A short lecture series on Contour Integration in the Complex Plane

Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 02
Cauchy Riemann Equations

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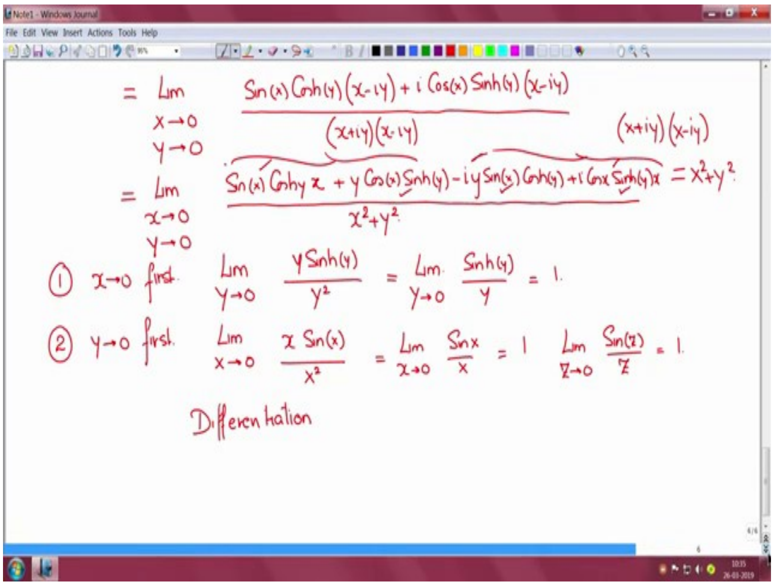

Good morning to all of you. Welcome to this series on complex variables with specific applications to contour integration. Now, we look at differentiation, derivatives or differentiation. Do they follow immediately? Let us see.

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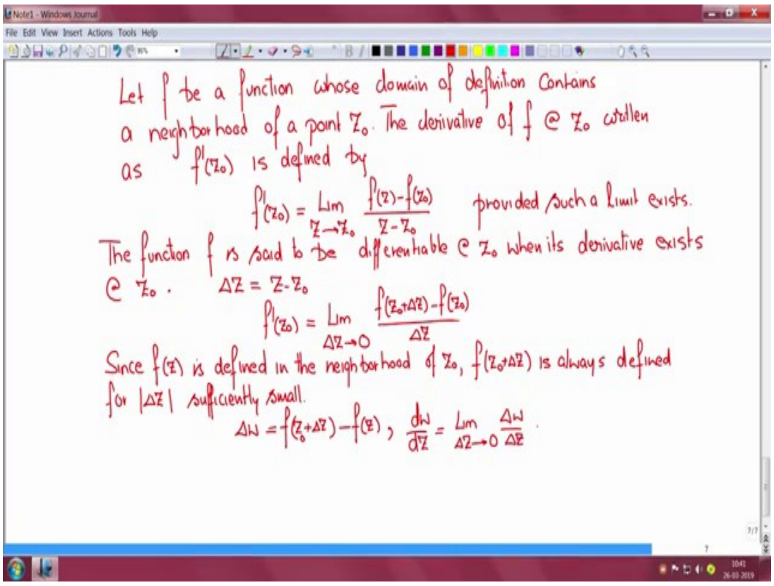

So, here I will give a few formal lines in mathematics. As I said, the treatment is largely engineering oriented and non rigorous, but occasionally will use the language that mathematics requires us to use.

So, we say let f be a function; let f be a function whose domain of definition contains a neighborhood of a point z 0. The derivative of f at z 0 written as f dash of z 0 is defined by: f dash of z 0, is equal to limit z tending to z 0, f of z, minus f of z 0, by z minus z 0. This is the definition. This is the definition of the derivative.

So, we say provided this limit exists, provided such a limit exists, such a limit exists. This function, the function f is said to be differentiable at z 0, when it is, when its derivative, when its derivative exists at z 0, ok. If we set delta z to be equal to z minus z 0, ok, then we have f dash of z 0, is equal to limit delta z tending to 0, f of z 0 plus delta z, minus f of z 0, by delta z, this is the familiar form from calculus.

Now, since f of z or f or f of z is defined, is defined in the neighborhood, in the neighborhood of z 0, f of z 0 plus delta z, is always defined for delta z sufficiently small and so we have delta w given by f of z plus delta z. Sorry, z 0 plus delta z, minus f of z and then we say, the derivative dw by dz is equal to limit delta z tending to 0, delta w by delta z. So, we will take a few examples.

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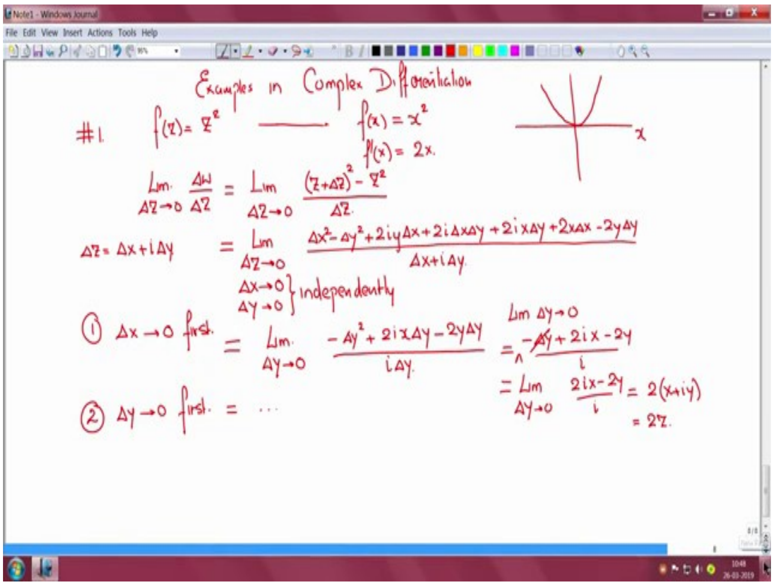

So, these are examples in, examples in complex differentiation. So, first example: take this function, f of z is equal to z square ok. The reason we take it is, that in the real variable calculus f of x is equal to x squared is a very nice function. It is a parabola, ok. So, it behaves like this; behaves like this. And it has a very nice derivative. The derivative f dash of x is given by 2 x, ok.

So, now, we want to check if the laws of the real variable calculus apply to my complex function f of z is equal to z squared. So, let us see, let us use the definition we have given about limit delta z tending to 0, delta w by delta z, is equal to limit delta z tending to 0, z plus delta z whole squared, minus z squared, divided by delta z. I will open this out and write it out, without so many details.

So, we have limit delta z tending to 0, the numerator will have delta x squared minus delta y squared, plus twice i y delta x, plus twice i delta x delta y, plus twice i x delta y, plus twice x delta x, minus twice y delta y. And the whole thing divided by delta x plus i delta y. Just to remind you, I am replacing delta z as delta x plus i delta y, ok. Now, here delta z tending to 0, again implies that delta x can go to 0 and delta y can go to 0, independently, ok.

So, we will do what we have done before. We will take case 1, where delta x is sent to 0 first, ok. If delta x is sent to 0 first, we get limit delta y tending to 0, ok, so delta x is gone. So, I will have minus delta y squared, plus twice i x delta y, minus twice y delta y, divided by i delta y, ok. Now, there is delta y in the numerator and denominator, so I can do some cancellations.

So, I will get minus delta y, plus twice i x, minus twice y, divided by i and of course, here, limit delta y tending to 0, ok. Now, here this delta y will go to 0 and I am left with limit delta y tending to 0, twice i x, minus twice y by i, which is equal to twice x plus i y, equal to twice z, ok. So, it indeed works out. So, we had f of x equal to x squared, its derivative is 2 x, we have f of z equal to z squared, derivative is indeed 2 z.

Now, we did that using, by sending delta x to 0 first. We can similarly send delta y to 0 first, then also you will get the same answer. A similar procedure you can follow, the answer then turns out to be twice z again ok, so I will not repeat the process. Now, we will take the next example. Does it always work. Will take another example.

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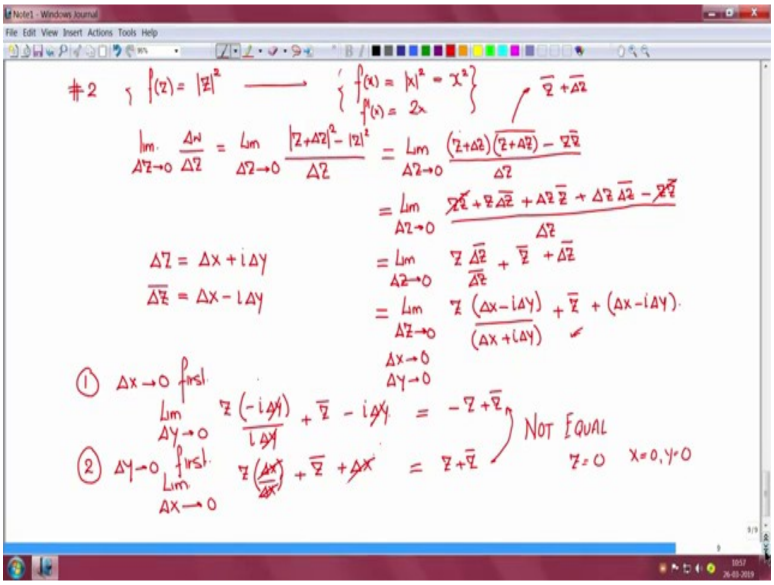

The next example is this, example 2; example 2, ok. The next example is f of z is magnitude of z squared, magnitude of z squared, ok. Again the reason for choosing this is, f of x in the real variable case, if we do magnitude of x squared, it is equal to x squared and it has a derivative everywhere. There is no problem with it. It is a very nice function in the real variable calculus. So, does that apply to magnitude of z squared is the question, ok.

So, we use the definition we gave earlier: limit delta z tending to 0, delta w by delta z, is equal to limit delta z tending to 0, z plus delta z magnitude squared, minus magnitude of z squared, divided by delta z. So, we will open this out, limit delta z tending to 0, z plus delta z, z plus delta z conjugate, which I show as a bar. The bar means a conjugate, minus magnitude z squared is z into z conjugate, divided by delta z. Now, z plus delta z bar is equal to z bar plus delta z bar, ok.

So, if we now use this, delta z tending to 0, we open this out, ok. So, we get z times z bar z; times z bar, plus z times delta z bar, this to that, then I have plus delta z into z bar, plus delta z into delta z bar, minus z into z bar, divided by delta z, ok. Here, there is some cancellation: z z bar goes with z z bar, ok. Then I have, let us see, limit delta z tending to 0. I will separate these terms, z into delta z bar by delta z, plus here we have delta z. So, we get z bar and here we have delta z cancellation again plus delta z bar. Now, delta z here is equal to delta x plus i delta y and delta z bar: the conjugate is delta x minus i delta y. So let us introduce these over here.

So, we get limit delta z going to 0, z; please do not confuse this with the bar ok. So, z into delta x minus i delta y, divided by delta x plus i delta y, plus z bar, plus delta x minus i delta y, ok. We still have to take delta z tending to 0, the limit and as you know we mean delta x goes to 0 and the delta y goes to 0, independently. So, again case 1, we set delta x to 0 first, ok, then what do we get? We get here, let me write here, then we get is equal to delta x is 0. So, we get remaining, let me use a little bit of space here, so I will write here below. So, we have limit delta y tending to 0, z, delta x is already 0.

So, I get minus i delta y by i delta y, plus z bar, minus i delta y. Now, if I apply the limit, so here delta y cancels, here delta y goes to 0. So, I have a minus i and i, so that brings in a minus, so I get minus z plus z bar, minus z plus z bar. Second case: we set delta y to 0 straight away first, then we are left with, limit delta x tending to 0, ok. So, in this expression over here, we have to set delta y to 0 first. So, we get z into delta x by delta x, plus z bar, plus delta x. This delta x cancels off, this delta x will go to 0. And so we have, this time z plus z bar, ok.

So, you can see these two are not equal; these two are not equal, not equal. In fact, the place, the only place where these two can be equal is when z is 0, z equal to 0 or x equal to 0 and y equal to 0, that is at the origin. It has a derivative only at the origin and nowhere else, ok. So, these two are not equal and therefore, what we had for this real variable calculus in terms of differentiation does not automatically translate to derivatives in the complex domain. So, there is some additional conditions that are needed, unlike the limits it is not that smooth a transition, ok.

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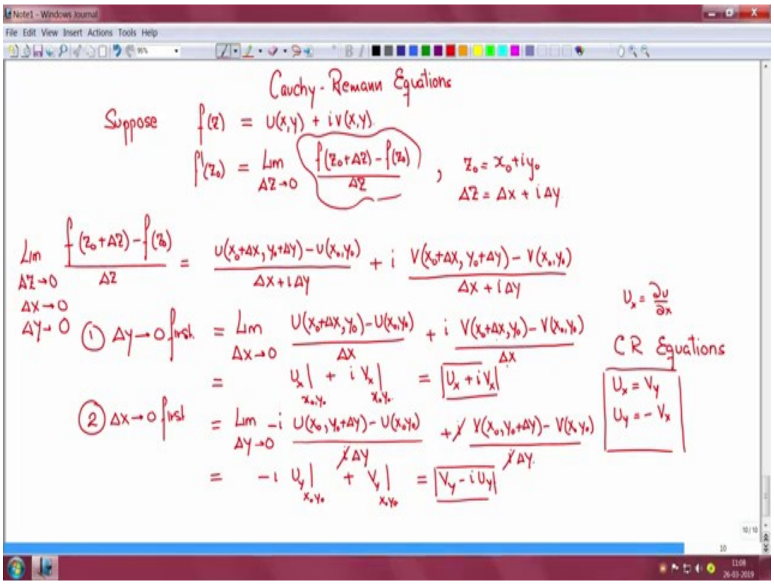

So, now what happens, what is the additional condition that we need to know to establish whether derivatives exist or derivatives do not exist. I am going kind of quickly, you can see that my pace is quite high. Again I said, I will assume some kind of familiarity in the formal course, ok. So, now, those conditions which establish whether derivatives of complex functions exist or not are called Cauchy Riemann equations or Cauchy Riemann conditions, ok.

Now, what are these? Suppose I have a function f of z, function of a complex variable and that is written as u of x comma y, plus i v x comma y. And now we define the derivative f dash of z 0 at a point, as limit delta z tending to 0, f of z 0 plus delta z, minus f of z 0, divided by delta z. We may now say where z 0 is given by x 0 plus i y 0 and delta z is given by delta x plus i delta y. So, now, we expand this; we expand this expression in terms of U x y and V x y, ok.

So, we have, start at the left here, it might be a long equation. f of z 0 plus delta z, minus f of z 0, divided by delta z, is given by u at x 0 plus delta x comma y 0 plus delta y, minus u at x 0 comma y 0, by delta x plus i delta y, plus i times V x 0 plus delta x comma y 0 plus delta y, minus V x 0 comma y 0, by again delta x plus i delta y. Now, there is this limit on this, limit of delta z going to 0 and therefore, delta x going to 0 and delta y going to 0, again independently.

So, we will take again the limiting process. We will set delta x goes to 0 first. So, we will say delta x is sent to 0 first, sorry. So, this time we do it the other way first ok. So, we do delta y is set to 0; delta y set to 0 first, it does not matter, but delta y set to 0 first. And then we end up with limit delta x tending to 0, U x 0 plus delta x comma y 0, minus U x 0 comma y 0, by delta x, plus i V of x 0 plus delta x comma y 0, minus V x 0 comma y 0, divided by again delta x.

So, now here you can see, the U function at x 0 y 0 and here incremented only in the x direction by delta x, the difference taken and divided by delta x. So, we know that this is del U del x, this is U at x evaluated at x 0 comma y 0. U x is equal to del U del x. Same here, this function initially at x 0 y 0 and evaluated with an increment only in x direction, y direction the argument is same, divided by delta x with a limit, that is now i v x evaluated at x 0 y 0. U x plus i V x evaluated at x 0 y 0. Next, here we set delta x to 0 first, straightaway, and then we get, limit delta y tending to 0, U x 0 comma y 0 plus delta y, minus U x 0 y 0, by delta x was already 0.

So, I get i delta y, plus i V x 0 comma y 0 plus delta y, minus V x 0 y 0 by again i delta y. Now, this i goes with this i here and this i in the denominator ends up as a minus i over here, ok. Now, what we have here is U evaluated at a base point x 0 y 0 and this an increment in the y direction with a delta y. So, I get here minus i U y evaluated at x 0 y 0 and I have here, plus, here the evaluation is V y at x 0 y 0. So, my U's and V's are looking similar, so I was getting worried.

So, we will write this clearly as U x plus i V x and this we write as V y minus i U y, ok. So, this is the derivative when we send delta y to 0 first. This is the derivative when we send delta x to 0 first. Now the real to real, imaginary to imaginary must be equal in order for the derivative to be unique. And you can see now, U x must be equal to V y, those are the real parts of this and then U y must be equal to minus V x, ok.

The partial derivative of U with x is equal to partial derivative of V with respect to y and partial derivative of U with respect to y is minus the partial derivative of V with respect to x these are called the Cauchy Riemann conditions or equations. We will write them as CR equations.

So, now the CR equations are necessary conditions. We will see what that means. So, we will continue this thought in the next class, we have derived the Cauchy Riemann conditions. So, we will see a few examples where Cauchy Riemann equations are satisfied, where they are not satisfied and then we will proceed to integrations in the complex domain. I will stop here today.

Thanks.