A short lecture series on Contour Integration in the Complex Plane

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Lecture - 21
L shaped branch cut continued

Hello good morning, welcome to this lecture on Complex variables. If you recall last time, I had started talking about an L shaped branch cut.

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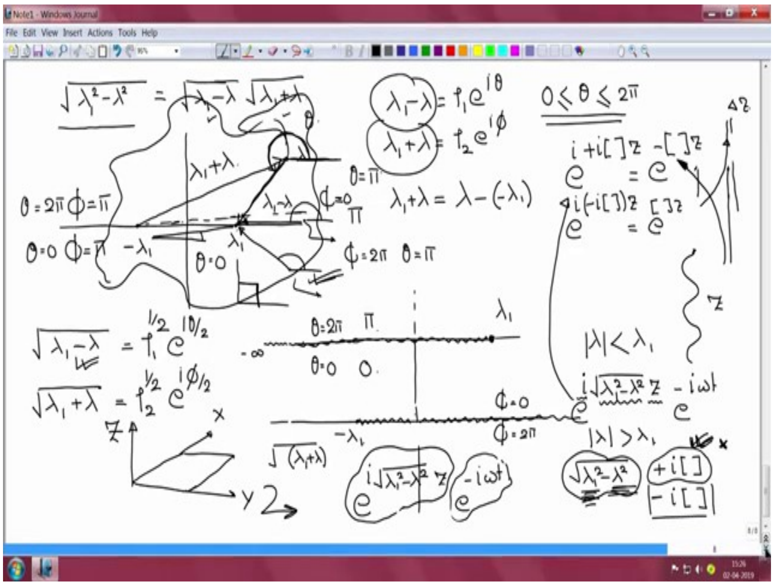

Continuing on that here, we have the square root function given by square root of lambda 1 square minus lambda square which breaks up into two pieces, ok. So, lambda 1 minus lambda is this phasor. Its starting fixed at lambda and now the theta gets defined in this manner. You draw a horizontal line from lambda and the angle it makes with the phasor from the horizontal is theta.

So, if I am here, if my lambda is over here, theta is pi. If lambda is over here, theta is less than pi, here theta is pi by 2 and it begins with theta 0. So, those are the limits on theta. Next lambda plus lambda 1, ok. So, lambda plus lambda 1 plus lambda can be written as lambda minus minus lambda 1, ok. So, then I have the base sitting at minus lambda 1 and I join lambda from here. So, that is my lambda 1 plus lambda.

This is very simple now. It begins here ok. Its argument is 0 over here. phi is 0 over here. Then, phi is pi over here. phi is equal to pi over here and then, below phi is equal to twice pi. With respect to theta, theta is equal to 0 over here and theta equal to pi over here. Theta is equal to pi over here and theta is equal to twice pi over here, ok.

Now, if we just look at square root of lambda 1 minus lambda which is equal to rho 1 to the power half e to the power of i theta by 2, ok. Let us say this is the origin and this is lambda 1 over here. Because the total argument theta goes from theta equal to 0 here to theta equal to twice pi over here. Half the argument goes from 0 to pi. e to the power i theta by 2 goes from 0 to pi and so one can see that with respect to lambda 1 minus lambda, this entire line is a branch cut, ok.

This entire line is a branch cut going off to minus infinity. With respect to square root of lambda 1 plus lambda which is rho 2 to the power half e to the power of i phi by 2, ok. I extend this down and let us say minus lambda 1 is here, ok. Here the theta phi argument is 0, above the line. Below, the phi argument is equal to twice pi and therefore, this entire line happens to be the branch cut for lambda 1 plus lambda square root, ok.

Now, without so much description of the problem, let me just say that this problem or this mathematics arises from radiation of a plate into a half space, ok. Let us say the plate lies in the X and Y plane and this happens to be the Z direction and the radiation, the function that defines radiation in the z direction or the wave vector that defines the wave expression that defines the radiation is given by e to the power of i square root of lambda 1 square minus lambda square, z e to the power of minus i omega t.

This is a temporal phasor, says we are running system at a single frequency, we are driving the system at a single frequency and this is the spatial phasor, ok. Now as long as long as lambda is less than lambda 1, ok. Lambda now takes complex values, let us say magnitude of lambda is less than lambda 1, then I have, square root of lambda 1 square minus lambda square is real and an i in front and z with an exponent minus i omega t, ok, says this is a propagating wave in the z direction, the wave propagating in the z direction. This value is a real number, I have an i in front, z is the coordinate.

Now, you can see that lambda is taking values all over the complex plane and it can go to infinity, real values complex values. So, there is a chance that lambda will exceed lambda 1, in which case square root of lambda 1 square minus lambda square, let us say becomes purely imaginary and it can be purely imaginary of the form i times a real number positive real number or it can be imaginary minus i times a positive real number. Now let us look at this here, where we have some room, if I take the positive i value.

So, I have e to the power of i and square root of lambda 1 square minus lambda square is positive i some real number, into z and this becomes equal to e to the power of minus a real number z, which is decaying in the z direction and that is my z direction, the sound field is decaying in the z direction. On the other hand, if I choose whenever lambda exceeds lambda 1 square, if I happen to choose negative imaginary value; watch what happens. e i negative imaginary a real number into z, then this is a e positive real number z.

This implies that along the z direction sound field is blowing of to infinity. We cannot allow that because physically that does not happen, ok. As we go further and further into the z direction, the sound field keeps on growing up, this does not happen in our experience. So, we cannot have, when lambda goes all over the complex plane. Lambda is going to go all over, wherever it is possible, it is going to go, ok. In the complex plane, we will take closed contours and drive it all over the place.

If it so happens that lambda square exceeds lambda 1 square. Therefore, we have to ensure that automatically the positive imaginary is chosen. We have to give definitions to lambda 1 minus lambda and lambda 1 plus lambda such that automatically the positive imaginary value is chosen. Otherwise we will have a non-physical answer, ok.

So, far we have come with the help of mathematics, we have come this far with the help of mathematics. Now, physics tells us something different, ok. So, now, we have to make adjustments such that nowhere when lambda exceeds lambda 1 square, we get a negative imaginary value. So, we have to modify our definitions. So, how do we modify that? The way to modify is this.

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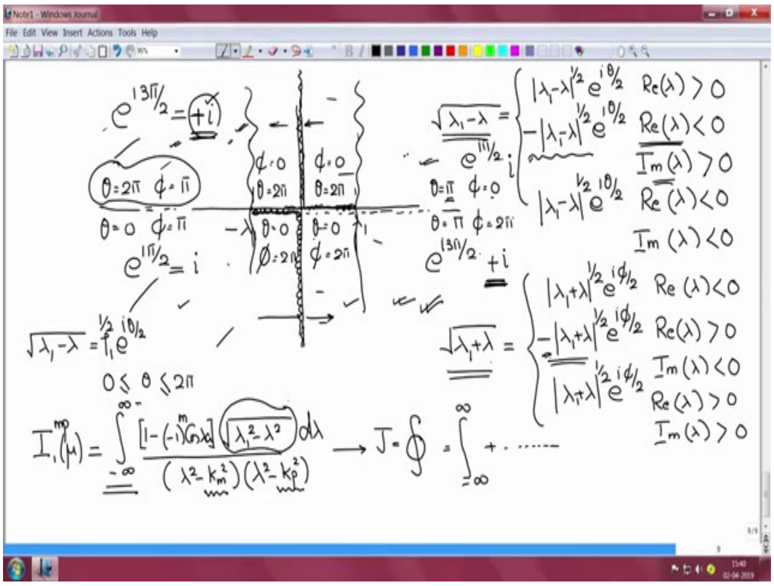

This is my lambda 1 and this is my minus lambda 1. Let me take it a little lower sorry. Let me take this a little lower. I will take it to the center, ok. So, here this is lambda 1 and here is minus lambda 1. So, for square root of lambda 1 minus lambda, we have this branch cut. It goes off to infinity and keeping it light now.

What now I do, is that I bend this branch cut into an L shape over here. I bend this branch cut. So, I this is my cut on the real axis. Then, my cut shifts 90 degrees that is a cut on the vertical axis, ok. Why is it so because this is what I do for square root of lambda 1 minus lambda, I change the definitions now to satisfy the non blowing-up condition in the z direction.

So, it is lambda 1 minus lambda magnitude to the power half, e to the power of i theta by 2, whenever real part of lambda is greater than 0; that means, here. Here I donot want to clutter the page here, ok. Then, it is equal to minus lambda 1 minus lambda to the power half e to the power i theta by 2, whenever my real value of lambda is less than 0 and imaginary value of lambda is greater than 0; that means, in this quadrant. That is why the cut. So, from here to here we have a jump that is why the cut, the sign change has occurred, ok. Further, for lambda 1 minus lambda to the power half, e to the power of i theta by 2, whenever real part of lambda is less than 0 and imaginary part of lambda is less than 0 in this quadrant, ok.

Now, what do we do with Square root of lambda 1 plus lambda, that cut is going from minus lambda 1 all the way to positive infinity, ok; all the way to positive infinity, we saw it last time. Now we bring it till the origin, we bend it 90 degrees. So, this take cut goes downwards now, ok. It goes down to infinity and hence the name L shaped cut, ok. So, now, what is the definition? For this, I need some room; I need some room. So, I have square root of lambda 1 plus lambda, is equal to lambda 1 plus lambda to the power half, e to the power of i phi by 2, whenever real part of lambda is less than 0, ok. So, in this part; in this part; in this part of the complex plane.

Next, it is minus lambda 1 plus lambda to the power of half, e to the power of i phi by 2 whenever the real part of lambda is positive and imaginary part of lambda is negative right here. So, as we move from here to here, there is a sign change and hence the cut then. The last bit is lambda 1 plus lambda to the power half, e to the power of i phi by 2 whenever real part of lambda is greater than 0 and imaginary part of lambda greater than 0, ok. Now watch what happens here. Earlier, Earlier when we started off with lambda 1 minus lambda as rho 1 to the power half, e to the power of i theta by 2, with theta limits twice pi; what happened at this point?

Here, my theta was equal to pi, ok. Here my theta was equal to twice pi. Here my theta was equal to twice pi and here my theta was equal to twice pi. Here, theta was equal to 0 ok; here, theta was equal to 0; here, theta was equal to 0 and here, theta was equal to pi. With respect to phi, here phi was equal to 0, ok; phi was equal to 0, phi was equal to 0; phi was equal to pi. Here, phi is equal to pi; here, phi is equal to 2 pi, phi is equal to 2 pi; phi is equal to 2 pi, ok.

So, here when lambda is greater than lambda 1; that means, we are here. This is the lambda 1 line, when we are here or lambda 1 is greater than magnitude lambda 1; lambda is greater than magnitude over here, let us see what happens. Here I have e pi plus 0 e to the power of i pi by 2 and here I have e to the power i 3 pi by 2. But if you see what we have done is we have brought in a negative sign to match. So, this is going to be equal to i and this is going to be equal to minus i and therefore, in this region lambda 1 plus lambda has a negative sign for real lambda greater than 0, imaginary lambda less than 0.

So, this sign gets corrected. So, whenever we have an imaginary argument, it is positive ok. So, this new definition on lambda 1 plus lambda in this quadrant corrects and makes this positive imaginary. Similarly here also from below we have e to the power of i pi by 2 and therefore, it is i positive imaginary. Whereas, here in this quadrant, we have e to the power i 3 pi by 2 which is equal to minus i and therefore, we modify the definition of lambda 1 minus lambda. So, that when we are real part is less than 0 that is here and imaginary is positive that is here, I have a minus sign.

So, this minus sign will correct and make it i. So, this cut this jump from here to here. We bring in a minus sign with respect to lambda 1 minus lambda and it corrects for the negative sign on the imaginary value. So, now, whenever my lambda exceeds lambda 1 on this side or minus lambda 1 on this side, the imaginary value is everywhere positive ok. So, this is the L shaped cut, ok.

Now, we still have the integral that we have to attend to. We still have this integral to attend to minus infinity to infinity, 1 minus minus 1 to the power m cos lambda a, square root of let me say lambda 1 square minus lambda square d lambda, by lambda square minus k m square, into lambda square minus k p square.

This is integral we have to do. As before we are going to replace with a J over a certain contour integral, where this real line minus infinity to infinity will be part of it. But there will be other pieces that come and now it is going to be a bit complicated because on the real line I have k m square, I have k p square. These are singularities sitting on the real line and I have this square root L shaped cut. ok. So, just to give you an idea how that integral could happen.

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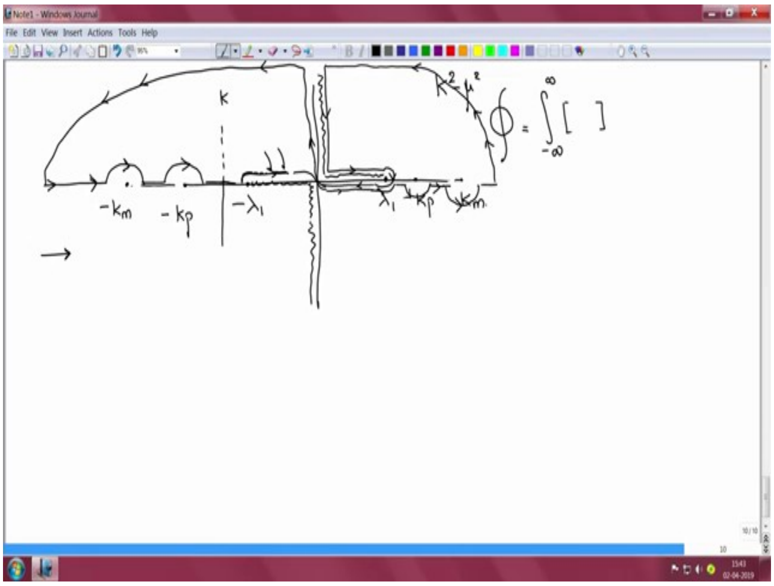

So, we have let us say lambda 1 over here minus lambda 1 over here. Let us say we have k square greater than mu square, ok. So, that is why we have it here, then lets say k is over here, k value is somewhere here. Then, perhaps k p is a pole here and k p is symmetrically or minus k p is a pole here and k p is here and may be k m is here, minus k m is here and plus k m is here, ok. Then it is important that I have minus infinity to infinity as part of the contour. This is our basic theme.

So, let us say I come for lambda from minus infinity. Here I have a pole. So, I circumvent the pole with an epsilon contour, I move forward, I circumvent the minus k p and move further. My branch cut begins over here. I get on to the branch cut, ok. So, now this is my branch and it goes below. This is my branch over here and it goes up to. So, now, I come from minus infinity, go around minus k m, go around minus k p, come forward, get on to the branch cut, cross the origin over here.

I am all on this cut over here, get off the cut, go forward, circumvent the k p pole, circumvent the k p pole, go forward, circumvent the k m pole, go forward to infinity; at infinity I take a circular contour, hit this infinity, I come down, I go around the cut, I drop off the cut, I get down to the lower cut, come back here, cross over go up to infinity; come back and join here, ok. That is how an L cut L cut will work.

Now, within this several cases will arise, it is possible that k p or k m or both will land up on the cut and then, just as we did for the earlier problem with the pole on the branch cut, the arguments have to be respected the branch cut defining arguments have to be respected. Only then, they will get the correct answer. So, this is not a full problem that I am going to show you, but L shaped cuts do arise and very rarely described in the literature. So, this is a; this is a way how an L shaped cut can arise and one has to go about doing it. The time is just right, time has come to an end we will continue in the next class.

Thank you.