A short lecture series on Contour Integration in the Complex Plane
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Lecture - 19
Pole on a branch cut

Hello, good morning. Welcome to this lecture on Complex Variables. Last time we did a problem which had a finite branch cut and we did it using two methods. We will reinforce those ideas, ok.

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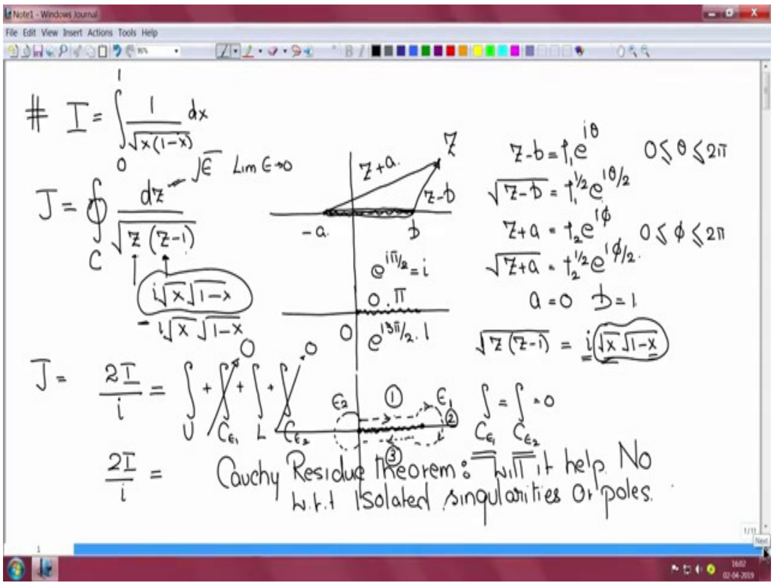

So, I have this integral having a square root function. So, we replace this again with, this time a clockwise integral over a contour, dz over square root of z, again z minus 1. This is equivalent to having branch points that are not symmetric, ok. I could have a branch point here and I could have a branch point here at minus a and here is my z, this is z minus b and this is z plus a.

So, if I write z minus b is equal to rho 1 e to the power of i theta, then square root of z minus b is equal to rho 1 to the power half, e to the power of i theta by 2. Similarly, if I say z plus a is equal to rho 2 e to the power of i phi and square root of z plus a is equal to rho 2 to the half, e to the power of i phi by 2, then we will see for this definition of theta and a similar definition of phi, the branch cut will lie between a and b, just like last time, ok.

In this case it happens that a is equal to 0 and b is equal to 1, ok. So, we have this situation 0 to 1, 0 ok. Now, because theta here at this point is going to be equal to pi and phi is going to be equal to 0, I have e to the power of i pi by 2 here, which is equal to i. And therefore, square root of z, into z minus 1, will have the definition of i square root of x, square root of 1 minus x to be a positive imaginary number. So, this x lies between 0 to 1, this x is less than 1, so this is a positive number and the i, ok.

So, now, if I decide on this contour that I will go let us say, I have 0 here, I have 1 here, I will go above take an epsilon 2 or epsilon 1 contour, come down here, come down back, taken epsilon 2 contour and join up. Just as last time, epsilon 1, integral epsilon 1 and integral C epsilon 2 will go to 0, because there is dz on top which has a full epsilon, there is z along with 1 which will not be comparable to 1 and I have a square root of epsilon; square root of epsilon and epsilon will cancel, I get a square root on the top and as limit epsilon goes to 0, the integrand goes to 0. So, C epsilon 1 and C epsilon 2 contours go to 0, ok.

But now I have for the positive side, i square root of x, square root of 1 minus x and similarly minus i, square root of x, square root of 1 minus x for the bottom side. Here it is e to the power of i 3 pi by 2. Therefore, my J ends up as twice I divided by i, because this is in the denominator, ok. If it is not clear, I have the upper leg 1, I have the C epsilon 1 leg 2, I have the lower leg 3 and I have C epsilon 2. And we have seen this goes to 0, C epsilon 2 goes to 0, so upper plus lower is, twice I divided by i.

Now, this is a branch cut, ok. So, what sort of theorems will help us? The question is; Cauchy residue theorem; Cauchy residue theorem. Will it help us? So, let me pose, will it help. The answer is no, because Cauchy residue theorem is with respect to isolated singularities, isolated singularities or poles, not with respect to branch cuts. So, we have to find some other means. So, what is that means?

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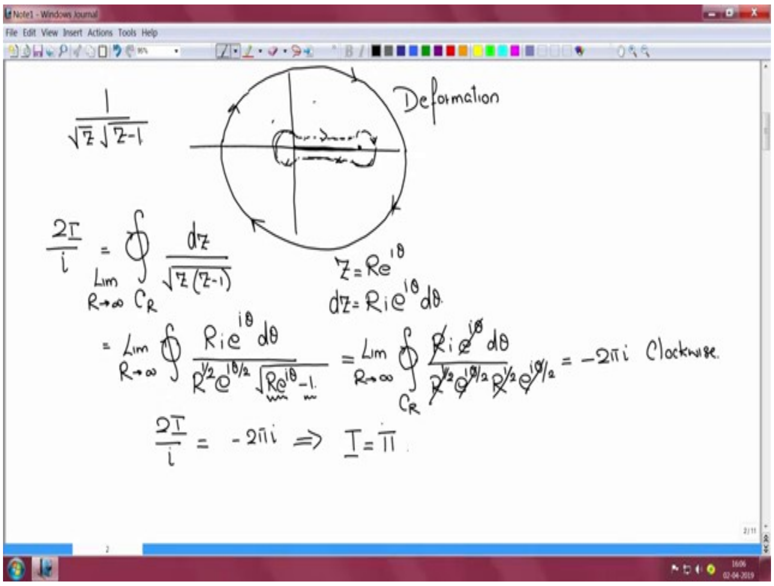

I have this integral that goes around this branch cut, and this function, I mean branch cut is a singularity, the function does not behave well around a branch cut, so it is a singularity, ok. But this function, 1 over square root of z, square root of z minus 1 is well behaved everywhere in the complex plane. Everywhere else except that this branch cut the functions well behaved. So, I can use the deformation theorem, deformation theorem, ok.

So, I deform this clockwise going contour, I expand it; expand it, I take it out, take it all the way to infinity, on a circular contour, of radius infinity. So, I have twice I by i is equal to this clockwise contour CR, limit R tending to infinity dz, over square root of z into z minus 1. On the CR, z is equal to R e to the power of i theta, dz is equal to R i e to the power of i theta d theta. So, I have limit R tending to infinity, a clockwise integral on theta, dz is R i e to the power of i theta d theta divided by R to the power half, e to the power of i theta by 2 and square root of R e to the power of i theta, minus 1.

As R tends to infinity, R will overwhelmed this value 1. So, we write as, limit R tending to infinity, integral in a clockwise sense, over 2 pi, R i e to the power of i theta d theta, by R to the power half, e to the power of i theta by 2, R to the power half, e to the power of i theta by 2. So, this cancels out. I get i d theta which is going to be minus twice pi i because it is a clockwise sense. So, I have twice I divided by i is equal to minus twice pi i, it implies my I is equal to pi, ok. Now, let us look at another problem.

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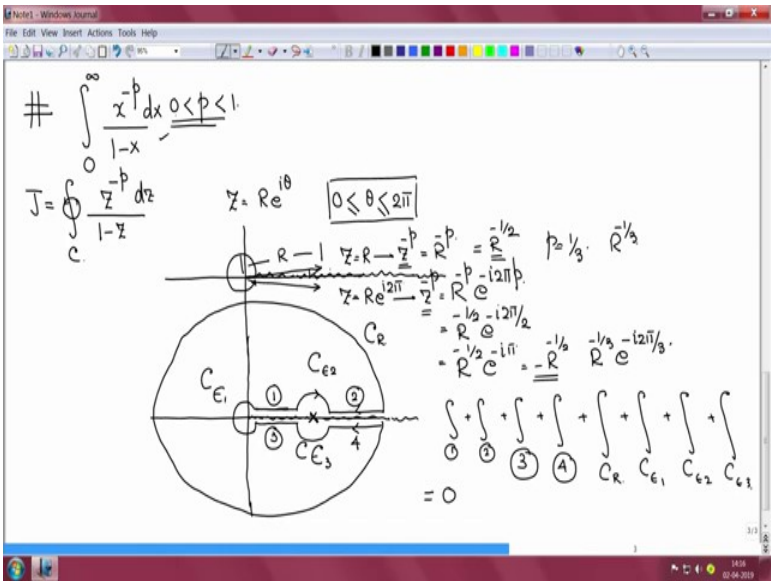

The next problem is this. I have an integral going from 0 to infinity, x to the power minus p, over 1 minus x, ok. p is a fraction less than 1, p can be anything between 0 and 1, dx. Now, if we replace this with a closed contour integral and say x is z minus p dz over 1 minus z. Now, if I make z for example, equal to R e to the power of i theta and lets say theta is given this definition, theta is given this definition, then look at this; if we have, if we start, if we start very close to the x axis, let us say this is R e to the power of i theta, theta is very close to 0 then z is equal to R over here. This distance is R.

If I go full circle and let theta become twice pi, then z is equal to R e to the power of i twice pi, ok. But here imagine, from here z to the power minus p is equal to R to the power minus p and from here, z to the power minus p is equal to R to the power minus p, e to the power minus i twice pi p, ok. Let us say p is half, p is half, then I get R to the power minus half and here I get R to the power minus half p is half.

So, e to the power of minus i twice pi by 2, which is equal to R to the power minus half, e to the power of minus i pi, which is equal to minus R to the power minus half. So, I go around, here my function does not come back to the same value. z to the power minus p is R to the power minus half above, z to the minus p is minus R to the power minus half below.

Similarly, if I choose p equal to one-third, then also they will be a jump, ok. Here because theta is 0, so I will get R to the power minus one-third on the top. Here I will get

R to the power minus one-third, e to the power of minus i twice pi by third. Again, the function jumps. So, if as long as p takes values greater than 0 less than 1, this line is going to be a line for a discontinuity. So, it is a branch cut. It starts at x equal to 0, Z equal to 0 and extends to infinity, ok.

So, now, I will choose a contour, we have to do this integral, so I will choose a contour, ok. I will choose a contour which starts with let us say some circle of radius epsilon 1, moves forward, ok, but now at z equal to 1, I have a pole, there is a pole on the branch cut. So, this is a cut, with this definition of theta I have a cut that goes off to infinity. But now z equal to 1 is a pole.

So, I have a pole, so I cannot go through the pole. So, I have to go a circle, half circle C epsilon go C epsilon 2 and then I go off to infinity, at infinity I take a CR contour go around, I come back around, then I start going back. But again, I encounter the pole below the branch cut. I go around the pole, I come straight back, join the epsilon 1 circle. Let us call this epsilon 3 circle, C epsilon 3, C epsilon 1.

So, how many portions do I have? I have portion 1, portion 2, portion 3, portion 4, then the CR portion and 3 epsilons, ok. So, if we write it, I have integral portion 1 plus integral portion 2, plus integral portion 3, plus integral portion 4, plus integral CR, plus integral C epsilon 1, plus integral C epsilon 2, plus integral C epsilon 3, ok. Now, within this contour there are no poles, so that is equal to 0, no residue contribution. That is equal to 0, ok. Now, let us see how to go about this.

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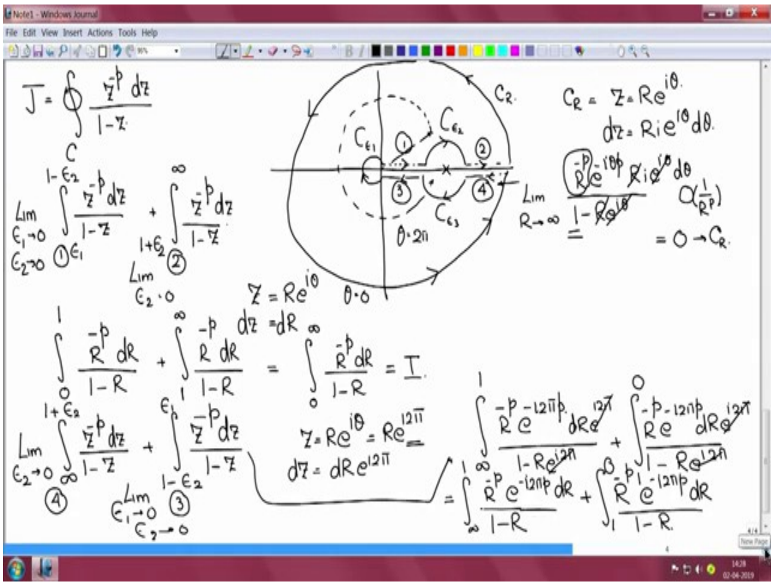

I have J is equal to integral over a full C, z to the power minus p dz, over 1 minus z. And let me take the contour at a distance, portion 1 pole at 1, go around the pole, portion 2, go CR, comeback, go around, go around the pole, back, C epsilon 1, C epsilon 1, C epsilon 2, C epsilon 3, 1, 2, 3, 4, CR, ok.

Let us see now on CR, on CR. Let z be equal to R e to the power of i theta, ok. Then dz is equal to R i e to the power of i theta d theta, ok. Then what do I have? R to the power minus p, e to the power of minus i theta p dz, R i e to the power of i theta d theta, divided by 1 minus z, 1 minus z is R e to the power of i theta and limit R tends to infinity. As R tends to infinity 1 can be dropped, this R cancels with that R, i theta with i theta and I have this as order 1 over R to the power p, ok. This R to the p minus p survives, so this goes to 0 as R goes to infinity. So, the CR integral is gone, CR integral has gone to 0, ok.

So, now, what about C 1 and C 2? C 1 and C 2, so integral, I mean integral portion 1 of z to the power minus p dz, over 1 minus z, plus integral portion 2, z to the power minus p dz, over 1 minus z, ok. Strictly these have limits, from let us say epsilon 1; epsilon 1 to 1 minus epsilon 2 then the limit here is 1 plus epsilon 2 to infinity. And in this limit, epsilon 1 has to go to 0, epsilon 2 goes to 0, same here, epsilon 2 goes to 0, ok.

So, what do we have? Z is along the x axis on the upper side, ok. So, if we said z equal to R e to the power of i theta, theta this case is 0, so z equal to R and dz equal to d R. So, I have integral 0 to 1, R to the power minus p, dR over 1 minus R, plus integral 1 to infinity, R to the power of minus p dz is dR, by 1 minus R, which gives me, integral 0 to infinity, R to the power minus p dR, over 1 minus R, which is the integral we wanted to do, this is I, in the principal value sense, Cauchy principle value sense, ok.

Now, for 3 and 4; for 3 and 4, I have integral, 4 which comes from infinity to 1 plus epsilon 2, ok. Limit epsilon 2 going to 0, z to the power minus p dz, by 1 minus z, plus integral, this is portion 3, ok. You should see the direction, here we are going forward, here we are coming back from infinity, ok. This is we are coming back from infinity, hence 4 comes first, then 3 comes, ok. Then 3 is 1 minus epsilon 2 to epsilon 1. Here also limit epsilon 1 goes to 0 and epsilon 2 goes to 0, ok, z to the power of minus p dz by 1 minus z, ok.

Now, let us see on this leg, z, ok. What is z equal to? z equal to R e to the power of i theta, R e to the power of i theta, but theta has now moved to the downside, the theta is equal to 2 pi, ok. So, here we have R e to the power of i twice pi, ok. Now, let us not be very quick in setting e to the power i 2 pi to 1. Let e to the power i 2 pi remain as it is, ok, but it is a constant, the angle is a constant, because we are coming along a straight line, ok. Now, dz is equal to d R e to the power of i twice pi, ok.

Now, watch what happens here. I have, so I am going to write this part here. I have integral, infinity to 1, z to the power minus p R to the power minus p e to the power of minus i twice pi p, dz is equal to d R e to the power of pi twice pi by 1 minus z, 1 minus R e to the power of i twice pi, plus integral 1 to 0, 1 to 0, z to the power minus p is R to the power minus p e to the power of minus i twice pi p, dR e to the power of i twice pi, by 1 minus R e to the power of i twice pi.

Let me write it on the same page, here I have infinity to 1, this gives me a 1, this gives me a 1, 1, 1. So, I get R to the power minus p, e to the power of minus i twice pi p, dR by 1 minus R, plus integral 1 to 0, R to the power minus p, e to the power of minus i twice pi p, dR by 1 minus R. Now, I can switch the signs, I have infinity to 1, I can make a 1 to infinity, I have 1 to 0, make it 0 to 1 and therefore, I get 0 to infinity. I put a minus, R to the power of minus p, by 1 minus R, e to the power of minus i twice pi p, dR ,ok.

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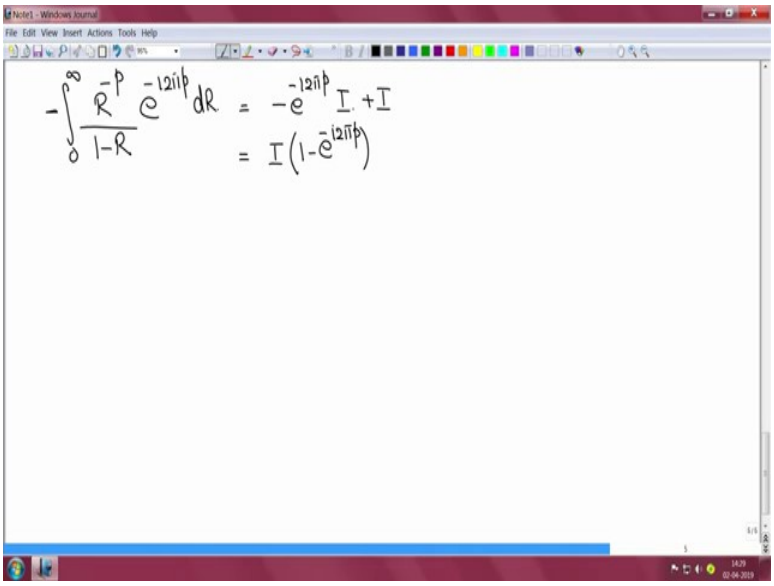

So, what is this equal to? This is equal to minus e to the power of minus i twice pi p into I, ok. And from the earlier case we got plus I, so, together we get I, into 1 minus e to the power of minus i twice pi p. Time has run out. So, I will stop at this point and continue from here in the next class.

Thank you.