A short lecture series on Contour Integration in the Complex Plane
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Lecture - 24
Additional material or corrections to lectures

Good morning to you all welcome to this lecture on Complex variables. Last class itself I had completed and closed the series; however, I felt that there are places in the entire series where I could have made a better statement or explained it a little better or made a more correct statement and so instead of going back there and editing those portions. I have made this lecture, where I will point you to that particular lecture and make the appropriate statement over here, ok.

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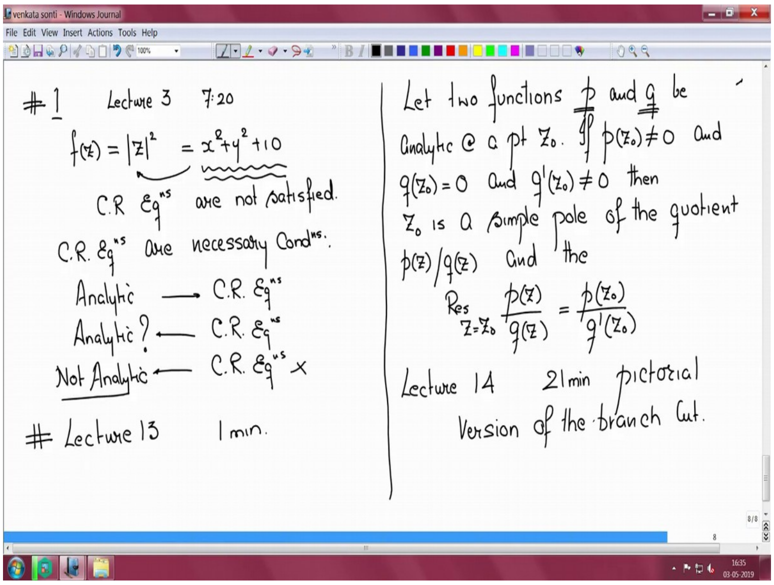

So, with regard to that, the first one I am going to say; this is in lecture 3 around 7 minutes 20 seconds, ok. I took an example of a function that was not analytic, f of z magnitude of z squared and that turns out to be x square plus y square, plus i 0 and we found that the Cauchy Riemann equations are not satisfied; are not satisfied. I just want to add that CR equations are necessary conditions. And therefore, if a function is analytic, then necessarily CR equations are satisfied.

However, if CR equations are satisfied we cannot say anything about the analyticity, ok. However, if CR equations are not satisfied, then the function is not analytic and so here you can see that if you took this function, the CR equations will not be satisfied and therefore, the function is not analytic. It was just a statement I wanted to add.

Now, next one is in lecture 13; lecture 13 around 1 minute; around 1 minute there is a theorem I stated; an alternative theorem in order to compute residues and there I made the statement that p and q were polynomials. In actuality its more general. So, let me state that theorem, let two functions; let two functions p and q be analytic; be analytic at a point; at a point z 0. If p of z 0 is not equal to 0 and q of z 0 is equal to 0 and q dash of z 0 is not equal to 0, then z 0 is a simple pole; is a simple pole, simple pole of the quotient p of z over q of z.

And here is the main thing the residue; the residue of this quotient the at z equal to z 0 is equal to numerator evaluated at z 0 derivative of denominator at z 0. It is very useful. Earlier I had stated p and q to be polynomials, it is more general than that, they could be any two analytic functions, ok.

Now, next is lecture 14; lecture 14 around 21 minutes, I had given a pictorial, I had given a pictorial version; pictorial version of the branch cut, ok. I was not very satisfied with the picture and explanation, so I would like to go over it. So, here it is branch cut.

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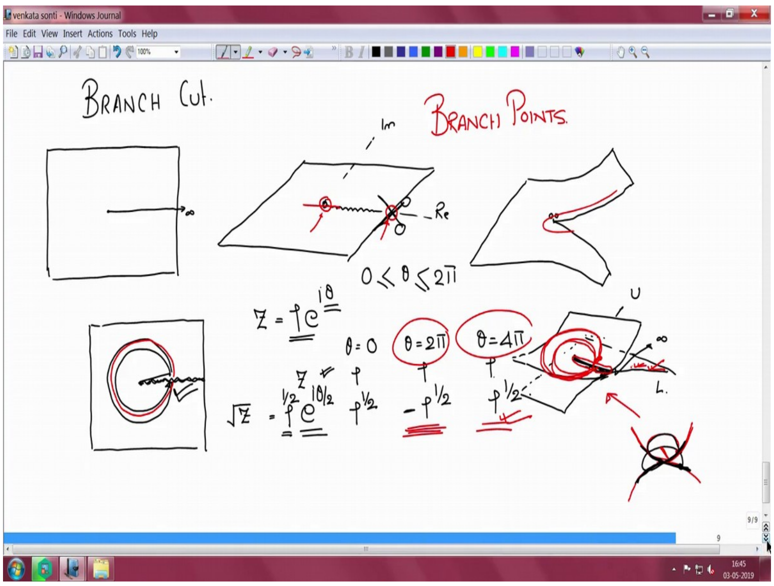

So, let us say originally this is your complex plane, this is your complex plane and let us say this is the origin and this is, by this time you get to infinity, that is infinity the x direction or the real axis, ok. Now, we take a 3D view of that; take a 3D view of that. This is my origin, that is my real axis, I am at infinity let us say here, that is my real axis and that is my imaginary. Now, I take a pair of scissors, ok; I take a pair of scissors, so that is my scissor and I cut it, I cut it from infinity upto the origin I cut it and how does it look? It looks like this; looks like this. This is still the origin two portions are separated out.

Now, I put them back together, the cut remains I put them back together; so I put them back together, here let us say or let me do this. So, now, I have let us see here, so there is a cut now, it has been cut and joined, but there is still a cut. Now, if I represent my z as rho e to the power of i theta and theta goes from 0 to twice pi, ok; so theta goes from 0 to twice pi, ok. Now, when theta is 0, theta equal to 0, then I have z equal to rho, z equal to rho. So, I start at some rho and theta equals 0, z equal to rho. When theta is equal to twice pi, theta is equal to twice pi, again my z is equal to rho, ok.

So, I have gone around once; I have gone around once, ok. Now, if I go around once again; if I go around once again, my theta goes to twice 4 pi, theta goes to 4 pi and again my z is back to rho, ok. So, z is reaching the same value at the same place, so that is analytic.

But if I take root of z, square root of z, which is equal to rho to the power half, e to the power of I theta by 2, rho to the power half, being a positive real number. And when theta is 0, I have rho to the power half, when theta is twice pi I have, twice pi by 2, e to the power I pi, so I get a switch in sign minus rho to the power half. So, the function has taken a jump over here, ok.

But again when I go one more round in z, at theta equal to 4 pi, then I am back to here 4 pi by 2, which is twice pi, so I get again rho to the power half, ok. So, its as though the complex plane has been divided into two separate planes, ok; it has been divided into two separate planes. How do we visualize that? So, let us say this is my lower plane; let us see, this is my lower plane and this is my upper plane. They have to be connected. This is my lower plane, that is my upper plane and this point is infinity and let this point be the origin, ok.

So, that is the lower sheet which is in the dotted line and it goes and meets the lower sheet here, meets the lower sheet here, that is the lower sheet. So, this is the upper sheet, this is the lower sheet and now here is the cut, from origin to infinity is the cut, ok. So, when I start and I start over here, theta equal to 0, theta equal to twice pi, I go theta equal to 0 and theta equal to twice pi by that time my function has switched sign. So, now, there is a hole here remember, this cut; this cut we made, there is a hole here, now through the hole when I reach minus rho to the power half, I inter enter the lower plane, I enter the lower plane.

And then I go around, now theta comes back to four pi, at 4 pi I am back to rho to the power half,, I enter the upper plane. So, I go around once theta 0 to theta twice pi, but my functional value is switch sine. So, I enter the cut that I made, get in I am in the lower plane. Now, theta is 2 pi to 4 pi, I go in the lower plane, by the time I come back to 4 pi I am back to rho to the power half, which is here on the upper plane, ok.

So, if I look at this from this angle, this side, it looks like this; this is the upper sheet and this is the lower sheet. So, I start; so this line is here, origin to infinity. I start over here go around, let me change. So, I start over here, I go around and I am ready to enter the lower plane, I enter the lower plane, this is a lower plane. I go around on the lower plane, I come over here, I am ready to enter the upper plane.

So, here to here is lower plane entry; here to here is upper plane entry, ok. So, that is how the branch cut works and it is a branch cut is always between two points, in this case its between 0 and infinity and they are called branch points; branch points, ok. In case of the finite cut it, was easy easier to see the cut would be between two finite points, ok. But here the cut is between 0 and infinity. So, that is what I wanted to say about branch cut ok.

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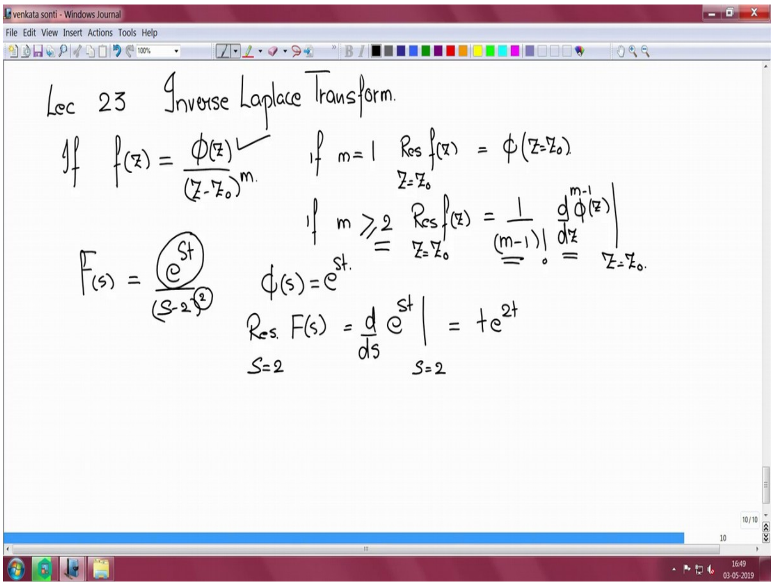

Now, one more point I wanted to make that is in lecture 22; lecture 22, lecture 22, lecture 22, actually would be lecture 23, my apologies it is 23. In lecture 23, where I did the inverse Laplace transform; inverse Laplace transform using contour integration.

You might need one theorem when you have multiplicity of poles. So, that theorem is this; if f of z is equal to phi of z divided by z minus z 0 to the power m ok, where phi of z is well behaved at z equal to z 0, ok. Then if m is equal to 1, then the residue of f of z at z equal to z 0 is simply phi at z equal to z 0, ok. However, if m is greater than or equal to 2, then the residue of f of z at z equal to z 0 is given by, 1 over m minus 1 factorial, d by d z, to the power m minus 1; m minus 1 derivatives of phi evaluated at z equal to z 0.

So, we will write it; so evaluated at phi evaluated at z equal to z 0, ok. So, in our example, Laplace transform example, my F of s was given by e to the power st, by s minus 2 whole squared, ok. So, phi of z in this case is e to power st, ok, so phi of s is equal to e to the power st and so the residue of F of s at s equal to 2 is given by, based on this, multiplicity 2, number of derivatives is 1, 2 minus 1 factorial. So, that is equal to d by d s once, e to the power st, evaluated at s equal to 2, ok, so that gives me t e to the power 2 t. So, that is about all the additional statements or corrections I wanted to make.

Thank you very much.