. So, this is a very powerful theorem.

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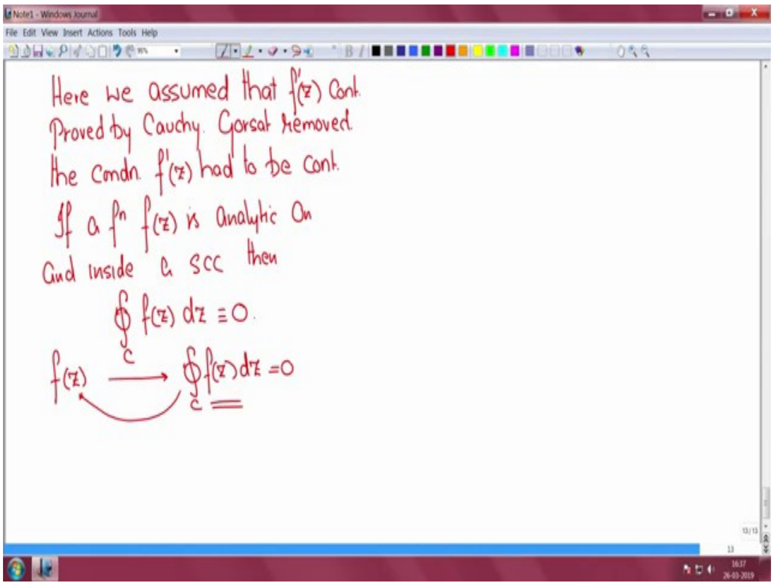

Now here, in here, we additionally assumed, we made an additional statement, assumed that f dash of z was continuous. So, up till this was supposedly proved by Cauchy making spelling errors, but then Gorsat came along and removed the condition that f dash z had to be continuous, ok.

So, now, we truly have: if a function f of z, the function f of z is analytic on and inside a simple closed contour, then this integral over the closed contour of f of z dz is identically equal to 0, ok. This is a very powerful theorem. So far it is a one way theorem; that means, f of z is analytic on an inside a closed contour then this integral is 0, ok. We have not said that if you find that this integral is 0, would it imply that f of z is analytic? No, we have not said that. So, it is a one way theorem ok.

So, again I leave you here. We will continue with these ideas in the next class.

Thank you.