A short lecture series on Contour Integration in the Complex Plane
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Lecture - 18
Example on finite branch cut

Good morning, welcome to this next lecture on complex variables.

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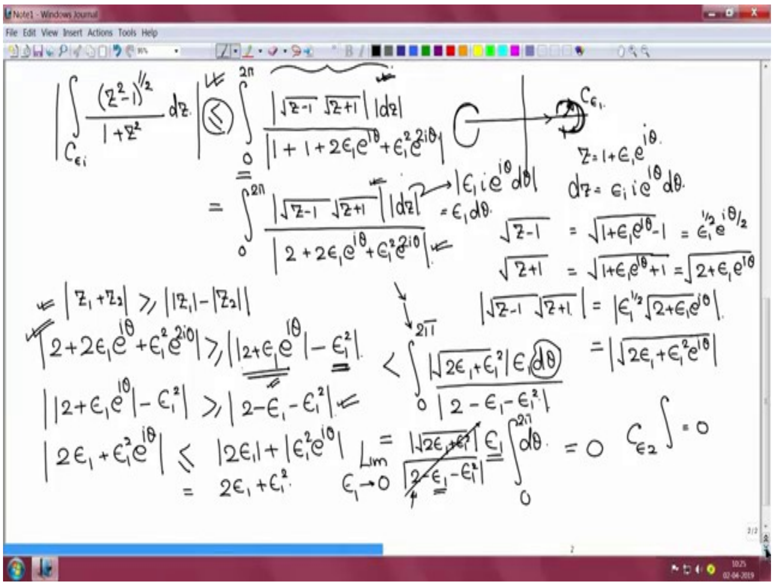

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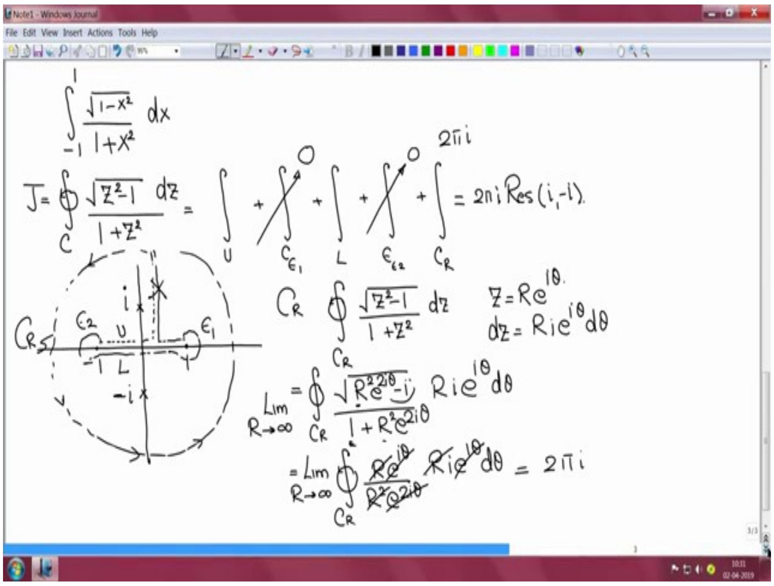

We were looking at this integral, which we want, which is minus 1 to 1, square root of 1 minus x square, by 1 plus x square, d x. And, we had proceeded a long way, we replaced this with J, which is equal to 0, integral over a closed contour, square root of z square minus 1 d z over, 1 plus z square.

And, this had portions of an upper integral, plus an integral over C epsilon 1, plus an integral over the lower portion, plus an integral over C epsilon 2, plus an integral over CR, which we said is equal to twice pi i times the residue and the two isolated singularities, ok.

Let us remind ourselves of the contour here, we have 1, we have minus 1, we have a singularity at i, a singularity at minus i. So, we come on the upper branch, we take a vertical turn, we go around this contour at infinite radius, we come down here, we take a turn, go around epsilon 1, come below take the lower branch, go around epsilon 2. So, this epsilon 1, epsilon 2, CR, upper, lower and these we said cancel out. And, last class we saw that C epsilon 1 and C epsilon 2 also go to 0. So, we are left with upper, lower, CR and the residue, ok.

So, let us look at CR now, CR is a counter clockwise integral. So, we are going on a now closed contour ok, CR and integral is square root of z square plus minus 1, by 1 plus z square d z. Now, on CR which is a circle, we will write z equal to R e to the power of i theta, R remains constant so, d z is equal to R i e to the power of i theta and d theta, ok.

And, so, let us write integral CR, limit R tending to infinity, we have square root of R square e to the power of twice i theta, minus 1, by 1 plus R square e to the power of twice i theta and R i e to the power of i theta d theta.

Now, we will do a hand waving approximation as R tends to infinity, 1 is much smaller than R square; this 1 is smaller than R square so, we write, limit as R tends to infinity CR. So, then this becomes R e to the power of i theta, the denominator becomes R square e to the power of twice i theta, by ignoring the 1, then R i e to the power of i theta d theta. This R, this R, R square cancel e to the power i theta, e to the power i theta, twice i theta cancel. So, I have an integral over twice pi of i d theta so, this becomes twice pi i. So, the CR integral ends up as twice pi I, ok. Now, let us see what happens to upper and lower.

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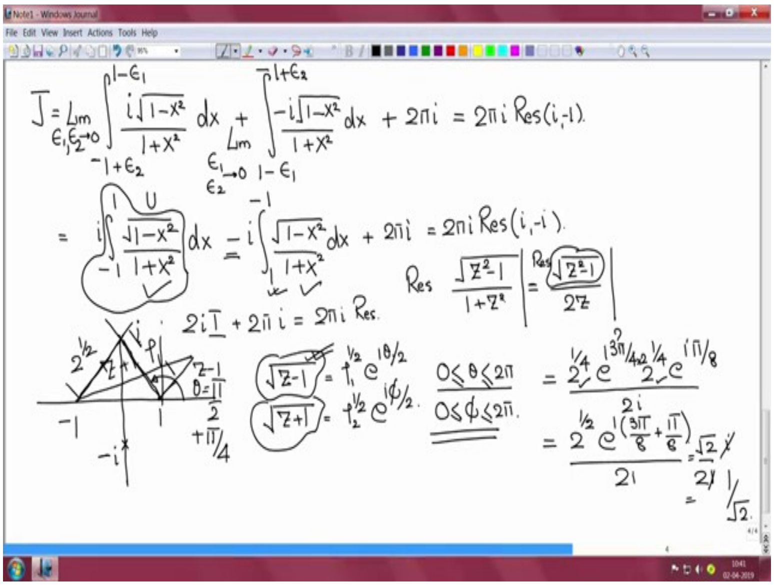

So, my integral now J finally, is like this, integral the upper portion is minus 1 plus epsilon 2 to 1 minus epsilon 1 and here I had i square root of 1 minus x square by 1 plus x square d x. If you see the definitions of the integrand in various parts of the complex plane.

So, this is my upper, then I had the lower, which is integral 1 minus epsilon 1 to minus 1 plus epsilon 2, minus i square root of 1 minus x square, by 1 plus x square d x. Then, C epsilons went 0 and I have twice pi i from CR and now I have to evaluate the residues.

Now, I have to evaluate residues, twice pi I, residues at i and minus i. Now, in here, limit epsilon 1 and epsilon 2 tend to 0, here also limit epsilon 1 and epsilon 2 tend to 0, ok. So, how does this look now, this looks like, integral minus 1 to 1, i square root of 1 minus x square, by 1 plus x square d x, minus I, integral 1 to minus 1, square root of 1 minus x square, by 1 plus x square d x, plus twice pi i, is equal to twice pi i times residue i comma minus i.

Now, you can see that if I take out the minus and switch the limits, this and this double. And, in fact, this part is the integral which I want, this is the integral which I want this my I. So, I get twice i I plus twice pi i as twice pi i times the residues. Now, let us look at the residues, ok. The integrand is square root of z square minus 1, over 1 plus z square ok, we have stated a theorem, where the residue can be evaluated as the numerator divided by the derivative of the denominator which is twice z, ok.

Now, evaluated at e to the power of i pi by 2. I am sorry just hold on, just hold on, we put the evaluation point later, ok. So, the residue of this at a certain point is the same as the residue of this the numerator and the derivative of the denominator at whatever points. Now, let us look at this here we have 1 minus 1, this is z minus 1, this is z plus 1 ok. And, my one of the singularities is at i, the other singularity is at minus i.

Now, z minus 1 or square root of z minus 1, was given by rho 1 to the power half e to the power of i theta, square root of z plus 1 was given by rho 2 to the power half, e to the power i phi, theta by 2; i phi by 2. And, theta definition 0, less than theta, less than twice pi, 0 less than phi less than twice pi, that was the theta and phi definition.

Now, it is very important that in evaluating the residue. The definition of square root of z square minus 1 and therefore, the definition of square root of z minus 1 and z plus 1 in terms of the limits of the argument are respected. I will show you what I mean, ok. So, now here we have to evaluate the residue at this point, for plus i we have to evaluate residue here. So, the numerator is square root of z square minus 1 and so, the numerator here evaluated is equal to rho 1 to the power half. So, rho 1 is here and rho 1 is root 2.

So, we get 2 to the power half, to the power half. So, 2 to the power one-fourth, and the angle theta is pi by 2 plus pi by 4, theta is equal to pi by 2 plus pi by 4, that is 3 pi by 4. So, e to the power of i 3 pi by 4, that is z minus 1, then z plus 1, this here is z plus 1, that distance is also root 2 and we get 2 to the power 1 by 4 and this angle is pi by 4. So, we get, just hold on a bit; just hold on, we get this has to be halved, further halved because is i theta by 2.

So, this will be into 2 and 2 to the power 1 4 e to the power of i pi by fourth by 2, which is pi by 8, divided by twice z, evaluated at I, z equal to is twice i. So, we get here 2 to the power half, because of one-fourth and one-fourth and e to the power of i 3 pi by 8, plus pi by 8, which is pi by 2, divided by twice i. So, this is root 2, by 2 i, into i which is equal to 1 by root 2, ok. So, we will see the next residue.

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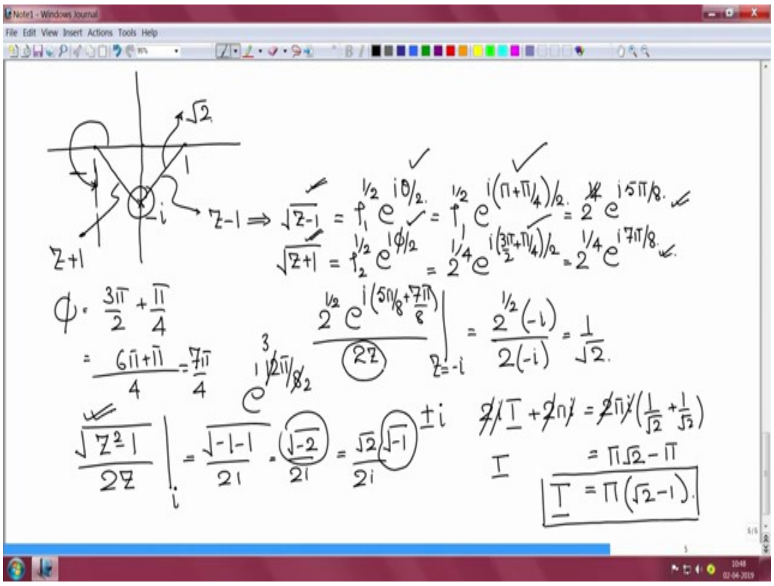

Let us see we will do it neatly here, this is my plus 1, this is my minus 1 and this here minus i is my singularity. So, this phasor is z minus 1, this phasor is z plus 1. And therefore, square root of z minus 1 is equal to rho 1 to the power half e to the power of i theta by 2. So, what is theta here? Up till here is pi, then I have pi by 4. So, rho to the power half, e to the power of i pi, plus pi by 4, by 2, which is and rho 1 is root 2.

So, this distance length is root 2, ok. So, I get 2 to the power one-fourth, e to the power i 5 pi by 8. And for square root z plus 1, this is rho 2 to the power half, e to the power of i phi by 2, which is equal to, let me write it rho 2 is again square root of 2 to the power half, 2 to the power one-fourth and here the angle is, I have up till here is 3 pi by 2 plus pi by 4.

So, 3 pi by 2, plus pi by 4 is my phi angle. So, which is equal to 4, 6 pi plus pi, which is 7 pi by 4. So, I get e to the power of i 3 pi by 2 plus pi by 4 by 2, which is equal to 2 to the power one fourth, e to the power of i 7 pi by 8, ok. And the denominator is 2 z as before. So, I have in the numerator, 2 to the power half, into e to the power of i 5 pi by 8, plus 7 pi by 8, divided by 2 z, evaluated at z equal to minus i.

Now, this is 12 pi by 8, e to the power of i 12 pi by 8, 4 2s 4 3s. So, e to the power i 3 pi by 2, which is minus i. So, I get 2 to the power half, minus I, by 2 z evaluated at minus I, minus I, hence 1 over root. So, I get 1 over root 2 in both cases. Now one thing, why we do this so, elaborately keeping the definition of theta and phi in mind is that, now I will tell you the mistake that one can do. You have square root of z square minus 1 by 2 z, while evaluating the residue at either i or minus i.

If, we did not account for the angles of root z minus 1 and root z plus 1 in this form, we will set z to be i and suddenly we have square root of i square minus 1, which is minus 1 minus 1 by twice i and square root of minus 2 by twice i.

Now, we do not know how to get square root of minus 2, because it is root 2 into root of minus 1. Now, this can be plus or minus i, this can be plus or minus i. So, and we do not know what to take? Ok. Where the square root problem is not there you can straight away take z to be i or minus i, but where you have this square root, ok. Because, it is not single valued and we have used a special branch for it, you have to use the theta definitions, otherwise you will make a mistake.

So, for example, here you do not know whether this is plus i or minus i, you do not take the right one you will get a wrong answer ok, divided by twice i. I can choose minus i in which case i get minus 1 by root. So, that is why this entire elaborate calculation has to be done, ok.

So, now what do we have, I have twice i I from last time, plus the twice pi I, which is coming from the CR integral, is equal to twice pi I, times 1 by root 2, plus 1 by root 2, ok. So, this is equal to, let me take out the i and 2s, i goes, i goes, i goes, 2. So, I is equal to pi into root 2 minus pi, which is equal to pi into, root of 2 minus 1, this is the answer ok.

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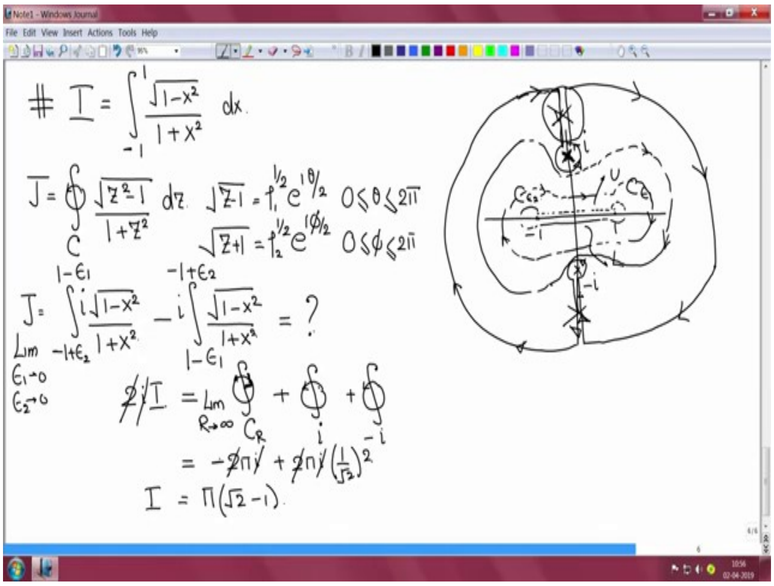

Now this integral, the same integral we will do it using another method. So, I is equal to integral minus 1 to 1, square root of 1 minus x square, by 1 plus x square d x, ok. We will replace it as before using a integral over a closed contour C and replace it with square root z square minus 1, by 1 plus z square d z, ok.

Now all definitions remain same z minus 1, z minus 1 square root defined as rho 1 to the power half e to the power of i theta by 2, 0 less than equal to theta less than equal to twice pi, then square root of z plus 1 is equal to, rho 2 to the power of half, e to the power of i phi by 2, 0 less than phi less than equal to pi they do not change.

And, the contour we now choose, is 1 minus 1 and we just go around the cut, this is the upper, go around epsilon, go down below, which is the lower and go around C epsilon 2. So, this is C epsilon 1, this is C epsilon 2, ok. Now, the function does have its isolated poles at plus i and minus i. Now, we will use the theorem which gives us the capacity to deform the contours, ok, where the function is analytic we are allowed to deform the contour. So, we will now so, let us write this first, I have the J given by, I will straight away take the expressions from last time.

So, it is going to be, i times square root of 1 minus x square, by 1 plus x square, ok. And, it was going from minus 1 plus epsilon 2 to 1 minus epsilon 1 and minus i, integral 1 minus epsilon 1 to minus 1 plus epsilon 2, square root of 1 minus x square, by 1 plus x square and the C epsilons were 0, ok. So, this is what I have, is equal to what is the question? So, now, the left side anyway since limit epsilon 1 and epsilon 2 go to 0, this is equal to 2 i times i. So, that part is set, ok.

Now, what I do is, I stretch this contour like a rubber band, ok, I start stretching this contour, keeping the direction in which I went in my mind, ok. I am stretching this contour, because I am allowed to stretch, ok. As long as I do not cross a singularity so, if a stretch like this, what I will get finally, these poles will behave like nails, I cannot cross them. So, it is like a stretching rubber band, this is like a stretched rubber band, but when I hit a pole, I cannot cross it, it is a nail. So, the string cast go beyond. So, ultimately, finally, I will come this way, come down this way, the pole prevents me from movement, I go around the pole, I come up, I go around, ok.

And, here again the pole prevents me from crossing. So, I come here, I come down straight, the pole prevents me from crossing, I come down and I go up and join up. So, now, see what happens? The function is well behaved here so, I have up and down integral, they cancel, the function is well behaved here, there is an up and down integral, they cancel. So, I am left with, this is equal to an integral over a circular contour of radius infinity going in the clockwise direction, sorry so, this is going in the clockwise direction, ok.

Plus integrations so, this is now in the counterclockwise direction, around the pole I, plus another integration around we come here so, this is also counterclockwise around minus i. So, last time we saw this integral in the counter clockwise gave me twice pi i. So, clockwise gives me minus twice pi I, ok, going around the poles give me residues. So, I get plus twice pi i, times residue, which is 1 by root 2 into 2, ok.

So, twice i, twice i, twice i, so, I get I is equal to pi into root 2 minus 1. So, I will stop here, there are a few comments I would like to make on this problem, I will make those ok.

Thank you we close here.