A short lecture series on Contour Integration in the Complex Plane
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Lecture - 03
Analytic Functions

Hello, good morning. Welcome to this next lecture on Complex Variables.

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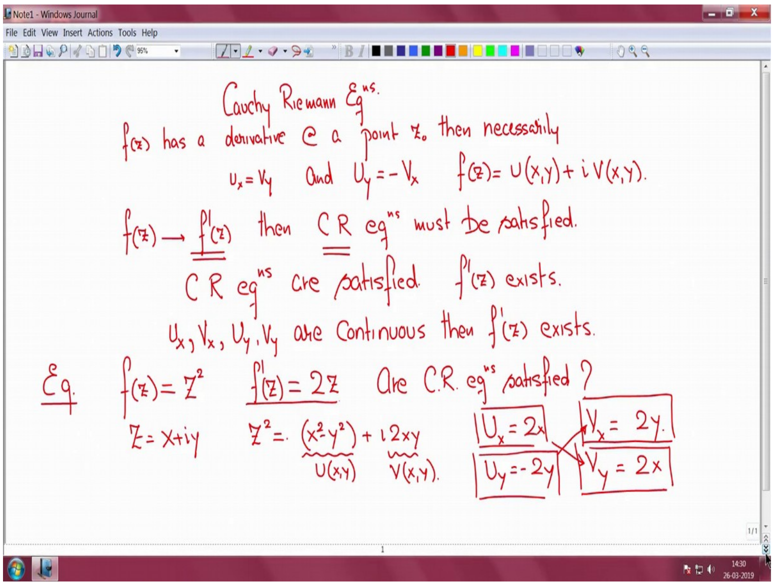

If you recall last time, we had stopped at the Cauchy Riemann equations. Which said, that if a function f of z has a derivative at a point Z 0, then necessarily it is true that U of x is equal to V of y and U of y is equal to minus V of x, where f of z was written as U of x comma y, plus i times V of x comma y, ok.

So, Cauchy Riemann equations or conditions are necessary conditions and necessary implies that if there is a derivative, if f of z has a derivative; that means, it has an f dash of z at a point, then the CR equations must be satisfied, must be satisfied. So, one way theorem, ok; if f dash exists then CR equations necessarily are satisfied. The other way is not true. If we find that CR equations are satisfied, are satisfied, it is not sufficient to say that f dash of z exists, ok. So, these are not sufficient conditions but these are necessary conditions.

Now, what makes a sufficient condition? In addition to CR equations, if the partial derivatives, in addition to the CR equations if the partial derivatives U x, V x, U y and V y are continuous then f dash of z exists. So, these are the sufficiency conditions. In addition to CR equations, partial derivatives U x, V x, U y, V y are continuous, then f dash z exists, ok. So, let us look at some examples. Let us look at some examples, ok.

So, we consider a nice function f of z is equal to Z squared, ok. It is a nice function. We know it has a derivative at every point in the complex plane, such that f dash of z is equal to 2 times Z. It is already known. So, it is analytic, it has a derivative. Then we check, are CR equations satisfied. Are CR equations satisfied. We check, ok. Now, Z is equal to x plus iy and therefore, Z square, is equal to x square minus y square, plus i 2 xy, ok.

Now, let us check the CR conditions. So, this function is U x comma y and this function is V, ok. So, let us do U of x. U of x is equal to twice x, then U of y is equal to minus twice y. Then V of x is equal to 2 y and V of y is equal to 2 x, ok. So, you see that U x is equal to V y, U x is equal to V y and U y is equal to minus V x. So, since this function has a derivative, necessarily the CR equations are satisfied, ok. Now, we take another example where it does not work, ok.

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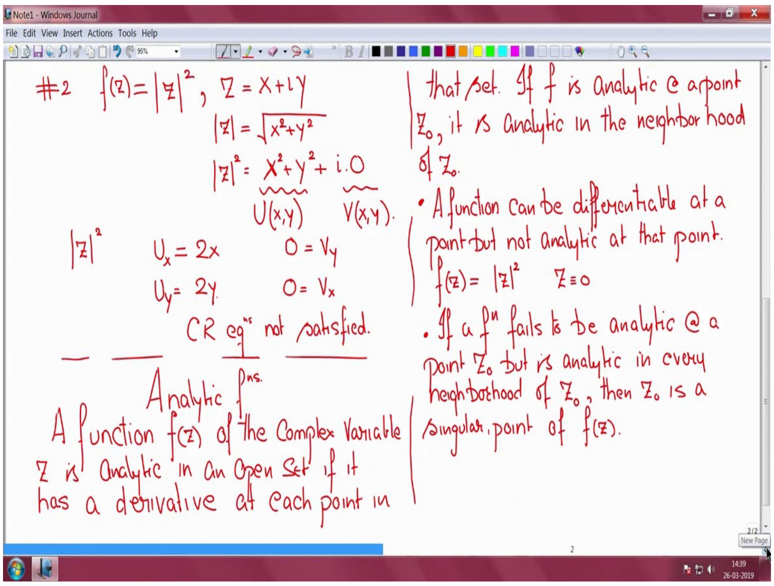

So, next example is, let f of z be equal to magnitude of Z square. Again, if we consider Z to be equal to x plus iy, then magnitude of Z; magnitude of Z is equal to square root of x square plus y square. And magnitude of Z squared is equal to just x squared plus y squared, and I will just add this plus i into 0. So, this part is U of x comma y and this part is V of x comma y.

We have already seen that magnitude of Z square is not a very well-behaved function, it is not analytic, it does not have a derivative, and so now, we will check if U x is equal to V y. Now, U x is here twice x and V the function itself is 0, so there is no question of V y. V y is 0. Similarly, U y is equal to twice y the other side V x is 0, ok. Therefore, there is no equation, there is no way this can be equal. So, CR equations are not as satisfied, CR equations are not satisfied, ok.

Now, we will move on to what are called analytic functions, ok. We will move to the next topic called analytic functions. A few definitions will come over here. So, I will partition the space this way, it seems convenient, analytic functions. So, here are a few definitions. So, there will be some writing to do, ok. A function f of z, f of z of the complex variable, of the complex variable z is analytic in an open set, in an open set, if it has a derivative at each point in that set. If f is analytic at a point z 0, at a point z 0, it is analytic in the neighborhood of z 0, ok.

So, here a few cases arise for our better understanding. So, one is a function can be differentiable; can be differentiable at a point, but not analytic, but not analytic, not analytic at that point, ok. We saw this case just earlier, this function f of z is equal to magnitude of z square, did not have a derivative anywhere except at z equal to 0, and nowhere else does it have a derivative. So, the fist function is differentiable at one single point z equal to 0, but not analytic at that point because in the neighborhood of z 0 it does not does not have derivatives, ok.

The other case arises, if a function fails to be analytic, fails to be analytic at a point z 0, but is analytic, but is analytic in every neighborhood of z 0. So, it is not well behaved at z 0 only, around it it is well behaved, then then z 0 is called a singular point, is a singular point of f of z, ok.

We will see such cases in the future when we solve problems, ok. So, that is as far as analytic function is concerned.

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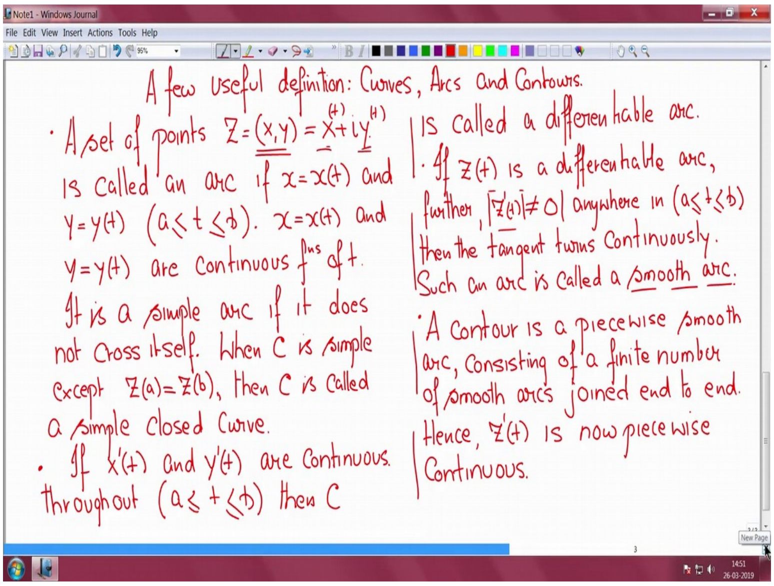

Now, we move to a few definitions which are required, ok. So, here, these are the definitions. So, let us call this a few useful definitions, useful definitions. What are they about? They are about curves, arcs and contours, ok. Now, let us see, what is in arc, ok. What is an arc? A set of points, a set of points, a set of points z is equal to x comma y, or this is also written as x plus iy. I have briefly introduced a new notation the complex variables that is written this way also.

So, a set of points x plus iy is called an arc, is called an arc, ok, if x is a function of this variable real variable t and similarly y is a function of the real variable t and t extends between a and b on the real axis, ok. So, such a set is called an arc, ok; t lies increasingly between a and b, x is a function of t, y is a function of t, then z which is x plus iy or x of t, plus iy of t, is called an arc, ok.

Now, x is equal to x of t and y is equal to y of t are continuous functions, must be, are continuous functions, continuous functions of t, ok, they cannot be discrete functions of t. x is a continuous function of t, y is a continuous function of t, then z is called an arc, ok. Now, this arc, it is a simple arc, it is a simple arc if it does not cross itself, does not cross itself, ok. However, the other case when C is simple is simple; that means, it does not cross itself, but yet, except that the end points meet, that z of a is equal to z of b, ok, then C is called a simple closed curve, a simple closed curve, then, the next definition, ok.

We have not said anything about how x dash and y dash, the derivatives of x and y with respect to t behave, ok. So, the next definition is if x dash of t; that means, the derivative of x with respect to t and y dash of t, are continuous, are continuous, the derivatives are also continuous throughout, as t varies between a and b, then C is called a differentiable arc, ok. We will take some examples a little later, so that this becomes clearer, ok.

Now, further if z of t is a differentiable arc as above, is a differentiable arc, differentiable, is a differentiable arc, and further, further this quantity: magnitude of z dash of t does not go to 0 anywhere, anywhere in t as t varies from a to b, this quantity never goes to 0, then the tangent x dash t plus iy dash t refers to the tangent. Then the tangent turns continuously, turns continuously, and such an arc is called smooth, ok, such an arc is called, such an arc, such an arc is called a smooth arc, a smooth arc, ok. Smooth arc will be very commonly you know will come across it very commonly in our course, ok.

So, z dash t magnitude going to 0 is a rare case, it is kind of a pathological degenerate case. So, it rarely happens. So, we will be dealing mostly with smooth arcs, ok. Further, ok, the next definition is a contour, ok. So, you will see how the smooth arc features in a contour. A contour is a piecewise smooth arc, piecewise, so most useful part of the set of definitions. A contour is a piecewise smooth arc, ok.

So, just to give you, ok, let me finish this, consisting of a finite number, of a finite number of smooth arcs, of smooth arcs joined end to end, joined end to end, ok. Then, what happens is, z dash of t, hence z dash of t becomes piecewise continuous, piecewise continuous, ok. So, we will take a look at this in the next page.

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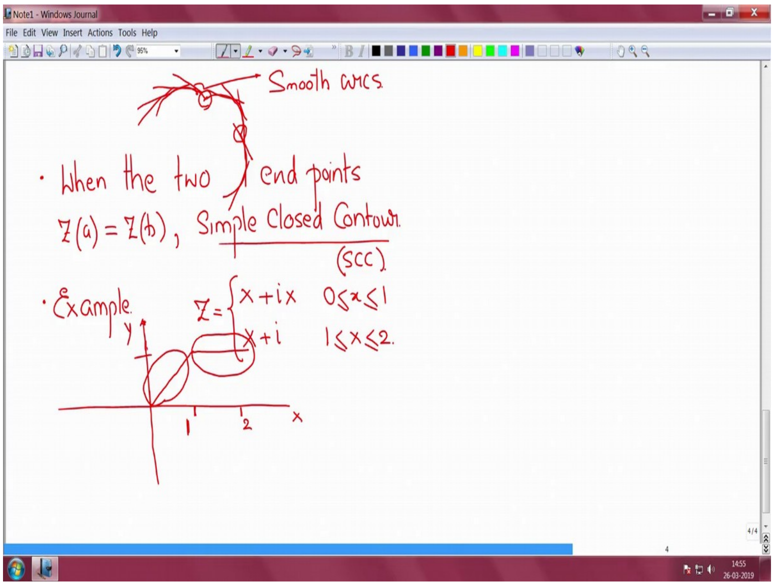

So, if I have a contour which is made up of smooth arcs joined end to end, ok. So, these portions are smooth arcs, these are smooth arcs, ok. So, that the tangent turns continuously, tangent turns continuously on them, here also the tangent turns continuously, the tangent turns continuously. Whereas, at these kinks, the derivatives z dash t will take jumps, ok.

So, z dash of t is now piecewise continuous, it is continuous on these smooth arcs, but at these kinks z dash t is piecewise continuous. It takes a jump, ok. Further, to that contour definition, when the two endpoints, when the two endpoints, the two endpoints, ok. It so happens that z of a is equal to z of b and the two endpoints meet then, we have what is called a simple closed contour, ok.

So, this is another contour we will come across, ok. You will come across this quite a bit and I will use this acronym, I will use SCC, will stand for Simple Closed Contour. I will not expand it all the time, ok. So, let us take now an example, let us take an example, ok. Suppose I have z, it is parameterized through x alone, ok, so I have x plus i x, when 0 less than equal to x less than equal to 1 and it is equal to x plus i, when one less than equal to x less than equal to 2, ok.

So, how does this thing look in the complex plane? I have 1 here, I have 2 here, this is real axis, this is the imaginary axis, ok. Now, z my curve is, I am sorry, why did I write 2 here. So, it looks like x plus ix, ok, this is x plus ix, so y is equal to x, ok.

And then it happens till 1, happens for values x between 0 and 1, and between 1 and 2, the imaginary part remains constant and we get this, ok. So, this is the example of a simple contour. So, this part is differentiable it is a smooth arc, this part is differentiable, it is a smooth arc and you have joined them end to end, it is a simple contour, ok.

So, we will continue with these ideas in the next lecture. I will close here.

Thank you.