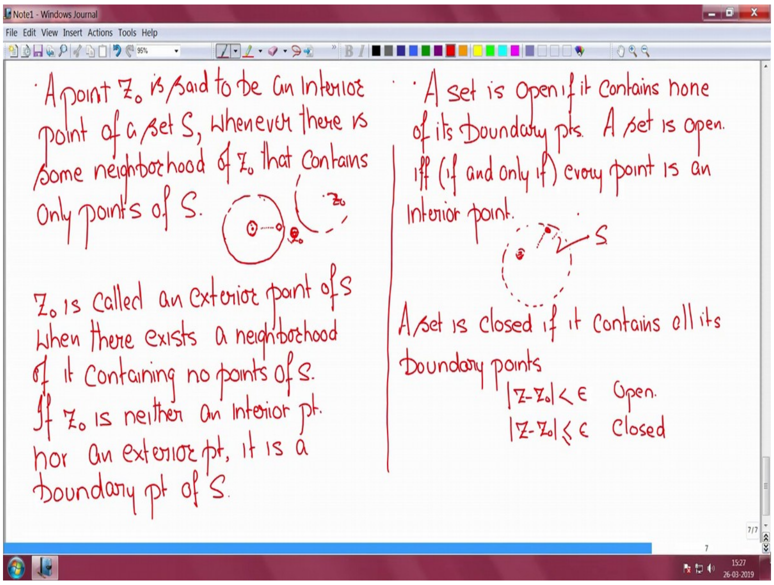
A short lecture series on Contour Integration in the Complex Plane

Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 05
Definition of sets, domains, theorem on antiderivative

Hello, good morning. Welcome to this next lecture on Complex Variables and Contour Integrations. In the last class, we left off while defining an open set, ok. So, we will continue over there. A set is open if it contains none of its boundary points. This is where we had left off last time, ok.

(Refer Slide Time: 00:44)

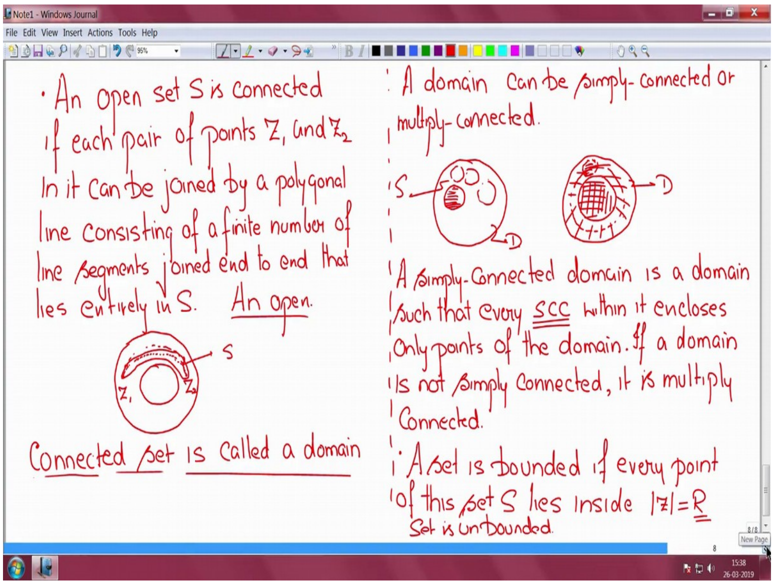


Another definition is: a set is open, iff: this is the double iff that is if and only if, if and only if and only if every point is an interior point, is an interior point, ok. So, if one wants to look at this. Suppose this is the boundary of S, and this is my set S. Now, every point is an interior point; that means, it has a neighborhood where all the points belong to S.

So, however, close I get close to the boundary I will find a smaller and smaller and smaller neighborhood of my point where all the points belong to S. So, that never ends, ok. I can move closer and closer and closer to the boundary and I will always find a smaller and smaller and smaller neighborhood of z 0 where all the points still belong to S. Such a set is called an open set, ok.

Now, a set is called a closed set, a set is called a closed set or a set is closed, a set is closed if it contains all its boundary points, ok. So, if I write magnitude of z minus z 0 less than epsilon, then this is an open set, ok, but if I write magnitude of z minus z 0 less than equal to epsilon, then that is a closed set, that is a closed set, ok.

(Refer Slide Time: 03:53)



Further on sets. An open set S is connected, ok, it is an open connected set, if each pair of points, each pair of points Z 1 and Z 2, if each pair of points Z 1 and Z 2 in it can be joined, can be joined by a polygonal line, by a polygonal line consisting of a finite number of line segments, line segments joined end to end, end to end, that lies, that lies entirely in S, ok.

So, what is this situation? We have this set over here, let us say I have this set, the set is between these two lines. It is the annular region. None of the boundary points are included an open set, ok. Now, I have a point Z 1 over here and I have point Z 2 over here, ok. Now this can be connected through a finites number of line segments joined end to end in so many ways, ok.

But the entire joined group of line segments: they still lie within the set S. So, this is a connected set, ok. So, next, an open connected, open connected set is called a domain. This is another important definition, ok. An open connected set is called a domain. We will be using the domain quite a quite frequently, ok.

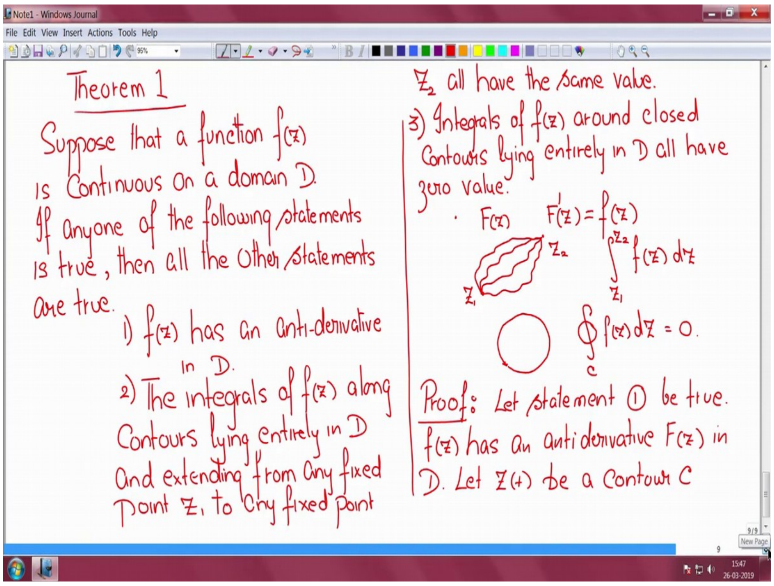
Further, the domain, a domain means an open connected set. A domain can be simply connected; can be simply connected, a simply connected domain or multiply connected, so you can have a simply connected domain. So, first of all it is a domain: it is an open, it is an open set. So, here let this be a set which is an open set: boundaries not included, ok. And this is a connected set. Now, here as we did before we took an annular region. This region between these two lines that is also an open connected set, but now, so both are domains. But one domain is simply connected the other domain is multiply connected, ok.

So, what is a simply connected domain? A simply connected domain, a simply connected domain, a simply connected domain, is a domain such that every simple closed contour, every simple closed contour, within it encloses only points of the domain, of the domain, ok. So, let us see here. Every simple closed contour in this set: I take a simple closed contour, I take another simple closed contour, I take another simple closed contour, any number I take, there are points inside, ok, and those points still belong to this domain D, they still belong to this domain D, ok.

So, such a domain is called a simply connected domain. Whereas, think of this if I consider a simple closed contour of this form then all the points inside belong to D. But, if I take this simple closed contour which is still going through all the points belonging to D, but that simple closed contour does not include you know all the points within do not belong to D. There is this region that is excluded from D, ok. So, such a domain is called multiply connected. So, the flip side is if a domain, if a domain is not simply connected, domain is not simply connected it is multiply connected, it is multiply connected, ok.

Then, last: a set is bounded, set consists of points in the complex plane or the set is bounded, if every point of this set S lies inside some circle, some circle, ok. What is that circle? Magnitude of z is equal to R, ok. Else, if this is not true. R is a finite number. If this is not true, the set is unbounded, set is unbounded, set is unbounded, ok. So, this more or less gives us a set of definitions with which we can work and move forward, ok.

(Refer Slide Time: 14:11)



Now, I said I was going to give a engineering sort of hand waving sort of treatment. For one should be familiar in a proper way with the theorems of complex variables, ok. So, I will go through some theorems in a sketchy manner, ok. So, we need these theorems. So, I will go through them in a semi rigorous manner. For a rigorous treatment you have to go through a proper course in complex variables, ok.

So, theorem 1, here is theorem 1. Suppose that a function, suppose that a function f of z is continuous, is continuous on a domain, here your domain comes: on a domain, D, either simply connected or multiply connected a domain D. Then, if any one of the, any one of the following statements, any one of the following statements is true, if any one of the following statements is true, then all the other statements are true, ok. So, we will be given a 2 or 3 statements, ok. When will any of the statements be true that will not be told to us; however, if any one of them you suddenly happen to find true with respect to a particular function then the other statements are also true, ok.

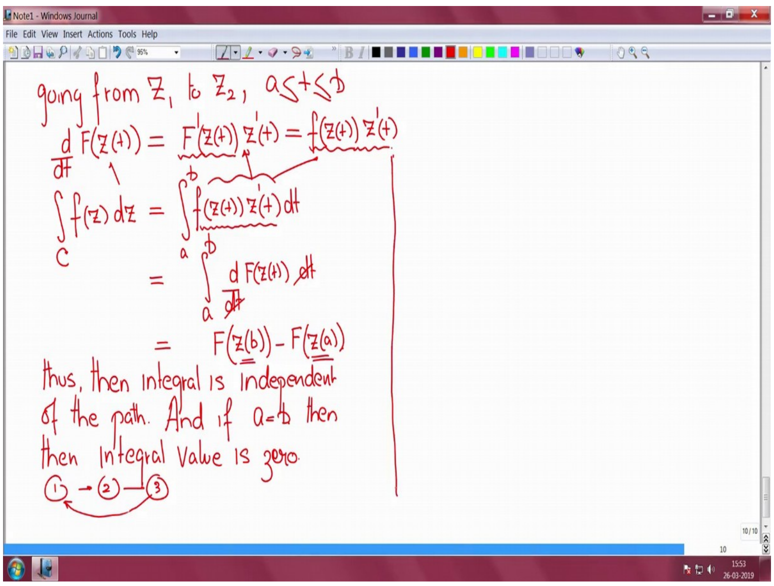
So, what are these statements? These statements are f of z has an anti-derivative, anti-derivative in D, f of z has an anti-derivative in D that is one statement, ok. The second statement is the integrals, the integrals, the integrals of f of z, of f of z, along contours, along contours lying entirely in D and extending, and extending from any fixed point, from any fixed point, any fixed point Z 1 to any to any fixed point Z 2, all have the same value, all have the same value, all have the same value, ok. This is second statement.

Then third statement; third statement is integrals of f of z, of f of z, around closed contours, around closed contours lying entirely in D, all have 0 value, all have 0 value, interesting set of statements, ok. One is f of z has an anti derivative which means there is a function F of z in that domain such that F dash of z is equal to f of z this is the meaning of anti-derivative. So, this there is an anti-derivative which is also equivalent to saying that if I have a point Z 1 and I have another point Z 2 and I integrate f of z along a contour and I do it along a different contour: on every contour I get the same value. Whatever contour I take between Z 1 and Z 2, this integral has the same value Z 1 to Z 2, f of z d z has the same value.

And lastly, in the domain if I start at some point and go around and come back to the same point, ok, I integrate over a closed contour, ok. This contour lies entirely in D, f of z is now 0, this value is now 0, ok. So, in the spirit of the course we will see some statements, proof of some statements and the others we will accept them as they are; so, the proof, so the proof of this, ok. So, how do we go about? Let 1 be true, let statement, let the statement 1 be true, let statement 1 be true, ok. What does that mean? f of z has an anti-derivative has an anti-derivative F, capital F of z, in D, in D, in the domain D, ok.

Further, we have to integrate around particular contours, so let Z of t parameterized in terms of t, be a contour, a contour, a simple contour C, ok. Let me change the page.

(Refer Slide Time: 23:52)



Simple contour C, going from Z 1 to Z 2 and t is I mean t extends between, t is a real number real variable extending between a and b, ok. So, now, what? Let us look at this d by dt of the anti-derivative, along this curve is equal to from the chain rule F dash with respect to the argument and the derivative z dash of t derivative of the argument, ok. Now, what is this? This is our function f of Z, ok, which has the anti-derivative. So, now, that is equal to f of z of t, Z dash of t. We had started off by saying that f of z has an anti-derivative, so this is anti-derivative, ok. F of z is the anti-derivative. So, the derivative is f of z of t.

Now, let us look at this integration over a simple contour between two points f of z dz, ok, in terms of t. Because t extends from a to b this looks like f of z of t z dash of t dt, ok. But this part, but this part, is the same as that part, ok. So, what happens now? I have integral a to b, d by dt of F of z of t dt, ok. This part here is equal to that, which is equal to this, which is equal to this, ok.

So, I have replaced this by d dt F of z of t dt. So, this goes away. So now, we have the integral value given by F of Z of b minus F of Z of a, ok. Thus, the integral is independent of the path, is independent, is independent of the path, ok. The path does not come into the picture. It is just a function difference of the endpoints, ok. There is no path mentioned over here. Why? Because it has an anti derivative, ok.

So, assuming first statement is true, the second statement turned out to be true, ok. And finally, and if the second statement is true and then we say if a is equal to b, a happens to be equal to b, then the integral is 0; obviously, the value is 0. Integral value, value of the integral is 0, which is statement 3, ok. Now, what we say here is we assumed 1 to be true, showed that 2 is true. Assuming or proving 2 to be true, 3 is true, ok, but 3 assuming 3 to be true 1 is true is a bit involved, ok. So, I will not go into it; well accept that, ok. So, that is our first theorem.

So, what we will do now is for today we close over here, and we will explore the implications of this theorem in the next class, ok. So, see you in next class.

Thank you.