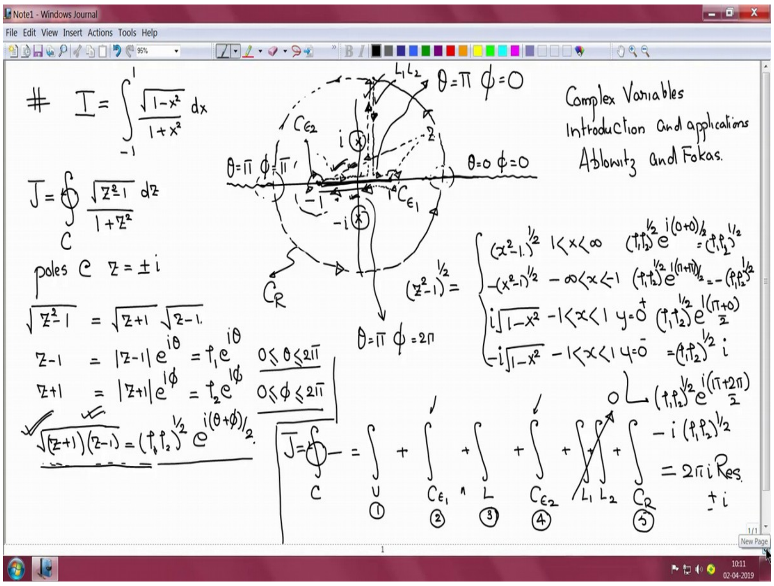
A short lecture series on Contour Integration in the Complex Plane

Prof. Venkata Sonti
Department of Mechanical Engineering
Indian Institute of Science, Bengaluru

Lecture - 17
Finite square root branch cut

Good morning, welcome to this next lecture on complex variables. We have been seeing a few examples involving Contour Integration and we even saw one example that involved a branch cut.

(Refer Slide Time: 00:49)



So, this next example involve a Finite branch cut and the integral we are interested in looks like this integral minus 1 to 1; square root of 1 minus x square, by 1 plus x square dx. Let me mention one more thing there is another good book on complex variables; its complex variables introduction and applications by Ablowitz and Fokas; two authors, some of the examples in fact, I have taken from this book, as this particular one. Now, as usual we are going to evaluate this real line integral using principles of complex variables.

So, as I have said before I will introduce a integral on a closed contour C and replace x with z, but make some slight change. So, I will make the numerator z square minus 1 dz, by 1 plus z square, so you will see why that small change has been done. Now, for one thing, this integrand has isolated poles at z equal to plus minus i, ok. And the contour I am going to choose, this maybe I will put it in the center; the contour I am going to choose is this.

So, I have 1 here, minus 1 here, I come from above. So, let me put my poles; one pole is here the other pole is here. So, I come from above, take a detour, go up to infinity; at infinity I take a left, go around, this is a circle at with radius infinity. I come back up here ok; I almost join up and I will join up, I come down take a detour, go around this, at an epsilon circle; come down below, go round here in an epsilon circle and join up. So, the directions are this, here I come down, ok.

So, this is a closed contour; if you see this is a closed contour that encloses this pole and this pole. And that is the only its one of the regions where the function is singular and there is going to be a cut over here, a finite branch cut over here, as discussed last time. So, let us see the definitions of z minus 1 and z plus 1, because we have a square root here, this involves the square root function, ok. So, let me write here perhaps let me write here.

So, let us say z square minus 1, square root, is given by square root of z plus 1 and square root of z minus 1. And z minus 1, I replace with magnitude of z minus 1 and e to the power of i theta, ok which is I will write as rho 1 e to the power of i theta and theta definition is twice pi.

Similarly, z plus 1 is magnitude of z plus 1; e to the power of i phi and I write this magnitude, it is a positive number, rho 2 i phi and phi definition is also twice pi. Now, we have seen last time that with this kind of a definition; the branch cut is going to be between plus z 0 and minus z 0; in this case plus 1 and minus 1 ok, that is the branch cut which we are not allowed to cross.

Now, the square roots now, ok; so what happens is square root of z plus 1, times z minus 1, are given by rho 1, rho 2 to the power half and e to the power of i theta plus phi by 2. Now let us see this function; what form it acquires in various portions of the complex plane as we move along the contour.

So, we have a region here greater than 1, we also have a region we have to think of less than minus 1; we will be moving just above the cut, we will be moving just below the cut. So, what is the form this function is going to take? So, let us see z square minus 1 to the power half is equal to ok; first let us think of 1 less than x less than infinity; so on the real axis here, on the real axis here ok.

Now, if I draw my phasors this is a point z, this is z minus 1, this is z plus 1; this is z. So, here if I look at square root of z minus 1; then theta here is 0, phi here is also equal to 0. So, I get rho 1, rho 2 to the power half; e to the power of i 0 plus 0 by 2 which is a positive real number rho 1, rho 2 to the power half. So, all over here I have a positive real number rho 1, rho 2 to the power half and therefore, this function will look like x square minus 1 to the power half.

What about minus infinity less than x, less than minus 1; what about here, what about in this range? Here again I have square root of z square minus 1 given by rho 1, rho 2 to the power half, but here theta is equal to pi, phi is also equal to pi. So, I get e to the power i pi plus pi by 2 and therefore, minus a real positive number. So, that would be x square minus 1 to the power of half, then minus.

So, now, what about the where we have minus 1 less than x less than 1. Just above the x axis y is equal to 0 plus, ok. Here rho 1 and rho 2 still positive real numbers , but right here we have seen last time theta is equal to pi; phi is equal to 0 and so I get e to the i pi plus 0 by 2. And therefore, rho 1 rho 2 to the power half a positive real number into i. So it will look like i times square root of 1 minus x squared. Then for the same part below the x axis; minus 1 less than x less than 1; y equal to 0 minus, we have again rho 1, rho 2 to the power half. But below, my theta is still pi, but phi is equal to twice pi, ok; the phi has to go a full round and come back here 2 pi.

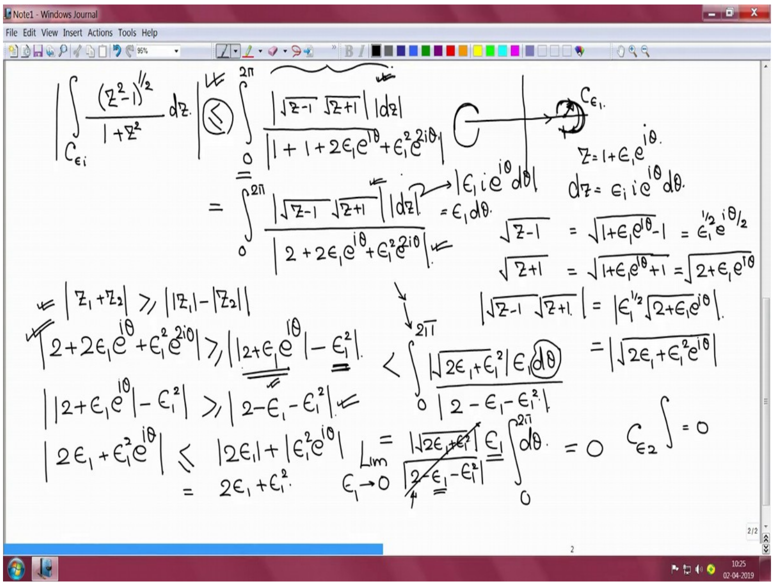
So, I get e to the power of i pi plus 2 pi by 2, it gives me minus i, a positive real number. So, here it will be minus i; square root of 1 minus x square because x is less than 1, ok. Now let us see how many portions or how many legs we have in doing this integral? I have J given by this closed contour C and the function which is equal to let us say the integral in the upper portion; we will call it 1, here as we move just above; just above the x axis, plus this I call C epsilon 1 and this I call C epsilon 2.

So, I have then integral C epsilon 1 which I will call it number 2, then I have integral going below the x axis. So, I will call thus that lower; lower portion 3, then I have integral C epsilon 2, which I have written there C epsilon 2. Then I have these vertical portions going up and going down; I will write them together, going up L 1 plus L 2 coming down L 2; so these are L 1 L 2. And then lastly this integral on a counterclockwise circle with radius infinity, I will call it CR.

So, plus integral CR that will be equal to twice pi i times, the residue at plus and minus i ok. So, now right away let me tell you that the function is not well behaved or not analytic at these isolated singularities plus minus i and around the branch cut; branch cut is where the function does not have unique values; so it is also a singularity, otherwise the function is analytic and well behaved everywhere.

So, now if in such a region I have an integral go goes up and comes down and these L 1, L 2 are very close or finally overlapping; then one going up and one going down they will cancel out. So, I will set them to 0, together they are 0; so we do not deal with them. Now, we have to deal with 1, 2, 3; C epsilon 2 we will call it 4 and CR we will call it 5; we will have to deal with 5 portions.

(Refer Slide Time: 18:29)



So, let us look at C epsilon 1 and C epsilon 2 in one shot. The C epsilon i in general; C epsilon i integral in general, ok; I will write it as C epsilon i, z square minus 1 to the power half, by 1 plus z square dz. Now, again we do the same argument, magnitude of this integral is less than or equal to the integral 0 to 2 pi of the magnitude. So, we have on top the magnitude of square root of z minus 1 and the square root of z plus 1, divided by ok, here also we have dz, divided by; now, let me just show it here, let us say we are moving, we are doing C epsilon 1.

So, I am here at one and I am looking at the C epsilon integral, ok; C epsilon 1 integral let us say, then z is moving on a circular contour. So, z will look like 1 plus epsilon 1 e to the power of i theta ok; z is 1; this 1 plus epsilon 1 e to the power of i theta, ok. So, we have 1 plus z square in the denominator; so it will be 1 plus z square is, 1 plus twice epsilon 1 e to the power of i theta, plus epsilon 1 square e to the power of twice i theta.

Now, one thing this angle here appears to be moving from pi to minus pi; here the angle will move from twice pi to 0. So, we should be careful to make this 0 to 2 pi; otherwise we may not get a positive answer over here; we are looking at the magnitude of this integral, so we should have a positive answer. And so I have set the limits to be 0 to 2 pi; it could be 2 pi to 0 here, it will be minus pi to pi to minus pi over here. So, we should be careful that the integral limits give this side a positive number; that we have to watch.

Now, in the denominator; so let me just say this is for now equal to, integral 0 to twice pi ,magnitude square root of z minus 1, square root of z plus 1 dz. And dz is equal to epsilon 1 i e to the power of i theta d theta, divided by magnitude 2 plus 2 epsilon 1 e to the power of i theta, plus epsilon 1 square e to the power of twice i theta, magnitude.

Now, in the denominator; so let us let us write the numerator and denominator separately here. So, square root of z minus 1, looks like, square root of 1 plus epsilon 1 e to the power of i theta minus 1. And square root of z plus 1, looks like, square root of 1 plus epsilon 1 e to the power i theta plus 1, ok.

So, this is equal to epsilon 1 to the power half; e to the power of i theta by 2 and this is equal to square root of 2 plus epsilon 1 e to the power i theta. So, together we have square root of z minus 1; square root of z plus 1, together and magnitude, we will get epsilon to the power half; magnitude, square root of 2 plus epsilon 1 e to the power of i theta magnitude.

And I will take the epsilon half inside; so I get magnitude square root of twice epsilon 1 sorry; epsilon 1, plus epsilon 1 square, e to the power of i theta, ok. Now, we have these inequalities in the complex plane with complex variables; which say that the magnitude of z 1 plus z 2 is greater than or equal to the magnitude of the magnitude of z 1 minus the magnitude of z 2.

So, the idea now is that we would strictly like to make this inequality a less than. So, we replace the numerator by something bigger and we replace the denominator by something smaller, ok. So, using this inequality, we say that the denominator; so we have 2 plus 2 epsilon 1, e to the power of i theta, plus epsilon 1 square e to the power twice i theta magnitude, is greater than or equal to the magnitude of the magnitude of 2 plus epsilon 1 e to the power of i theta close minus epsilon 1 square, ok.

Further, if epsilon 1 square is a small quantity, 2 plus epsilon 1 e to the power i theta is bigger than this, bigger than epsilon 1 square. If I replace this 2 plus epsilon 1 e to the power i theta by one more smaller quantity, then the difference will be further smaller. So, what I say, is that magnitude of magnitude 2 plus epsilon 1 e to the power of i theta minus epsilon 1 square is greater than or equal to magnitude 2 minus epsilon 1 e to the power of i theta minus; I am sorry minus epsilon 1 square.

So, I have replaced the denominator with a smaller expression. Similarly for the numerator, I have twice epsilon 1, plus epsilon 1 square, e to the power of i theta magnitude, ok; that is less than or equal to the magnitude of 2 epsilon 1 plus the magnitude of epsilon 1 square; e to the power of i theta ok, which is equal to twice epsilon 1 plus epsilon 1 square. So, here dz now is magnitude epsilon 1 i e to the power of i theta d theta and therefore, equal to epsilon 1 d theta.

Now, if I write it here; we get that equal to integral, 0 to I am not equal to strictly less than, 0 to twice pi, root of twice epsilon 1, plus epsilon 1 square; epsilon 1 d theta, divided by 2 minus epsilon 1, minus epsilon 1 square, can put magnitude or you cannot put does not matter, ok.

This way only d theta is part of the integral all the epsilons can be taken out. So, we get that this equal to magnitude square root of twice epsilon 1, plus epsilon 1 square, by magnitude 2 minus epsilon 1, minus epsilon 1 square, into epsilon 1 d theta and I have not been putting it but actually the limit epsilon 1 tends to 0.

And therefore, because there is this epsilon 1sitting over here and in the denominator 2 is the biggest term; this goes to 0, integral goes to 0. So, the magnitude of C epsilon 1 goes to 0, similarly the C epsilon 2 also goes to 0, the C epsilon 2 integral also goes to 0. Time has run out, so we will complete the problem in the next lecture; we will look at the other portions of the integral, ok.

Thank you, we close here.