A short lecture series on Contour Integration in the Complex Plane
Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 23
Inverse Laplace Transform

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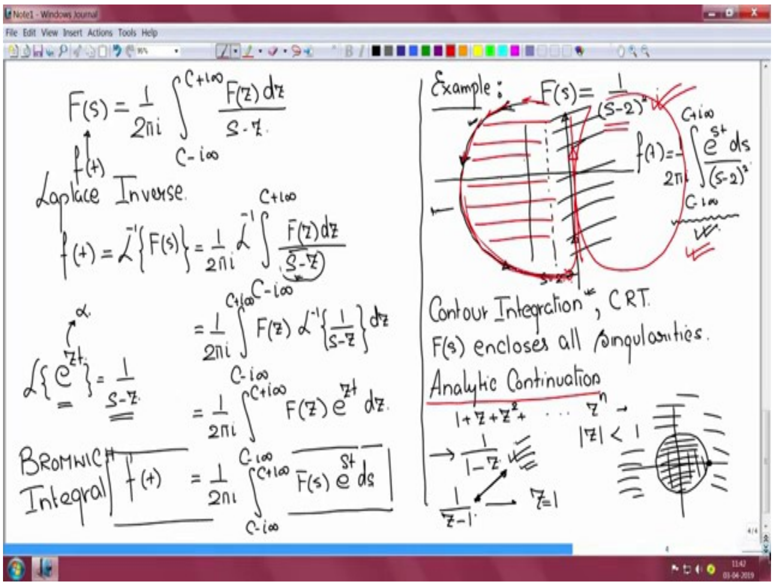

Good morning, welcome to this lecture on complex variables, the last topic we were considering was inverse Laplace transform and inverse Laplace transform also requires contour integration. So, we were doing an example, we had presented the theory and the example was F of s is 1 over s minus 2 whole square and this is the inverse Laplace transform formula the Bromwich integral, ok.

In order to do that we were we are supposed to go in this vertical, on this vertical segment and we closed the region using a circular arc with infinite radius. And we were now looking at this function on this side of the plane and we had spoken about analytic continuation; how it is possible. So, it is the same function that is well behaved, analytic over here, that exists over here also, except at its singularities, except where it is singular in the same function, ok.

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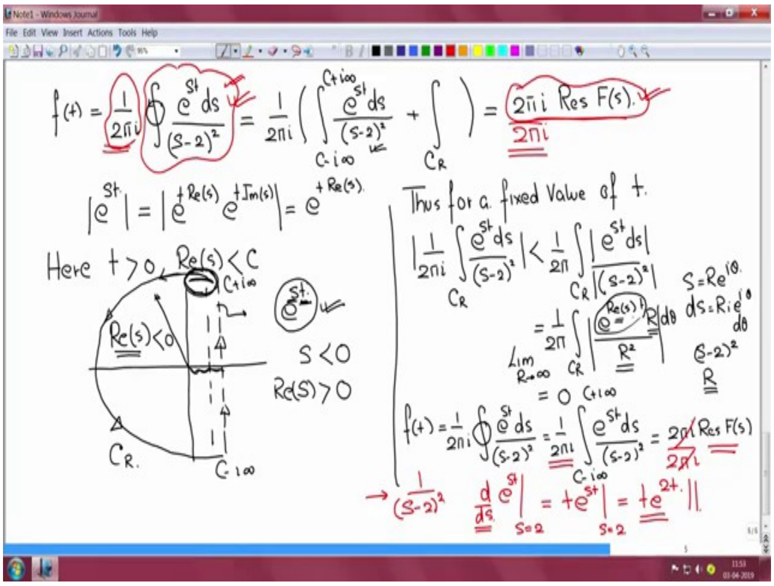

So, now let us see how this contour integration is done, f of t now, f of t is equal to 1 over twice pi i , integral, again counterclockwise, e to the power st ds, over s minus 2 whole squared, which is equal to 1 over twice pi i ,the vertical portion C minus i infinity to C plus i infinity, e to the power st ds, over s minus 2 whole squared, plus the arc, plus integral over the arc, circular arc, ok, that is now equal to twice pi i times the residues of F of s, ok.

Now, the arc portion has e to the power st in it, st in it, e to the power st, which is so, if you look at the magnitude of this, ok, then we have magnitude of e to the power t real part of s, e to the power t imaginary part of s, which is equal to e to the power of t real part of s, t is positive and the real part of s is strictly less than C, ok.

So, now, if we look at the contour we have; we have these singularities, we have the arc over which we are going, C minus i infinity, C plus i infinity and we close it with a CR and here we have e to the power st, ok. Thus, for a fixed value, fixed value of t, the range where s real part of s is positive is a very short range and t is a finite time, ok.

So, here the exponent does not have the capacity to overwhelm functions, this is a finite value, where s is negative there is no problem ok; however, where s still has some positive value or the real part of s has some positive value we have to worry about it, but that real part of s is less than C ok; that real part of s is a small region and t is a finite quantity. So, this is not unbounded that is the idea in this small region, if we cross the imaginary axis here real part of s is negative and it is not a problem at all, ok.

So, thus for a fixed value of t, I have 1 over twice pi i ,integral over the arc, e to the power of st ds, by s minus 2 whole squared, is less than 1 over twice pi, integral over CR, the magnitude of e to the power of st ds, over s minus 2 whole squared, is equal to 1 over twice pi integral over CR e to the power real part of st R.

Because s is equal to R e to the power of i theta and ds is equal to R i e to the power of i theta d theta, by s minus 2 whole squared over R, which is going to infinity is dominantly R square. And as I said for most of the region real part of s is negative, here real part of s is bounded for a finite time, ok. So, this is not a very large number; so, the denominator R will overwhelm it, ok.

So, this now goes to 0; this now goes to 0 as R tends to infinity, limit R tends tending to infinity. So, the integral over this arc is 0. So, thus f of t is equal to 1 over twice pi i, the integral counter clockwise, e to the power st ds, over s minus 2 whole squared is equal to 1 over twice pi i, integral C minus i infinity to C plus i infinity, e to the power of st ds by s minus 2 whole squared, which is equal to twice pi i times the residue of F of s. Now, here there is a slight tricky part the contour integral itself gives me a this contour integral.

This part of let us see, this contour integral itself gives me a residue on the right hand side which is twice pi i times residue of F of s. So, there is a slight error here. We should bring in the 1 over twice pi i perhaps later. So, this contour integral itself gives me a twice pi i times residue of F of s. So, this 1 over twice pi i actually takes out this twice pi i, that is correct, ok. So, here I should put it twice pi i in the denominator because of this ok. So, this closed contour integral itself gives me a residue which is twice pi i times this.

So, when I have this additional twice pi i there should be this twice pi i in the denominator, ok. So, that twice pi i, so this goes away, so I have residue of F of s ok. Now, those who know, there is a double pole at s equal to 2, ok. So, the residue is d by d s of e to the power st evaluated at s equal to 2, that gives me t times e to the power st at s equal to 2, I have to use the rule for residues when there is a double pole, ok. So, the derivative comes in and that answer is t e to the power 2 t. So, that is the inverse Laplace transform.

With this I close the short lecture series on contour integration using complex variables. The whole attempt was not to get into the depth of complex variables and theorems, but to present the usefulness of contour integration and branch cuts, that part which typically is not emphasized on complex variables courses and which is very useful to the engineers. I hope you find this very useful.

Thank you.