A Short Lecture Series on Contour Integration in the Complex Plane
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Lecture - 20
L shaped branch cut

Good morning, welcome to this lecture on complex variables and we were in the middle of a problem. So, I will straight away continue from there.

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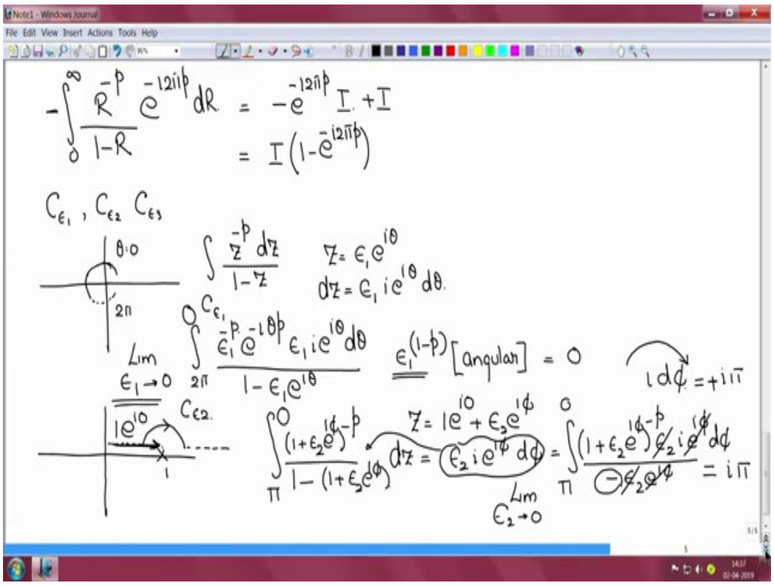

We are now left with 3 C epsilons that we have to do; C epsilon 1 C epsilon 2 and C epsilon 3, ok. C epsilon 1 is right here, we have to come from the bottom, go around in a clockwise direction. So, from theta equal to 2 pi to theta equal to 0. So, if we examine this integral, we have an integral over C epsilon 1, z to the power of minus p, d z over 1 minus z; z we will write as epsilon 1 e to the power of i theta, then dz, epsilon 1 is a constant, epsilon 1 i e to the power of i theta d theta, ok.

So, if I plug it in, theta has limits twice pi to 0, z to the power minus p, epsilon 1 to the power minus p, e to the power minus i theta p, dz is epsilon 1 i e to the power of i theta d theta by 1 minus epsilon 1 e to the power of i theta, limit epsilon 1 going to 0, ok. So, without rigor, we will just see here that we have epsilon 1 to the power 1 minus p in the numerator all others are angular functions. So, there is a part of epsilon 1 which survives in the numerator, limit epsilon 1 tending to 0, this goes to 0. Now, let us look at C epsilon 2; C epsilon 2 is here, we have a pole at 1, we come here we go around and we move forward in the clockwise sense C epsilon 2.

So, we now have integral pi to 0; pi to 0 and what is z? This part is very important, z now is equal to 1 e to the power of i 0, I am here, this is 1 e to the power of i 0, plus epsilon 2 e to the power of let say i phi, dz now is equal to epsilon 2 i e to the power of i phi d phi. So, if we plug it in, we get, 1 plus epsilon 2 e to the power of i phi to the power minus p and dz; dz comes in here; d z comes in here, divided by 1 minus z, which is 1 minus epsilon 1, plus epsilon 2 e to the power of i phi .

So, what do we have here, let me just clarify, I have integral 0 to pi to 0; pi to 0, 1 plus epsilon 2, e to the power of i phi, minus p, epsilon 2 i e to the power of i phi d phi divided by minus epsilon 2, e to the power of i phi, ok. So, then we cancel this, we cancel i phi and I have 1 plus epsilon 2, e to the power i phi minus p and limit epsilon 2 going to 0. So, I am left with i d phi; i d phi which is going in the clockwise sense from pi to 0. So, I get a minus i pi, but I have a minus in the denominator also, so I get plus i pi, so, this is plus i pi.

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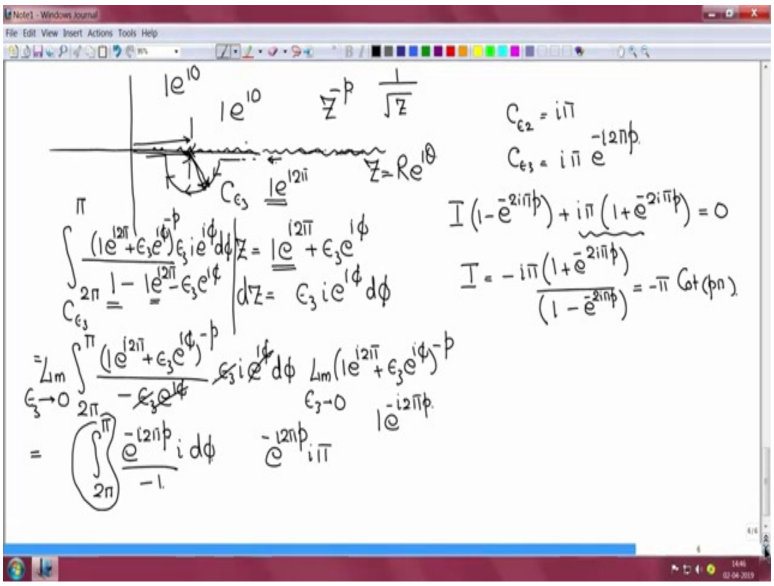

The last case; the last case is here C epsilon 3. So, we are coming from infinity I have a pole at 1, I circumvent the pole and I proceed, I am coming from infinity, this is C epsilon 3 this is 1, ok. This is the important part, now I have this integral on C epsilon 3 which extends from 2 pi to pi angle is now 2 pi, ok.

So, what is z here? z here is; z here is 1 plus epsilon 3 e to the power i phi, but it is 1 e to the power of i twice pi, this is important, when we were above, the 1 here was 1 e to the power of i is 0, when we are below, because this is a cut, because this is a cut we have to respect the z definition of the cut, ok. The cut is because z happens to have minus p, just like having square root of z or 1 by square root of z.

So, we have to respect the definition of z; z goes as Re to the power i theta and above theta is 0, below theta is 2 pi. So, z is now here, 1 e to the power twice pi, plus epsilon 3 e to the power of i phi. z is moving on this epsilon 3 semicircle. what is the definition of z, it is 1 plus epsilon 3 e to the power i phi, but this 1 is not purely 1 it is 1 e to the power i twice pi.

Now, dz is equal to epsilon 3 i, i e to the power of i phi d phi. So, we let us put the values here z to the power minus p. So, I have, 1 e to the power of i twice pi, plus epsilon 3 e to the power of i phi minus p, d z is epsilon 3 i e to the power of i phi d phi, divided by 1 minus 1 e to the power of i twice pi, minus epsilon 3 e to the power of i phi, that is equal to phi goes from twice pi to pi.

Then I have, 1 e to the power of i twice pi, plus a vanishing epsilon 3 e to the power of i phi, to the power minus p, limit epsilon 3 going to 0, divided by this is 1, that is 1. So, I get minus epsilon 3 e to the power of i phi, the numerator epsilon 3 i e to the power of i phi d phi. So, epsilon 3 goes, e to the power i phi goes.

Now, the movement is in the again in the clockwise direction and epsilon 3 goes to 0 and so I have a movement in the clockwise direction on a circular contour and this is e to the power of minus i twice pi p, ok. I hope you get this part, 1 e to the power of i twice pi, plus epsilon 3 e to the power i phi minus p, with limit epsilon 3 going to 0, ends up as 1 e to the power of minus i twice pi p, that is the idea into i d phi by a minus 1, this is going clockwise the difference is pi.

So, I will get, e to the power of minus i twice pi p i, minus pi, but they have a negative in the denominator, so I get pi. So, what do I have now? The sum of these two; the C 2 and C epsilon 2 and C epsilon 3, I forgot what we got. So, C epsilon 2 gave me i pi, ok. So, C epsilon 2, C epsilon 2 gave me i pi, C epsilon 3 gave me i pi e to the power of minus i twice pi p, ok. Now, what else do we have? We have I times 1 minus e to the power i minus i twice pi p ok.

So, we have; so, we have, I times 1 minus; 1 minus e to the power of minus twice i pi p, plus i pi, into 1 plus e to the power of minus twice i pi p, is the full contour, the full integral, now it is not enclosing any singularities. So, that residue contribution is going to be 0 ok. So, now, if I take this to the right hand side, I get I is equal to minus i pi, 1 plus e to the power of minus twice i pi p, by 1 minus e to the power of minus twice i pi p, that gives me minus pi cot of p pi.

The main idea in here is that you should respect the branch definition of z when you were here, 1 was 1 e to the power of i 0, when you are here it is 1 e to the power i twice pi, otherwise you will make a mistake. It is not one below and one above because there is a cut over here. Now, this sort of gives you a an idea of the kind of problems you can have. I have mainly shown you branch cuts related to square roots., you can have branch cuts related to other functions for example, like log.

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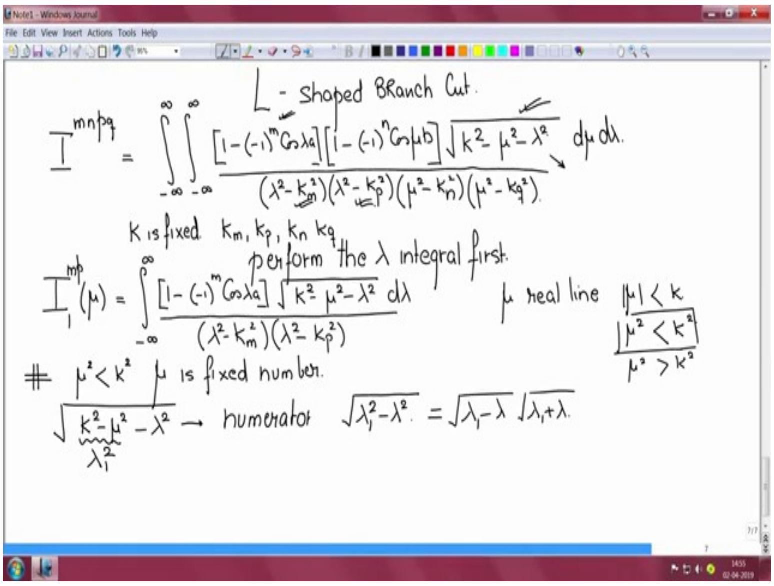

Now, what I want to show you is. This is a bit of an advanced problem, ok, but one does come across it and there is no better place than this lecture here to give you an idea. Now, I am going to talk about an L shaped branch cut, L shaped branch cut. How do we get an L shaped branch cut? In my line of work where it is sound and structure interaction, we come across a certain integral let us say I mnpq that is the name of the integral, ok.

Its a Fourier transform type integral, minus infinity to infinity, again minus infinity to infinity, 1 minus, minus 1 to the power m cos lambda a, 1 minus, minus 1 to the power n cos mu b, square root of, this is where you get a square root, k square minus mu square minus lambda square, ok. k is the acoustic wave number for those who are interested and d mu and d lambda these are the Fourier transform variables. In the denominator, I get lambda square minus k m square, into lambda square minus k p square, mu square minus k m square, mu square minus k q square.

Here k is a fixed number all are fixed numbers, k is a fixed number, is fixed a given number, wave number depends on frequency and speed of sound. Then k m is a fixed number, k p is a fixed value, k m k p sorry k m, ok, then k n; then k n, k n; k n is a fixed value and k q is a fixed value, ok. Now, the plan is to perform, plan is to perform the lambda integral first; lambda integral we want to do first. So, once we do the lambda integral how does it look like, I get this part, now lambda is here; here and here and here and lambda gets mixed up with mu here.

So, we write I 1, mp mu and that is equal to integral, minus infinity to infinity, 1 minus minus 1 to the power m cos lambda a, square root of k square minus mu square minus lambda square, d lambda, divided by lambda square minus k m square, into lambda square minus k p square, ok, this is the integral we want to do first.

Now, we know that mu, mu is moving on the real line, from minus infinity to infinity. So, there is a chance that somewhere mu magnitude is less than k; k is a fixed positive real number. So, that mu square is going to be less than k square, there is a region where that that happens and there is a region where it exceeds, there is also region where mu square will now become greater than k square. So, let us not worry about this region we will worry about this region.

Let us take the case where mu square is less than k square. So, then what happens, so we take the case where mu square is less than k square, mu is real number, we do not need to put this, mu square is less than k square and mu is fixed; mu is a fixed number, mu is fixed; fixed number.

So, now, this particular square root function; k square minus mu square, now k being greater than mu, I will put this as lambda 1 squared and this as minus lambda square. Now, what do I have in the numerator, in the numerator I have other things, but I have square root of lambda 1 square minus lambda square. Which means what now; which means this is broken into lambda 1 minus lambda and square root of lambda 1 plus lambda.

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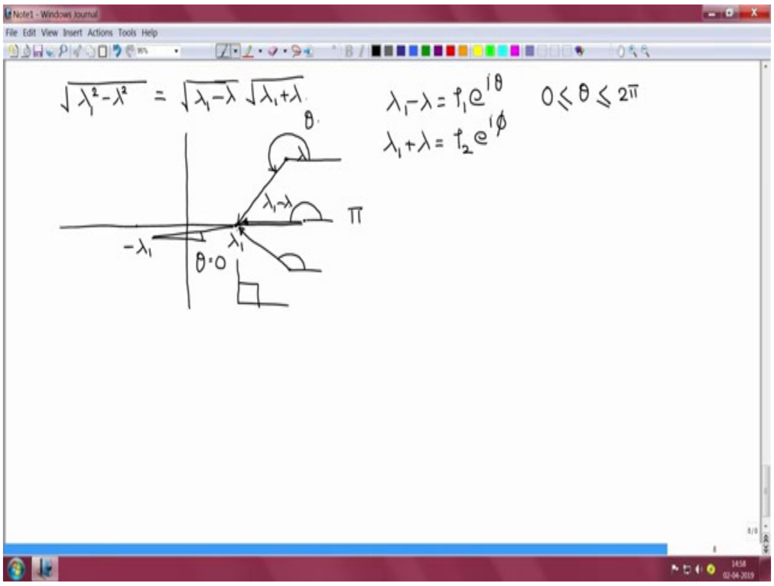

So, let us get to the next page for some room, ok. So, let me write this, I have square root of lambda 1 square minus lambda square which is equal to square root of lambda 1 minus lambda into square root of lambda 1 plus lambda, ok. So, let us say let me say that let lambda 1 minus lambda be equal to rho 1 e to the power of i theta and lambda 1 plus lambda is equal to rho 2 e to the power of i phi. Now, let us examine this on the complex plane, here is my lambda 1, here is my minus lambda 1, here is my lambda, ok.

So, lambda 1 minus lambda; so lambda 1 minus lambda is this, the arrowhead is here, the tail end is here, this is lambda 1 minus lambda, ok. So, the amplitude is rho 1 and theta is this angle from the horizontal to where we are is theta, that is theta, ok. So, what are the limits on theta, if lambda ends up here on the real axis, then theta is equal to pi, if lambda ends of ends up here, then theta is less than pi, if lambda ends up right here, then theta is pi by 2 and if lambda ends up here theta goes to 0. So, theta limits we will say are 0 less than twice pi. Now, time is running out, I will continue the problem from here in next class.

Thank you.