A short lecture series on Contour Integration in the Complex Plane

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Lecture - 04
Simple Definitions

Hello good morning, welcome to this short series on Contour Integrations using complex variables ok.

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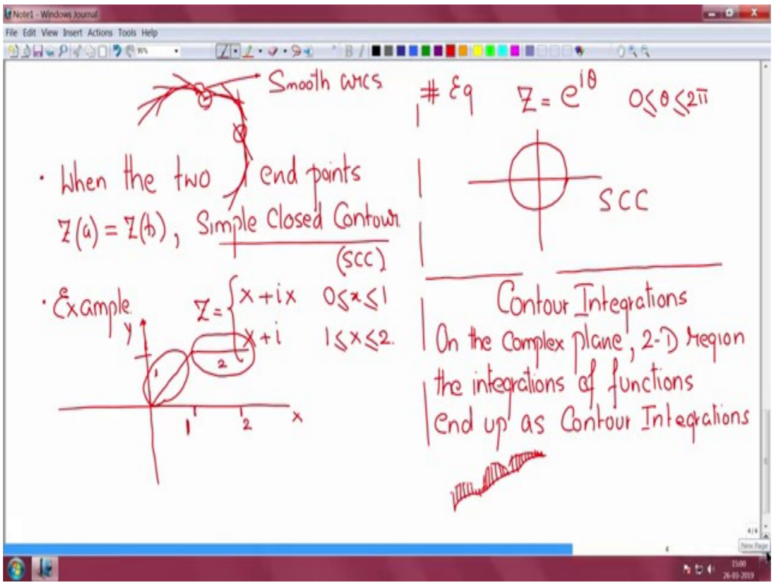

Last time we stopped while defining a few curves, arcs and contours. So, we will continue from there. I was giving an example of a simple contour. A simple contour was z equal to x plus i x and x plus i within these limits.

So, we had seen last time that the first branch is a straight line, the second branch is a horizontal straight line. So, each of these pieces 1 and 2 is a smooth arc. It is differentiable and they are joined over here to make a simple contour. So, here there will be a kink in z dash, the derivative of z will have a kink over here. So, this is a simple contour.

The next example the next example is again Z parameterized in this following way e to the power i theta for 0 less than theta less that 2 Pi. This is very simply a circle it is very simply a circle, ok. Now, the endpoints are equal. The z a and z b are equal. So this is a simple closed contour, simple closed contour, ok. It happens to be differentiable all around. It is no more bits and pieces addition of smooth arcs. A simple closed contour is differentiable everywhere. Now, we want to talk about integrations because that is our goal. We are going to do contour integrations, ok, contour integrations.

So, what about contour integrations? Since we are in a 2D region on the complex plane, on the complex plane, on the complex plane. it is a 2D or 2 dimensional region, 2D region, 2 dimensional region. The integrations, the integrations of functions end up as contour integrations. End up as, end up as contour integrations which means I have functions defined on lines. I have a line in the complex plane, ok.

And there is a function defined on this line that is sitting in the complex plane. Now the integration of this ends up as a contour integration ok. Hence, the value depends not only on the contour, but also on the function both of them get involved in telling us what is the value of the integral, so we will see that ok.

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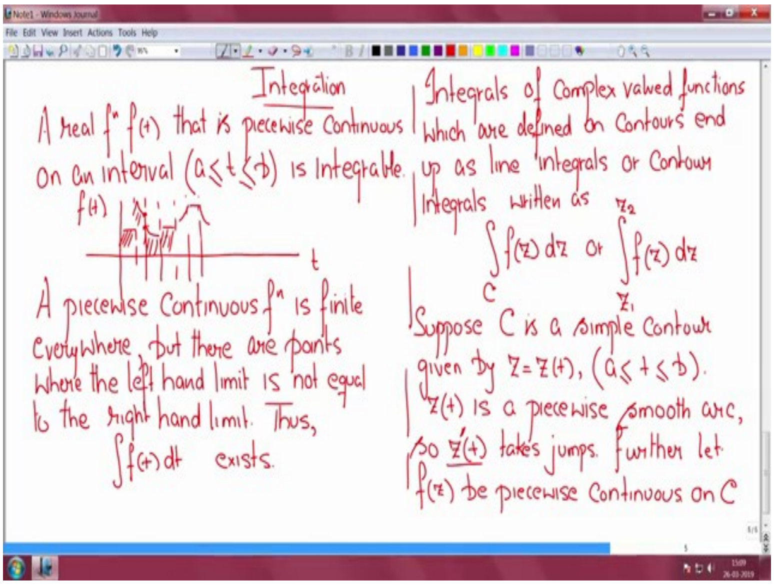

So, more about integration. A real function, a real function f of t that is, that is piecewise continuous on an interval, on an interval a less than equal to t less than equal to b is integrable, is integrable, ok. So, if I have the independent variable as t and I have this function which behaves like this. Say, this piecewise continuous. Nowhere does it blow up ok, but it takes short jumps, short finite jumps. The function takes more short finite jumps this is f of t. So, now, we can find the area under the integral and find the area under the curve. Nowhere is it going back to infinity that is what is piecewise continuous.

So, a piecewise continuous, a piecewise continuous function, piecewise continuous function is finite everywhere, is finite everywhere, is finite everywhere, but there are points, there are points where the left limit, where the left limit, left hand limit, left hand limit, left hand limit is not equal to, is not equal to the right hand limit, right hand limit, ok. So, somewhere here if I approach from let us say the right side, I have this value. I approach from the left side I have this value. So, you have these jumps. But the function itself is finite, it is bounded everywhere.

Thus, as we see from the picture, integral f of t dt exists, is a finite number, exists ok. Now, we will use this idea with respect to complex functions. So, integrals of complex valued functions, of complex valued functions which are defined, which are defined on contours, these functions are defined on contours end up as line integrals, as line integrals or contour integrals or contour integrals, ok. And they are written as, they are written as integral over some simple contour f of z dz or to be even more specific moving from one point z 1 to another point z 2 on some particular contour fz dz.

So, now, we will connect this idea of line integral or contour integral to the earlier idea of the real line integral, where the function was piecewise continuous. We will connect the two ideas. So, suppose now, suppose now C is a simple contour; C is a simple contour; C is a simple contour and it is given by let us say, it is given by z equal to z of t, where as before t has these limits ok. Now, z of t being a simple closed contour, z of t is a piecewise, piecewise smooth arc. We said that before, piecewise smooth arc.

And so, z dash of t takes jumps. It is not continuous, but takes short jumps, takes jumps. It is piecewise continuous ok. So that is the nature of z dash of t ok. Further, let f of z, let us see further, let f of z be piecewise continuous, be piecewise continuous on C, then let us see what happens, ok.

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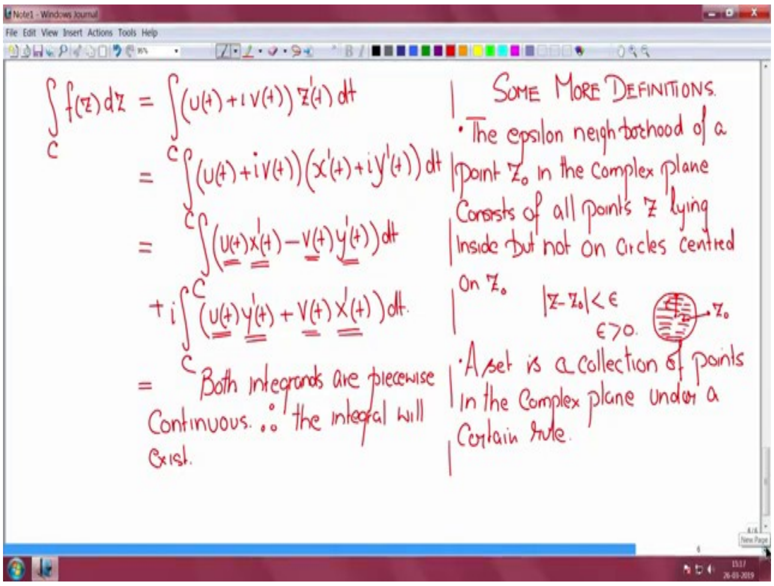

Then what happens is, we first write the integral on C, f of z dz, ok. Now, z is parameterized in terms of t, so we will write this further as U of t plus i V of t. Now dz is z dash of t dt, ok. So, what is the use? Use is this: I have integral C, U of t plus i v of t, z dash of t is again, x dash of t plus i V dash of t, dt, ok. Now, if we write it in terms of real and imaginary, ok, so I get this integral C, the real part will be U of t, x dash of t, minus v of t, what happened here, this will be y, sorry, this will be y, minus v of t, y dash of t, dt, plus the imaginary part, integral over C, U of t y dash of t, plus V of t x dash of t, dt.

Now, x dash t and y dash t are piecewise continuous. Why? because z of t is a simple contour composed of many smooth arcs. Same here, y dash t and x dash t are piecewise continuous. The function f of z itself was stated as piecewise continuous on this arc. So U f t and V of t are piecewise continuous.

So, the product of you know the real component and the imaginary component they themselves are both components are piecewise continuous. So, both integrals, both integrals are piecewise continuous, or both integrands, actually both integrands; both integrands are piecewise continuous. And therefore, the integrals will exist, ok. So, therefore, the integral will exist, will be finite, ok.

So, we will be dealing with such functions, such nice functions which are piecewise continuous, except occasionally, there will be singularities and because of singularities we will have to invoke some theorems and deal with them. Now, that is one set of definitions that are related to integrations, ok. Now we need more definitions, some more definitions, some more definitions.

What is an epsilon neighborhood of a point, ok? What is an epsilon neighborhood of a point? The epsilon neighborhood, the epsilon neighborhood; the epsilon neighborhood of a point z 0 in the complex plane, in the complex plane consists of all points z, consists of all points z, lying inside, lying inside, ok, but not on circles, but not on circles, centered on z 0, centered on z 0 and it is given mathematically by z minus z 0 magnitude less than epsilon.

So, if you imagine z 0 as the center and there is an epsilon radius circle around it ok, this radius is epsilon and this point is z 0. Now, the epsilon neighborhood of z 0 is all the points in here excepting the points that are on this circle. And epsilon is a positive number ok. Now, what is a set in a simple sense? A set is a collection of points; a collection of points; is a collection of points in the complex plane under a certain rule, under a certain rule, under certain rule, it is a rough definition.

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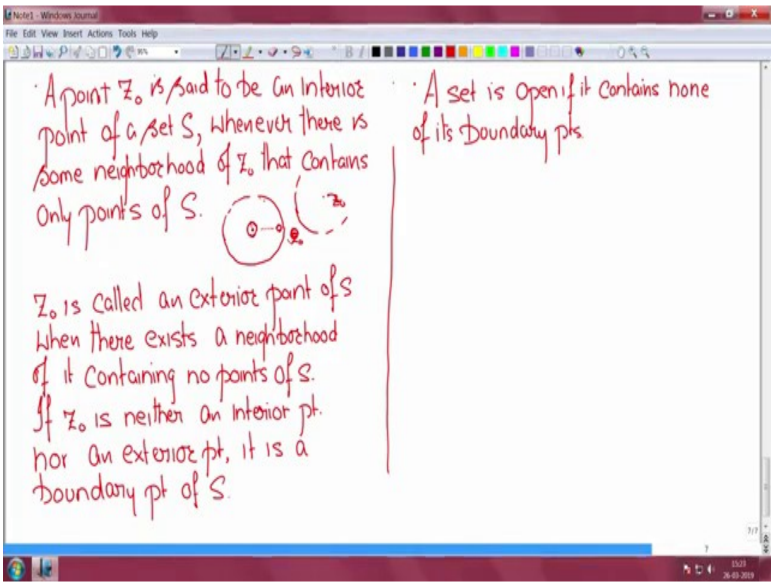

Next a point, a point, a point z 0; a point z 0 is said to be an interior point, an interior point, an interior point of a set S, of a set S, ok, whenever there is some neighborhood, there is some neighborhood; some neighborhood, there is some neighborhood of z 0, of z 0 that contains, that contains only points of S only points of S; of S, ok. So, there is some set, there is a set and z 0 is some point. It is an interior point how do we say, we take a neighborhood of 0; however, small it may be then all the points in that neighborhood are belonging to S.

So, you can move this points z 0 slowly towards the boundary, ok, now the neighborhood of z 0 may be shrinking, becoming smaller and smaller and smaller and smaller, but at any instant around z 0 there is a small neighborhood around it such that all the points belong to S, ok. So, that such a point is called an interior point. Now z 0, z 0 let me write it here; z 0 is called an exterior, an exterior point of S, ok, when there exists, when there exists, when there exists a neighborhood of it, a neighborhood of it containing no points of S, no points of S, ok.

So, now, we are outside, z 0 is outside. Let us say it is far off, then there is a big neighborhood. This is z 0, it is a big neighborhood. None of these points belong to S. Now, let us say that z 0 starts coming closer and closer and closer and closer, still; however close it comes, I will find a small neighborhood such that points within it still do not belong to S, so such a point is called an exterior point.

Now, if z 0 is neither, is neither an interior, an interior point, nor an exterior point; neither an interior point nor exterior point ok, then it is a boundary point. It is a boundary point of S. It does not satisfy either definition completely. It is a boundary point of S ok, so that must be obvious. Next, what is an open set, ok; what is an open set? A set, a set is open, set is open if it contains none of its boundary points. We will close this session over here. We will continue at this point in the next lecture.

Thank you.